



SYSTEM

NATURAL PHILOSOPHY

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O F
NATURAL PHILOSOPHY.

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O F
NATURAL PHILOSOPHY,

BEING
A COURSE of LECTURES
IN
Mechanics, Optics, Hydrostatics, and Astronomy;

Which are read in St Johns College Cambridge,

By T. RUTHERFORTH D.D. F.R.S.
CHAPLAIN TO HIS ROYAL HIGHNESS THE PRINCE OF WALES.

IN TWO VOLUMES.

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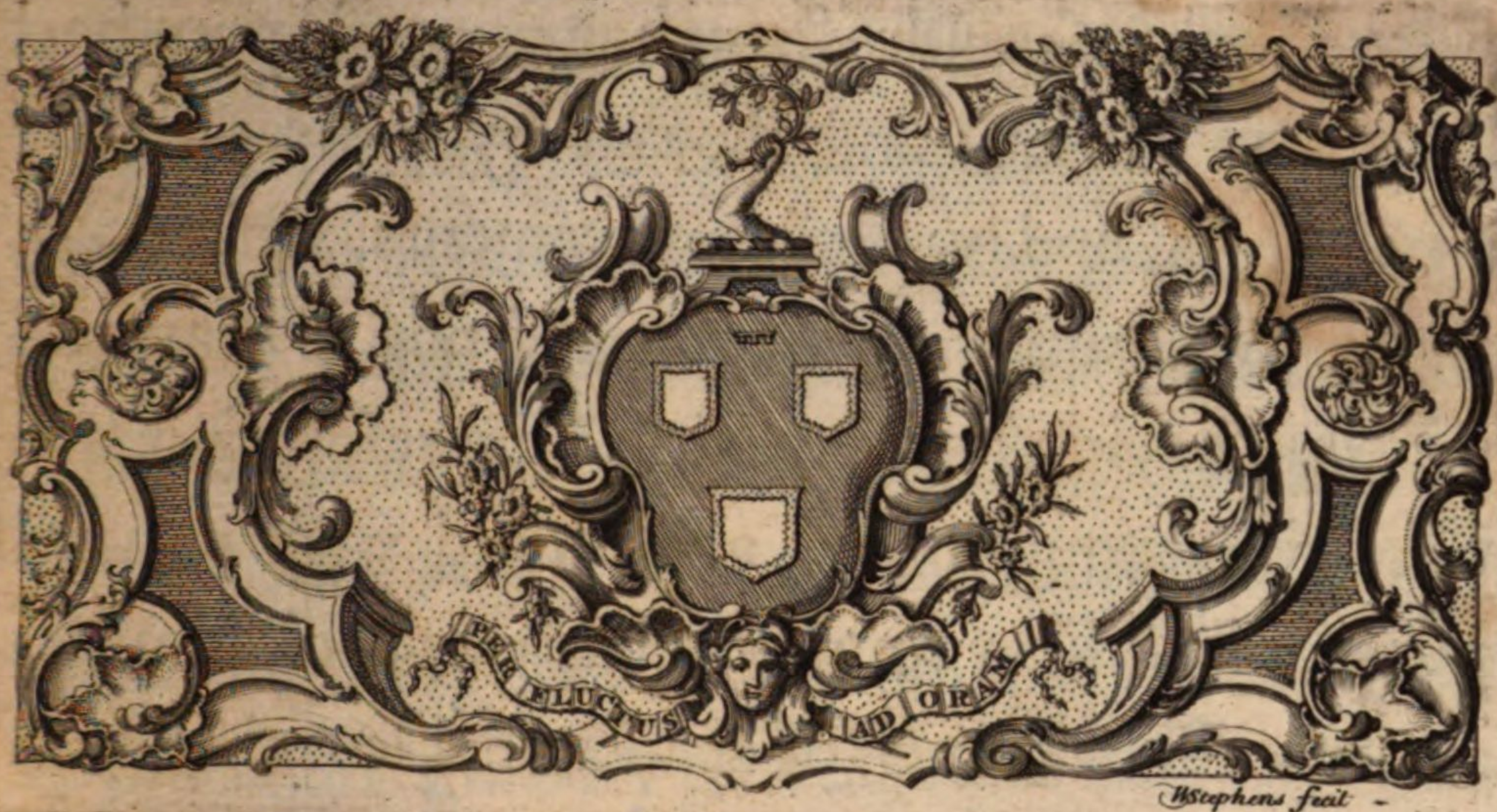
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OF
A COURSE OF LECTURES

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IN TWO VOLUMES

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T O

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SIR,

THE credit, which you did to yourself and to me, when you were examined for a degree in the university, was one motive that determined me to address the following sheets to you. The public honours, which you then received, were well known to be the reward of your merit, and not an unmeaning compliment, that was paid to your rank and fortune. You
were

DEDICATION.

were always sensible for what purpose you had been sent hither, and shewed by your diligent application, and by the improvements, which you made, that you were determined not to leave us without having answered it. This resolution was what first brought you acquainted with natural philosophy. You found that it was looked upon as a necessary branch of academical learning, and therefore thought yourself obliged to employ some part of your time in the study of it. There were indeed many other branches of learning, in which you excelled; and some of them, to a person in your station of life, may be more ornamental and more useful. Yet I am convinced that the subject of this book, as well as your friendship for the author, will make it an agreeable present to you: because you have often told me, that you were particularly entertained with this science, and that you wished I would give you an opportunity of keeping up your acquaintance with it by publishing my lectures.

But

DEDICATION.

But I had another motive for giving you this public testimony of my esteem and affection; besides the reputation, that you gained by the study of natural philosophy; and besides your kind partiality to my instructions: I mean, Sir, your regular and virtuous behaviour, whilst you were under my care in the university. If your virtue and regularity at that time had not been owing as much to your own good disposition, as to any restraints, that were laid upon you here; those restraints would have been disagreeable to you, and would have given you a distaste to the place, where your inclinations had been checked: we should not have found you so warmly attached, as you now are, to the interests of the university; nor should I in particular have received so many marks of your friendship for having endeavoured to discharge my duty faithfully, when I was entrusted with a share in your education.

I am sensible, Sir, that I shall be more likely to weaken your regard for me by this address,
than

DEDICATION.

than to confirm it: because those, who are the most careful to deserve well, are commonly the most displeased at being told of it, especially in so public a manner. However I shall willingly run this hazard: for I had rather offend your modesty, than not endeavour to do justice to your character: the former is a fault, which your good nature may forgive; but the latter is a crime, which I could not forgive myself.

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The reader is desired to correct proposition 50 of optics in the following manner.

When parallel rays pass out of a rarer medium into a denser through a convex surface, the semidiameter of the sphere, of which that surface is a segment, is to the distance of the focus whether affirmative or negative, that is, whether real or virtual, as the sine of refraction to the sine of incidence.

A SYSTEM OF NATURAL PHILOSOPHY.

INTRODUCTION.

IN explaining the appearances of nature, that alone can be called true philosophy, which is deduced from fact and experience. For as the causes of these appearances are not arbitrary notions of the minds own making, but are qualities, which really belong to bodies, and which exist without us; our knowledge of nature will all be fantastical; unless we find out by experiment and observation what these qualities are, and so make our knowledge conformable to the reality of things. Indeed some bodies are so remote, and others are so minute, that it is not in our power to make observations upon them, and discover their qualities: and amongst those, which are neither removed out of our reach by their distance nor hidden from our observation by their minuteness, there are but few that we can make experiments upon, in comparison of that almost infinite multitude, with which we are surrounded. So that if, in reasoning about natural things, we were to be confined wholly to experience, our knowledge must fall short of being general; for it would be confined to those particular bodies only, that we had made experiments upon.

The natural philosopher therefore, in order to make his knowledge real and universal, should observe the three following rules.

RULE I.

No more causes of natural appearances are to be admitted, than what are real, and what are sufficient to account for those appearances.

True philosophy can admit of no qualities in bodies, which have not been found by experience and observation really to belong to them: if we go beyond this, if in order to explain the appearances of nature we introduce such qualities, as no experiments have ever discovered; however we may amuse ourselves with ingenious conjectures and plausible suppositions, our knowledge will be fantastical, and our best digested systems will be only philosophical romances.

But

INTRODUCTION.

But if we are to admit of no causes but such as are proved by experiments to have a real existence; then we are necessarily to admit of no more than what are sufficient to account for the appearances of nature. When we have any appearance of nature to explain, and have by experiment found out a cause that is sufficient to explain it, here we must stop: for as the appearance does not require us to go farther, if we multiply causes beyond this, they must be needless, the appearance can be explained without them, and therefore we cannot from that appearance, collect the reality of their existence.

RULE II.

Natural effects of the same sort are to be accounted for by the same causes.

The descent of a heavy body towards the earth in Europe is an effect of the same sort with the descent of a heavy body towards the earth in America. We should not therefore assign different causes for such similar effects as these. In like manner the ascent of water into small tubes of glass and into the pores of a sponge are effects of the same sort: and when we have found out what cause produces one of these effects, we are to conclude that the other effect is produced by the same cause. The motion of the moon round the earth, of the satellites round jupiter or saturn, and of the primary planets round the sun, are effects of the same sort: therefore whatever we find to be the cause of one of these motions, the same we may conclude to be the true cause of the others.

This rule is deduced from the latter part of the foregoing one. Effects of the same sort may be accounted for by the same causes: for this reason we must always assign the same causes for effects of the same sort; because otherwise we should multiply causes without reason and should introduce more than are sufficient to account for the appearances of nature.

RULE III.

Such qualities, as are found invariably to belong to all bodies that we can make experiments upon, are to be looked upon as qualities belonging to all bodies whatever.

Though there are many bodies, upon which we have not, and many, upon which we never can make any experiments; yet if we have frequently made experiments upon other bodies, that fall immediately under our notice, and find them to be invariably endued with certain qualities,

INTRODUCTION.

lities, we may be allowed by induction to extend our conclusion to all other bodies, and so to make it universal.

All bodies that fall under the notice of our senses are extended; and though the constituent particles, of which larger bodies are composed, are too minute to be perceived either by the sight or touch; yet it is a fair conclusion, that these particles are endued with the same property. We find that all bodies, upon which we can make experiments, have solidity; and though there are other bodies, upon which we have never made the experiment, though the sun and planets are so remote from us that we cannot make it, yet we may be allowed to conclude that these bodies have likewise the same property. All bodies, that we have observed, are found to gravitate towards one another; they endeavour, if they are near the earth, to descend by their weight towards the earths center. Now though we have not tryed the experiment upon every stone, or upon every piece of lead, that we see, yet from what we have tryed we conclude that all of them have this same quality. And if this conclusion is just, when it is extended thus far beyond our own observation, we may likewise extend it still farther. No one, after what he has experienced in all sorts of bodies, that he has been used to, can reasonably doubt whether other bodies of different sorts have not gravity as well as these: and from what he has observed in all bodies where he can make experiments, he may conclude that this property of gravity belongs to all bodies universally, and that the moon gravitates towards the earth. This is what we call reasoning by induction, when upon finding a thing to be true in all instances where we have an opportunity of examining it, we conclude it to be true in all instances whatever. And it is by the help of this reasoning that in natural philosophy we are enabled from a sufficient number of particular experiments to draw general conclusions.

A SYSTEM

A SYSTEM OF NATURAL PHILOSOPHY.

M E C H A N I C S.

CHAP. I.

Of matter and its properties.

- I. *By matter is meant an extended, solid, and moveable substance, that has a force of inactivity.*

TO know what extension is, we must be sent to our senses for information. It is a simple idea, and therefore cannot be defined. All we can do is to express the same thing in different words. Thus instead of telling our hearers what extension is, we only endeavour to shew them, where they may find it; when we say, that the extension of matter is its length, breadth, and thickness.

From the resistance, which we find in any one body to the entrance of any other body into the place it possesses, till it has left that place, we get the idea of solidity. This is likewise a simple idea and incapable of being defined: the best and indeed the only way of making it understood is to describe, as well as we can, the manner in which we acquire it; and then our senses will lead us to the true meaning of the word. But we must be careful to distinguish between solidity and hardness. These two words are used promiscuously in common life, as if they both stood for one and the same property. Whereas solidity is a property of all bodies; hardness only of some. Indeed hard and soft are names, that we give to things only in relation to the constitutions of our own bodies: that being called hard by us, which will put us to pain, sooner that change figure by the pressure of any part of our bodies; and that on the contrary, soft, which changes the situation of its parts upon an easy and unpainful touch. Therefore the solidity of matter consists in an utter exclusion of all other matter out of the space it possesses; but

hardness in a firm cohesion of its parts, making up masses of sensible bulk, so that the whole does not easily change figure. We never say that air or water are hard, nor do we in our common way of speaking say that either of them are solid. But yet solidity as now explained is as much a property of air or water as it is of a flint or a diamond. The softest body in the world will as invincibly prevent the coming together of any two other bodies, if it be not put out of the way, but remain between them, as the hardest, that can be found or imagined. He, that thinks otherwise, may easily satisfy himself by trying whether he can bring his hands together, when he has nothing between them but a full-blown bladder. The two flat sides of two pieces of marble will indeed more easily approach each other, when there is nothing but a drop of water between them, than they would have done, if there had been a diamond between them. But this is not because the parts of the diamond are more solid than those of the water; but because the parts of the water being more easily separable from each other, will by a side motion be more easily removed, and give way to the approach of the two pieces of marble. But if they could be kept from making way by that side motion, they would eternally hinder the approach of these two pieces of marble, as much as the diamond; and it would be as impossible by any force to surmount their resistance, as to surmount the resistance of the parts of the diamond. Mr. Locke, to prove that water has solidity, mentions an experiment that was made at Florence with a hollow globe of gold filled with water and exactly closed. For the globe thus filled, being put into a press, which was driven by the extreme force of screws, the water made itself way through the pores of that very close metal, and finding no room for a nearer approach of its particles within, got to the outside, where it rose like a dew, and so fell in drops, before the sides of the globe could be made to yield to the violent compression of the engine that squeezed it. This experiment will indeed prove that gold, as close and compact a metal as it seems to be, is very full of pores. Or it will prove that water is an incompressible fluid; for no force, that could be applied, was sufficient to bring the particles nearer together. But how does it demonstrate that water is solid? What, because the sides of the globe could not be made to yield to the compression, till the water had made itself a way through the pores of the gold? I suppose then, we must have granted, that water has not this property, if the force of the screws had made the sides of the globe yield, whilst all the water continued within it? And yet, if the globe had been filled with air instead of water, notwithstanding the solidity of the
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the air, the sides would have been pressed inwards, though none of the air had escaped through the pores of the gold. Not because the air has not the property, that we are now describing; but because it is a compressible fluid, so that the particles may be squeezed closer together, and the same quantity of it may be thus made to fill a lesser space. The experiment is plainly applyed by this author to an improper purpose, and is so far from clearing up the idea of solidity, that it has only served to perplex and mislead his readers.

We may venture to say that all matter is, by the very essence of it, a moveable substance, without giving ourselves the trouble to enquire with the metaphysicians, whether it is not possible for matter to be infinite. For though, if it was infinite and so filled all place, it could not be moved from one place to another: yet we are to remember, that all our ideas of substances are fantastical, which are made conformable to no pattern existing, that we know of, and that consist of such a collection of ideas as no substance ever shewed us united together. This sort of ideas being made in reference to things existing without us, and being intended to be representations of substances, as they really are, can be no farther real than as they are combinations of simple ideas, which are really united and coexist in things without us. The metaphysicians, who determine the essences of things by fancy and supposition without regarding what matter is, perplex themselves with endeavouring to find out what it might be; and, if they think they can conceive a possibility of its being infinite, they immediately determine that a capacity of being moved from place to place is not an essential property of it. But our notions in natural philosophy are not formed by our own imaginations, but are taken from fact; we endeavour to make our ideas conformable to the reality of things, as we find them, and do not amuse ourselves with fancying what possibly might have been. Therefore as matter is finite, we reckon a capacity of being moved among its essential properties, and never give ourselves the fruitless trouble of enquiring, whether the infinity of matter is a possible supposition.

The force of inactivity is that property of matter, whereby all bodies, that are at rest, resist being moved, and all that are in motion resist being stopped in proportion to the quantity of matter contained in them. Matter is not simply inactive; it is not merely unable, when at rest, to put itself in motion, and, when in motion, to stop itself: for if this was all; bodies would only be indifferent as to motion or rest, and any force however small applyed to any body however large, would be able to change the state of the body; that is, would be able either to move

it, if it was at rest, or to stop it, if it was in motion. But bodies, we find, are not only incapable of producing motion or rest in themselves, but likewise make such a resistance to the production of either, that there is some difficulty in causing them to change their present condition; some force is required both to produce motion in them, and to destroy it. And this force must be so much greater as the quantity of matter is greater in the body, to which the motion is to be communicated; or in which it is to be destroyed. If a force that we will call A was sufficient to move a body as B with a certain degree of velocity; then to move 2B with the same velocity 2A or double the force would be requisite; and 3B or three times the quantity of matter could not be moved with the same velocity, unless 3A or a triple force was applied. In the same manner; if the force A could stop the body B, when it was moving with a certain degree of velocity; to stop 2B or twice the quantity of matter moving with an equal velocity the force must be doubled; and 3B or three times that quantity, could not be stopped unless the force, was tripled. The counter force which the bodies exert, the resistance, which they thus make to a change of their present state either of moving or resting, is the property of matter, that we are now describing, and is called the force of inactivity.

To this definition of matter it might be expected that I should add a definition of motion; but I rather chuse to transcribe a passage out of Mr. Lockes essay concerning human understanding as an excuse for omitting it. The names of simple ideas are incapable of being defined. The reason whereof is this, that the several terms of a definition signifying several ideas, they can altogether by no means represent an idea, which has no composition at all. And therefore a definition, which is properly nothing but shewing the meaning of one word by several others not signifying each the same thing, can in the names of simple ideas have no place. The not observing this difference in our ideas and their names has produced that eminent trifling in the schools, which is so easy to be observed in the definitions they give us of some few of these simple ideas. For as to the greatest part of them, even those masters of definitions were fain to leave them untouched, merely by the impossibility they found in it. What more exquisite jargon could the wit of man invent, than this definition, The act of a being in power as far forth as in power? Which would puzzle any rational man, to whom it was not already known by its famous absurdity, to guess what word it could be the explication of. If Tully asking a dutchman what beweegeing was, should have received this explication in his own lan-

language, that it was *Actus entis in potentia quatenus in potentia*; I ask whether any one can imagine he could thereby have understood what the word *beweeginge* signified or have guessed what idea a dutchman ordinarily had in his mind and would signify to another when he used that sound. Nor have the modern philosophers, who have endeavoured to throw off the jargon of the schools, and speak intelligibly, succeeded much better in defining simple ideas, whether by explaining their causes or any otherwise. The atomists, who defined motion to be a passage from one place to another, what do they more than put one synonymous word for another? For what is passage other than motion? And if they were asked what passage was; how would they better define it than by motion? For is it not at least as proper and significant to say passage is a motion from one place to another, as to say motion is a passage from one place to another? This is to translate, and not define, when we change two words of the same signification one for the other: which, when one is better understood than the other, may serve to discover what idea the unknown stands for; but is very far from a definition, unless we will say, every english word in the dictionary, is the definition of the latin word it answers, and that motion is a definition of *motus*. Nor will the successive application of the parts of the superficies of one body to those of another, which the Cartesians give us, prove a much better definition of motion, when well examined.

I have not included divisibility in the definition of matter, because though all matter is divisible, yet as this cannot be reckoned a primary property, but is only a consequence of extension, there was no more reason for making this a part of the essence of matter than innumerable others, which might be named. For since a definition of matter would fill a volume, if it was to take in all those properties of it, which follow from the primary ones; many of them must be omitted, and most of them commonly are omitted. I saw no reason therefore for including this in particular in the definition of matter; the essence of it may be understood without this as well as it can without the other properties of the same sort, which instead of being made parts of the definition are usually deduced as consequences from it. Thus having defined matter to be an extended substance we may from thence collect the following proposition relating to its divisibility.

2. Matter is infinitely divisible.

The meaning of this proposition seems to have been but little understood even by those, who have demonstrated the truth of it. They
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have not attended to the idea, which the word *infinitely* stands for, and by that neglect have laid themselves open to objections which they knew so little how to answer, as to make a very judicious and thinking writer say that the infinite divisibility of a finite extension, involves us, whether we grant or deny it, in consequences impossible to be explicated, or made in our apprehensions consistent; consequences that carry great difficulty and some apparent absurdity.

But if we attend to the idea of infinite as the same writer has explained it, we shall be less perplexed in our enquiries concerning the divisibility of matter than he seems to have been himself. Whoever thinks on a cube of an inch diameter, has a clear and positive idea of it in his mind, and so can frame one of $\frac{1}{2}$ an inch, or $\frac{1}{4}$, or $\frac{1}{8}$ and so on, till he has the idea in his thoughts of something very little; but yet reaches not the idea of that minuteness, which division can produce. For after he has continued this halving in his mind and has diminished his idea as much as he pleases, he has no more reason to stop, nor is one jot nearer the end of such division, than he was at first setting out. Nor is it here an insignificant subtilty to say, that we are carefully to distinguish between the infinite divisibility of matter, and matter infinitely divided. The first is nothing but an endless progression of the mind in dividing the parts of matter as long as it pleases: but to talk of matter as infinitely divided; is either taking it for granted that in our division of it we must come to an end at last, and this is begging the question; or else it is supposing the mind has already passed over and has actually a view of all those parts of matter, which an endless division can never totally represent to it; which carries in it a plane contradiction.

Plat. I. Fig. 1. will shew us that any part of extension is infinitely divisible, or that after we have divided as long as we please we have no more reason to stop nor are one jot nearer the end of such division than we were at first setting out. For let AB be drawn perpendicular to TM, and FG parallel to it: then on C as a center with the opening of the compasses CT strike the arc STL and it will divide the line FG at p. If the semidiameter had been longer the arc would have approached nearer to the right line AB and would have divided the line FG nearer to F: thus if E is the center RTD passing through o will be the arc. And in the same manner if M was the center the arc would still approach nearer to F. Since therefore as the radius lengthens the line PF is divided into constantly less and less parts by the constant approach of the arc to F. And since the mind has a power of lengthening the line TM at pleasure without being ever obliged to stop; it follows that

that the division of PF is endless; unless we could at last come to some arc which the same right line AB would touch in two points one at T the other at F. But this is impossible; Euclid. book III. prop. 2. corol. Therefore any part of extension, and consequently matter, because it is extended, is infinitely divisible.

Nor can we well conceive it to be otherwise. For if any particle of matter is placed upon a plane, the bottom of the particle only can touch the plane and the top will not touch it: because, if it were otherwise, the particle would have no thickness, and so must be a mathematical surface and not matter. But where there are two parts an upper and a lower, a division is always possible. And since this would be the case, however small the particle was, it follows, that when we have divided as long as we please, a farther division will be possible. Unless division could cause that matter should cease to be matter by destroying the extension of it in some one of its dimensions.

3. *The earth attracts all bodies, that are near it: and the force, with which bodies when they are so attracted, tend toward the earths center, is called gravity.*

I have not reckoned gravity amongst the essential properties of matter, though we find it in all bodies, upon which we can make experiments. For this I had two reasons. The first is, that no particle of matter can gravitate unless there be some other particle, which attracts it: so that gravity is a relative property, as it carries the mind beyond the particle that is possessed of it, to something else, which acts as if it was the cause of gravity, and towards which this gravitation is made. My other reason for leaving gravity out of the definition of matter is, that I had reckoned the force of inactivity amongst the essential properties of it: and at the same time to have reckoned gravity amongst them would have been making the parts of the definition inconsistent. It would have been saying, that matter makes a resistance to motion in all possible directions, and yet that in one direction it will begin to move of its own accord. We must therefore leave either gravity or the force of inactivity out of our definition; and as the gravity of bodies is a particular exception from the universal force of inactivity; there can be no difficulty, when we are defining matter, to determine, which of the two properties we should take into the definition, and which we should leave out of it.

4. *The*

4. *The attraction of cohesion is that force, by which the smallest, or constituent, particles of all bodies tend towards each other, and are united together.*
5. *The repulsive force of matter is that, by which its particles, when they are too far asunder to attract, are found to repel each other.*

There is a remarkable instance of this attractive and repulsive force between the edge of a knife and the rays of light; as they pass by it, when they are let into a dark chamber through a hole in the window-shutter. For those rays, which pass close to the knife, are bent downwards by it; those, which pass at greater distances being attracted less by it, are less bent downwards; and those, which pass at still greater distances, are repelled by it, and in their passage are bent the contrary way. That there are in nature such forces as these is evident enough. The several particles, which compose a stone, or a piece of iron, are united by some force or other, so as to form one compact mass: and this force is what we call attraction or cohesion. The same force is called by the same name in all other bodies: and the attraction of cohesion considered as a primary law of nature means that force, by which the particles of matter are united to each other. A repulsion likewise appears in some instances. It seems to be such a force as this which shakes off the particles of light from the sun, a candle or any other body which emits light. The same force appears, when any oily substance is placed on the surface of water; for by a repelling force between the two substances, the surface of the water will be depressed about the oil, that is laid on it.

But these attractive and repulsive forces, though they may be proved to exist in nature, are frequently applied in philosophy to cases, which they do not sufficiently explain. Aqua fortis will dissolve iron. If we ask why? the answer immediately is, because the particles of iron are more attracted by those of aqua fortis, than either the particles of iron are by each other or the particles of aqua fortis by each other. For which reason the particles of the iron are separated and so are those of the aqua fortis by means of this stronger attraction: and by the same mutual attraction these two substances will be mixed and blended together. But then this aqua fortis will not dissolve gold? And since gold is of a looser texture; that is, since its parts cohere less forcibly together than those of iron do, ought not the same force, which will separate the particles of iron, to be even more than sufficient to separate those of gold? Here it is answered; there is an attraction between iron and aqua fortis; but none
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between gold and aqua fortis. If we ask why aqua fortis, which will not dissolve gold, will dissolve silver? the answer is ready and is the same as before; aqua fortis attracts silver more than the particles of silver attract each other; but either does not attract gold at all, or else the mutual attraction of the particles of gold amongst one another is greater than the force, with which they are attracted by the aqua fortis: and this weaker force cannot so overcome the stronger as to separate or dissolve the gold. Ask again, why aqua regia, which will dissolve gold, does not dissolve silver? they answer again that aqua regia attracts gold, but does not attract silver. These answers have something in them very like the occult qualities of the Peripatetics. For as they used to assign as many different occult qualities as there are different appearances to explain; so the philosophers, who give these answers, introduce as many different sorts of attractions, as there are bodies to be dissolved and fluids to dissolve them.

We may shew still farther how unsatisfactory a method this is of accounting for dissolutions; by asking those, who maintain it, what they mean by the word *attraction*. Put the word *drawing* instead of it, and then we shall find that what they say may amuse, but does not instruct. I enquire why aqua fortis, for instance, will dissolve iron, or draw the particles of a mass of iron asunder. I am answered because aqua fortis attracts those particles more than they are attracted by each other, which is just the same as to say that aqua fortis draws them asunder with a greater force than they draw one another. But this is no better than assigning the fact as a reason for itself: it is saying that aqua fortis will draw the parts of iron asunder, because it has a force sufficient to draw them asunder: which is a very unsatisfactory account of the matter, when I wanted, not to be informed that aqua fortis has such a force, for that my senses prove to me, but to know what that force is.

I would here enquire of these philosophers what attraction this is, which dissolves hard bodies in fluid ones? Is it the attraction of cohesion? If they say it is not this sort of attraction, but quite a different one; I do not see how they will get rid of the charge of introducing so many different sorts of attractions into nature, as to make their explanation of things by attraction little better than the Aristotelian explanation of them by occult qualities. But if they say that the attraction, to which dissolutions are owing, is the attraction of cohesion, it will be very easy to shew that they are mistaken. I suppose it will be readily granted that a body is so much the harder as the parts of it are more firmly united to each other, or cohere more strongly together. From

hence it follows, that those bodies will be the hardest, whose parts attract each other the most forcibly. Now in this account of dissolution the particles of iron are said to be attracted by aqua fortis more forcibly than they are one amongst another. Therefore a dissolution of iron in aqua fortis, that is, a mixture or mass consisting of particles of iron and aqua fortis, ought to be a harder body than iron, or than a mass, which consists of none but particles of iron. Whereas in fact the dissolution, instead of being harder than iron, is as much a fluid as aqua fortis itself.

We may make this argument still more forcible by another instance. For as in a dissolution of aqua fortis and iron there is more of the fluid body than there is of the hard one; it might perhaps be objected, that this is the reason why the mass is fluid. This objection is indeed of very little weight: because the mass or mixture, if it does not become harder than iron, ought at least to be much harder than aqua fortis alone. But we may quite avoid the objection, if we use the instance of Epsom salts and common water. For any quantity of common water will dissolve its own weight of Epsom salts. So that in the dissolution or mixture of these two bodies, there are just as many particles of the salts as there are of the water. And as the attraction of cohesion is supposed more strong between each particle of salt and each particle of water, than it is between any two particles of salt, it follows that the mixture ought to be harder than salt. Whereas it continues as fluid as common water. From whence I think we may conclude that this dissolution is not owing to the attraction of cohesion.

6. *A body is called an hard one, if its parts make so forcible a resistance to every impression, that can be made upon them, as not at all to yield to it.*

7. *A body is called elastic, if its parts yield to the impressions, that are made upon them; and then restore themselves again to their former situation with a force equal to that, which made them change it.*

8. *A fluid is a body, whose parts yield to any force impressed upon them, and in yielding are easily moved one amongst another.*

CHAP. II.

Of the three laws of motion.

9. *The velocity of a body moving uniformly is directly as the space described by it, and inversely as the time of its motion.*

THAT is, the line, which a body runs over in any certain time, will be so much the longer, as the body moves faster, and so much the shorter as the body moves slower. And the time, which it takes up to run over a line of a certain length, will be so much the shorter as the body moves faster, and so much the longer as the body moves slower. Since therefore the velocity is directly as the space described, when the time of motion is given, and inversely as the time, when the space is given, it will, where neither of them is given, be in a ratio compounded of both, that is, directly as the space described and inversely as the time of motion.

Let the velocity of a body $= v$, the time of motion $= t$, and the space described $= s$; then from what has been said it follows that v , is always as $s \times \frac{1}{t}$ that is as s multiplied into $\frac{1}{t}$ or the reciprocal of t , or as $\frac{s}{t}$, or the velocity is as the space described divided by the time of motion.

10. *The spaces described by bodies moving uniformly are as the velocities and times of motion conjointly.*

That is, the line run over by a body in any certain time will be so much the longer as the body moves faster, and so much the shorter as it moves slower. And this line, where the velocity is given, will be so much the longer as the body moves for a greater time, and so much the shorter, as it moves for a less. Therefore the space described being as the velocity, when the time is given, and as the time when the velocity is given, will be always in a ratio compounded of both or as the time and velocity conjointly. So that if the velocity of a body A is to the velocity of any other body B as 2 to 3, and A moves for 6 minutes and B for 8, then will the spaces described by these two bodies be in a ratio compounded of 2 to 3 and 6 to 8, that is, they will be as 2×6 and 3×8 , or as 12 to 24.

This proposition may be deduced from the foregoing one, for since v is as $\frac{s}{t}$ it follows that, multiplying both sides by t , vt is as s .

The spaces described by bodies moving uniformly, that is, the proportions, which the lines they run over bear to each other, may be represented geometrically. If the time of A's motion is to the time of B's motion as CD to GH Plat. I. fig. 2. and A's velocity to B's as DL to HR , then will the line described by A be to the line described by B as the area of the rectangle under CD , DL , to the area of the rectangle under GH , HR . For the lines described by the two bodies are to each other in a ratio compounded of the times and velocities, that is, of CD to GH and of DL to HR : and the areas of these rectangles being in a ratio compounded of their two sides are likewise as ED to GH , and as DL to HR conjointly. Euclid, book VI. prop. 23.

11. The first law of motion is, that all bodies continue in the same state of resting or of moving uniformly in a strait line: unless some external force impressed upon them makes them alter that state.

The first part of this law, that all bodies continue in a state of rest, till some external force puts them in motion, follows from the very nature of matter. For whatever is capable of moving itself, must be called by some other name, since it is a part of the essence of matter not only to be inactive or unable to produce motion in itself, but even to make a resistance to motion, when communicated to it by any external cause. It will be as easy to prove the second part of this law to those, who understand what this force of inactivity means. Because as all bodies make a resistance to being stopped, they cannot be conceived more capable of stopping themselves, when they are in motion, than they are of moving themselves, when they are at rest. Some external force is equally necessary to produce either of these changes in them; and such a force too as is proportionable to the resistance, which they make. This continued motion of bodies would, if they were left to themselves, be uniform; that is, bodies can neither be accelerated nor retarded, but by some external force. For they cannot move faster without some additional motion, nor can they move slower without losing a part of what they had before; therefore since they cannot of themselves either produce motion or destroy it, they will persevere in a state of moving uniformly, till some external force makes them change that state. And for the same reason they will of themselves move on in a right line, because changing their direction is the same thing as moving themselves another way.

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If a vessel full of water is suddenly moved forwards upon a horizontal plane, the water, which hangs loose in the vessel or does not adhere to it so as to be a part of it, will not have the motion of the vessel communicated to it immediately, but persevering in a state of rest will begin to rise up against that side of the vessel, where the moving force acts, and being by this means made to swing in the vessel will dash over. But when once the water has the motion of the vessel communicated to it, and begins to move with a velocity equal to that of the vessel, then it will persevere in a state of motion, and if the vessel is suddenly stopped, it will run against the opposite side and by rising up there will be made to swing again and to dash over as before.

When a horse is running with his full speed, the rider moves along with him the same pace: and if the horse stops at once, the rider, unless he sticks close, so as to make in a manner one mass with the horse, will continue to move forwards and will be thrown out of the saddle towards the horses head. But if a horse from standing still sets out on a sudden to run with his full speed, the rider continuing in a state of rest will be thrown out of his saddle the contrary way.

Whilst a stone is whirled round in a sling it describes a circle; for at that time it is acted upon by two forces. But if the sling was to break, by which means one of the forces is removed, the stone would then fly off, as by this law it ought to do, in a line, which is a tangent to the circle, that it described before. The stone does not indeed continue to move for ever, for its own weight makes it fall, and by striking against the ground it will be stopped: but this is agreeable to what the law affirms, for it does not stop of itself, but is stopped by the earth, which is an external force. It is its weight, another external force, which changes the direction of its motion, and makes it, instead of going on in a right line describe that curve which brings it to the ground.

12. *The second law of motion is, that the changes made in the motion of bodies are always proportional to the moving forces impressed, and are produced in the same right line, in which those forces act.*

A change is made in bodies in respect of their motion, when they are moved, or stopped, or accelerated, or retarded, or have their direction altered: and this law relates alike to all these changes. By a moving force, we are not here to understand a force that actually produces motion, but a force which might produce motion and which applied to a body at rest would produce it. For in this law not only those forces, which move a body or accelerate it, are called moving forces, but those
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likewise which stop or retard it. That is in short, all forces are here called moving ones, which produce any change whatever in the motion of bodies.

That all effects are proportional to their adequate causes, is an axiom that every one allows to be true. And as the first part of this law is nothing but an application of this general proposition to the particular case of motion, we may take the truth of it for granted without giving ourselves the trouble to find out any formal proof of it.

By the first law of motion every moving force produces its effect in a right line, and no moving force can act where it is not, from whence it follows that all changes made in motion, because they are the effects of a moving force, will be produced only in that right line, in which the force is impressed.

But before I leave this law, it may not be improper to take particular notice of two consequences that may be deduced from it, which I shall endeavour to explain and establish in the two following propositions.

13. *The moments of bodies, that is, their quantities of motion, are as the velocities of the bodies, and as the quantities of matter contained in them conjointly.*

The quantity of motion, by the law just explained, is always proportional to the moving force. But whatever degree of our strength is required to throw a ball of lead that weighs 1 pound with a certain velocity, we know that double the force is requisite to throw a ball of 2 pounds weight with the same velocity, and triple the force to throw one of 3 pounds. Therefore, when the velocity is given, the force impressed on any body to move it, and consequently the motion produced in it, is as the quantity of matter, which it contains. We know likewise that whatever degree of our strength is required to throw a ball weighing 1 pound with such a degree of velocity, as we will call 1, twice the force would be requisite to throw the same ball with a double velocity, and thrice the force to throw it with a triple velocity. Therefore, when the quantity of matter is given, the force impressed on a body to move it, and consequently the motion produced in it, is as the velocity with which it moves. Since then the moment is as the quantity of matter, if the velocity is given, and as the velocity, if the quantity of matter is given; it follows, that if neither of them are given the moment will be as both of them conjointly, or as the product arising from the quantity of matter multiplied into the velocity.

14. *The moments of bodies are equal, if their velocities are reciprocally as the quantities of matter contained in them.*

For, by prop. 13. if the quantity of matter is called q the velocity v and the moment m then m is always as $q \times v$, that is as the quantity of matter multiplied into the velocity. But here we suppose that v is as $\frac{1}{q}$, that is reciprocally as q or the quantity of matter. Therefore $q \times v$ is as $q \times \frac{1}{q}$ or as $\frac{q}{q}$, and since $\frac{q}{q}$ expresses an invariable quantity the moment must be invariable in the case supposed, or when the velocity is reciprocally as the quantity of matter.

Let the weight of a body A be 4 ounces, and that of a body B 10, then if the former moves with 5 degrees of velocity and the latter with 2 their velocities are inversely as their weights or quantities of matter: for 5 is to 2 as 4 to 10 inverted, or as 10 to 4. The moment of A in this case $= 4 \times 5 = 20$ and that of B $= 10 \times 2 = 20$. Where it is obvious, that just as much as the moment of A is less than that of B, on account of the less quantity of matter in A; just so much the moment of A is greater than that of B on account of the greater velocity, with which A moves: so that taking both accounts together the moment of one must be equal to that of the other.

15. *The third law of motion is, that reaction is always contrary and equal to action; or, the mutual actions of two bodies upon each other are equal and in contrary directions.*

From the force of inactivity it follows, that whatever change is made in the state of one body A, whether at rest or in motion, by the action of another B, the same change is produced in the state of B by the resistance, which A makes to that change. Thus, if a man in a boat A draws another boat B by means of a rope fastened to it: the boat A will be equally drawn towards B and they will approach each other with equal quantities of motion; that is, by the last proposition, with equal velocities, if their weights are equal; or, if their weights are unequal, with velocities reciprocally as those weights. From whence we may easily see the reason why by pulling the rope, when it is fastened to a tree or a post upon the bank we shall draw the boat up to the shore. For the velocities produced on each side, that is, in the boat and in the bank will be reciprocally as the quantities of matter. And since the quantity of matter in the boat is as nothing compared with that in the earth
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of which the bank is a part, it follows that the velocity in the bank will be as nothing; and consequently all the velocity that is produced, will be in the boat.

A person, who is in a boat by pulling with ever so great a force at a rope, which is tied to the head of it, will not be able to stop it, when in motion, nor to move it, when at rest. For just as much as he acts one way with his hands, just so much he reacts the other way with his feet: and these two equal and opposite forces destroying each other, will leave the boat in the same condition of resting or moving, that it would have been in, if he had not pulled at the rope at all. Whereas, if he had been upon the shore, when he pulled the rope, he would then have drawn the boat towards him. For then his reaction would have been exerted upon the ground, where he stood; and the boat sustaining only the force of his action would have moved in the direction, in which he pulled it.

If a horse draws a stone forwards, the stone draws the horse back equally: that is, whatever motion the horse communicates to the stone, he will lose the same quantity himself. For suppose him able, if he had no stone behind him, to push forwards with his breast a weight of a 1000lb. it is evident, that when he has a stone of 400lb. to draw, so much of his strength will be laid out that way, as to leave him able to push forwards with his breast no greater weight than $1000 - 400 = 600$ lb. He has therefore lost a quantity of motion forwards exactly equal to what he has communicated to the stone; or as much as he draws the stone one way, so much it draws him the contrary, and, these equal and contrary forces destroying each other, he will go on with the difference.

CHAP. III.

Of the composition and resolution of forces.

16. *A body, that is acted upon by two forces together, will describe the diagonal of a parallelogram in the same time that it would have described the sides, if the forces had acted upon it separately.*

IF there was a force which acting alone upon the body A Plat. I. fig. 3. in the direction AB would carry it to B in a certain time; and if there was another force, which acting alone upon the body A in the direction AC, would carry it to C in an equal time: then, making AB and AC the two sides of a parallelogram and drawing the diagonal AD,

I say, that these two forces acting together will make the body describe this diagonal in the same time that one of them alone would have made it describe A B, or the other of them alone would have made it describe A C.

Let the force, which acts in the line A B, be called X, and that, which acts in the line A C, be called Z; then in a certain time the force X will impell the body through a space equal to A B, or, since A C and B D are parallel, through a space equal to the distance between those two lines. And because the direction A C, in which the force Z acts, is parallel to B D; therefore by the second law of motion, this force will not hinder or destroy the velocity of the body, by which the force X makes it tend towards the line B D. Consequently at the end of the time the body will be found somewhere in that line B D. In like manner the force Z will impell the body in the same time through a space equal to A C, or since A B and E D are parallel, through a space equal to the distance between those two lines. And, because the direction A B, in which the force X acts, is parallel to C D, therefore, as before, this force will not hinder or destroy the velocity of the body, by which the force Z makes it tend towards the line C D. Consequently, at the end of the time the body will be found somewhere in that line C D. But it cannot be found at the same time both in B D and C D unless it is at D the intersection of these two lines. And by the first law of motion it must move in a right line from A to D. Therefore it will have described the diagonal A D.

If two men set upon the opposite sides of a boat, whilst the boat is going on, and toss a ball backwards and forwards to each other, they will catch the ball in their turn, just as they would have done, if they had both stood still upon the shore. The ball is here acted upon by two forces: it goes along with the boat, whilst it flies through the air, just in the same manner as it would have done, if it had not been thrown, but had laid still in the boat. This motion common to the ball, the boat, and the men is one of the forces. The other force is that, with which one of the men throws it across to the other. By these two forces together the ball will describe the diagonal of a parallelogram, one of whose sides is the line that the boat has described, whilst the ball is flying across, and the other side is a line drawn from one man to the other. Let A C be the breadth of the boat; let A be the place where one of the men sets, and C the place where the other sets: and whilst one side of the boat is moving in the direction A B, and the other side of it in the direction C D, let the ball be thrown from A to C with such a velocity as will carry it to C just when the boat has got into the situation B D. Now

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if A had not thrown the ball at all, but had continued to hold it in his hand, then it would have moved on along with him and the boat from A to B. This motion, which is common to the boat, the man, and the ball, will continue the same when the ball is thrown, that it was whilst A held it in his hand. For though it might at first be produced in the ball by being held in his hand, as he was carried along with the boat; yet, when once produced it will continue; by the first law of motion. But whilst this force is carrying the ball from A to B, the force with which it is thrown will carry it from A to C or directly across the boat. And at the instant it has flown across, the man, who was at C when it was first thrown, will be got to D, because at that instant the boat will be at BD. Therefore, since the ball has gone on in the direction AB, CD, as fast as the boat has done, and in the mean time has flown across the boat; just when the man arrives at D the ball will be there too, and he will catch it in the same manner that he would have done, if both he and the person who threw it had stood still. And any one, who stands upon the bank, will in the mean time see the ball describe the diagonal AD.

Suppose a ruler AC to be carried with an uniform motion parallel to itself, and with such a velocity as would bring it into the situation BD in four minutes time. Then a fly standing upon one end of the ruler at A and carried along with it, would in the first minute describe the line AE, in the second EF, in the third FG, and in the fourth GB. But if the fly was in the mean while to walk upon the ruler with an uniform velocity, which would carry it in four minutes the whole length of the ruler or from A to C; then at the end of the first minute, or when the ruler has moved one quarter of the way from A to B and is in the line EM, the fly will have walked down one quarter of the length of the ruler and will be at S. In the same manner, when the ruler has moved half way and is at FN; the fly will have walked half down it and will be at T. When the ruler is at GO, the fly will be at V. And when the ruler, at the end of the four minutes, is in BD, the fly will have walked the whole length of it and will be at D. But A, S, T, V, and D are all of them in the diagonal AD. And, in the same manner, if we had divided the line AB and ruler AC into ever so many more parts than four; we might have shewn that the fly would at the end of every such part have been in some point of the same line AD. Therefore the fly, acted upon by two forces, one of which is common to itself and the ruler, and the other the force, by which it walks upon the ruler, is made to describe the diagonal of a parallelogram AD in four minutes time; that is, in the

the same time that one of the forces alone would have made it describe AB or one side of that parallelogram, and the other alone would have made it describe the other side AC.

Here we must observe that the velocity produced by the compound action of these two forces X and Z, that is, of the two forces, of which the former acts in the direction AB and the latter in the direction AC, is proportional to the diagonal or space described in a given time. Thus the velocity produced by X is to the compound velocity as AB to AD, and the velocity produced by Z is to the compound one as AC to AD. By proposition 9.

18. *If a body is impelled at the same time by two forces, that act in the same direction; its velocity will be as the sum of those forces.*

If two forces, which are to each other as AB to AL, Plat. I. fig. 5. act upon a body at the same time; then by the last proposition the velocity produced will be as AR the diagonal of a parallelogram, whose sides are AB and AL. Suppose the direction of the force that acts in the line AL to continue the same, whilst the direction of the other force approaches to it; then the velocity produced by the two forces, when they act in the lines AL and AC, will be as the diagonal AF. And when they act in the lines AL and AD, the velocity will be as the diagonal AS. Thus we find that the diagonal lengthens, that is, the velocity of the body increases, as the angle at A contained between the directions of the two forces becomes more acute. But when AD lies in the direction AL, or when the two forces act in the same direction, then AL and LS will lie straight out, and the diagonal AS by coinciding with them will be equal to AL added to LS, or to AL added to AB, since AB is equal to LS. (Euclid. book I. prop. 34.) That is, the diagonal, and consequently the velocity produced, will be as the sum of the two forces AB and AL, or as these two forces added together.

19. *If two forces, that act in contrary directions, impell a body at the same time; the velocity produced in it will be as the difference of those forces.*

By the joint action of two forces, as AB and AC, Plat. I. fig. 6. the velocity produced in the body impelled by them will be as the diagonal AD. Let the direction of the force AC continue; then as the direction of the other force grows more oblique to AC, the diagonal shortens, that is, the velocity produced by the two forces diminishes. Thus, when they act in the directions AC and AR, the diagonal is AO; and when in the directions AC, and AS, it is AG. Till at last, when AS and AC lie straight out from each other, that is, when the two forces act in contrary directions,

rections, then, making CE equal to CG, CG will coincide with CE, and the diagonal becomes AE. But because $CG = AS$, (Euc. book I. prop. 34) therefore $CE = AS$, and this diagonal AE is equal to $CA - AS$; that is, to CA with CE or AS taken from it. From whence it appears that the diagonal being the difference of the two forces, the velocity of the body will likewise be as that difference.

The body, by the second law of motion, must necessarily move in the direction of the stronger force, if one be stronger than the other: but if the two forces are equal, that is, if their difference is equal to nothing, then the velocity likewise $= 0$ and the body acted upon by them will by their joint action be made to rest.

20. *A motion, though ever so simple, may be resolved into the action of two forces: and the directions of the two forces, which might have produced such a motion, may be infinitely varied.*

Thus a motion in the line AM, Plat. I. fig. 7. may be considered as the effect of two forces acting in AL and AN the two sides of a parallelogram of which AM is the diagonal. For, by prop. 17. these two forces would have produced such a motion. It may for the same reason be considered as the effect of two forces acting in AB and AI the two sides of another parallelogram, of which the same line AM is the diagonal. And in the same manner the direction of the forces, which might have produced this motion, may be varied as much as the direction of the two sides of a parallelogram may, whilst the same line AM continues the diagonal.

The third law of motion alone is sufficient to explain the manner in which a bird by the stroke of its wings is able to support the weight of its body. For if the force, with which it strikes the air below it, is equal to this weight; then the reaction of the air upwards is likewise equal to it: and the bird being acted upon by two equal forces in contrary directions, that is, by the reaction of the air and its own gravity, will rest between them. If the force of the stroke is greater than its weight, then the reaction of the air upwards being made greater than the force of gravity downwards, the bird will rise with the difference of these two forces. And if the force of the stroke be less than its weight, then it will sink with the difference. By prop. 19.

But to explain the manner, in which a bird by striking its wings downright in the lines DH, ES, OR, IG, Plate I. Fig. 4. perpendicular to the horizon, is able to move itself along the line MT parallel to the horizon, we must, besides the third law of motion, have recourse to the pro-

proposition just now explained. For the wings of a bird are not like the paddles of a boat, they are not struck backwards in order to produce a motion in the contrary direction. But the stroke being made in the lines DH, IG, &c, the wings yield to it, and turning round a little, one on the joint I and the other on a joint of the same sort on the other side, the ends of them A and F are made to approach each other, and the planes of them become oblique to the line MT. Suppose therefore LN to be perpendicular to the surface of one wing, when it is bent in the manner we have been describing. Then, since by the second law of motion, the action of the wing upon the air is in lines perpendicular to its surface, the reaction, by the third law, will be in the line LN or oblique to the line MT, in which the bird is to move. But this oblique force may be resolved into two others LV and VN, of which LV is perpendicular to the horizon and supports the bird, the other VN is parallel to the horizon and will carry it along in the line MT.

21. *If three forces act upon a body at the same time, and the body rests between them; then those forces are to each other as the three sides of a triangle, that are respectively parallel to the directions, in which they act.*

If a body at D, Plat. I. fig. 8. that is acted upon by three forces in the directions DA, DC, and DB, yields to none of them, but is by their joint action made to continue at rest: then, if we continue AD to E, and draw EF parallel to BD, and EG parallel to CD, the three sides of the triangle EDF are parallel to the directions of the three forces. For DE is parallel to DA: DF is parallel to DE because it is a part of it: and EF is parallel to BD. I say therefore that the forces, which act in the lines DA, DC, DB, are to each other as the sides of this triangle DE, DF, and EF respectively. For since the body D rests, it follows, from proposition 19, that, if the two forces acting in the lines DB and DC are compounded into one in a direction DE opposite to DA, then this compound force must be equal to the simple one DA. But the three forces DE, DC, and DB, that is, the compound force and the two out of which it is compounded are to each other as DE the diagonal of the parallelogram DGEF, and the two sides of it DF, and DG = EF respectively. By prop. 17. Therefore since the compound force DE is equal to the simple one that acts in the direction DA, the three forces acting in the directions DA, DC, and DB, are to each other as the three lines DE, DF, and EF, respectively, that is as the three sides of a triangle, which is formed by lines drawn parallel to those directions.

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We have an instance of a body resting, whilst three forces act upon it, in a boys kite as it rests in the air: for it is acted upon by the wind, by its own weight, and by the string that holds it. There is another instance, to which we shall have occasion to apply this proposition, when we come to shew what force is sufficient to support a heavy body upon an inclined plane.

CHAP. IV.

Of gravity.

22. *Any force, which is propagated through right lines in all possible directions from one and the same center, is, at different distances from that center, inversely as the squares of the distances.*

LET A represent the center, Plat. I. fig. 9. from which the force is propagated; HLB a great circle in a sphere, through whose surface the force is spread at the distance AB; HGC a great circle in the sphere, through whose surface the same force would spread itself at the distance AE; and let ALG, ACF, ABE, be some of those lines, through which this force is propagated. The greater the space is, through which this force is diffused, so much the weaker will be its action: that is, the action of it at the distance AE will be every where so much weaker than it is at the distance AB, as the space, through which it spreads itself at the distance AE, is greater than the space, through which it spreads itself at the distance AB. Thus if A was a candle, and the rays of light proceeding from it were to describe the lines ALG, ACF, ABE: it is evident, that the light would be stronger at B or L than at E or G; because at E or G the rays are spread farther asunder, and consequently fewer of them will fall upon any given space than at B or L. Now at the distance AE the force is spread through the surface of a sphere, whose semidiameter is AE, and at the distance AB it is spread through the surface of a smaller sphere, whose semidiameter is AB. Let $AE = D$ and $AB = d$, then since the force is greater as the space through which it spreads itself is less, it will be as much greater at the point B than it is at the point E, as the surface of a sphere whose semidiameter is d is less than the surface of one whose semidiameter is D , that is, the force will be as those surfaces reciprocally. But the surfaces of spheres are to each other as the squares of their semidiameters. The surface of the sphere whose semidiameter is d , is to the surface of the sphere whose semidiameter is D , as the square of d to the square of D ; that is, as d^2 to D^2 , or as dd to DD ,

DD; for these two squares are expressed either of these ways. Therefore the force at the distances AB and AE, being reciprocally as the surfaces through which it is spread, is reciprocally as the squares of those semidiameters, or as $\frac{1}{dd}$ to $\frac{1}{DD}$, or as the reciprocals of dd and DD.

These two fractions $\frac{1}{dd}$ and $\frac{1}{DD}$, are called the reciprocals of dd and DD. Because all fractions, whose numerators are given, are inversely or reciprocally as their denominators. For where the numerator is given, the value of the fraction decreases as the denominator increases, and vice versa. It will be easy to understand this in numbers. For of two fractions $\frac{1}{2}$ and $\frac{1}{4}$, which have the same numerator 1, the latter $\frac{1}{4}$ is just as much less than the former $\frac{1}{2}$, as 4 is greater than 2. That is, by increasing the denominator from 2 to 4, the fraction decreases from $\frac{1}{2}$ to $\frac{1}{4}$; or by doubling the denominator, the value of the fraction becomes but half what it was before.

Or otherwise. Every fraction expresses a quotient arising from the division of the numerator by the denominator. And in all division, where the dividend is given, the quotient is greater as the divisor is less, or less as the divisor is greater: that is, the quotient is reciprocally as the divisor. Therefore a fraction, where the numerator is given, is reciprocally as its denominator.

23. *The force of gravity from the surface of the earth upwards is, at different distances from the earths center, as the squares of those distances inversely.*

That is, call the gravity of a body 1 when it is at the earths surface or at the distance of one semidiameter from the center: then at the several distances 1. 2. 3. 4. &c. the gravity of it will be as the squares of those distances inverted. The squares are 1. 4. 9. 16. &c. if we invert them, or, which amounts to the same thing, take the reciprocals of them, they are $1. \frac{1}{4}. \frac{1}{9}. \frac{1}{16}.$ &c. and in this proportion the force of gravity decreases, whilst the distances from the center increase in the proportion 1. 2. 3. 4. &c. For gravity is a force propagated through right lines in all possible directions from one and the same center: because heavy bodies, which are made to descend by this force, describe right lines, and gravitate in all parts of the earth, or on all sides of the earths center, and by the force of gravity are made to tend towards this center. But by the last proposition every force of this sort, and consequently the force of gravity, is, at different distances from the center, in a reciprocal duplicate ratio of those distances, or as the squares of them inversely.

versely. The force of gravity below the earth's surface is varied by a different law from this: and in order to find out that law, we must prove the following proposition.

24. *If in a concave spherical surface the particles that compose it attract with forces, which, at different distances from them, are as the squares of the distances inversely; that is, if they attract according to the above-mentioned law of gravity; then a body within the sphere will remain at rest wherever it is placed.*

It is self-evident that a body, Plat. I. fig. 10. placed at C the center of the hollow sphere IHQLK would rest, because it would be equally attracted all round, and those equal and contrary attractions would destroy each other. If it was placed any where else, as at P, then indeed it would be nearer to one side HI of the sphere, than to the other side KL, and would, by the last proposition, be attracted more towards H than towards K, upon account of the distance PH being less than the distance PK. But the quantity of matter or number of attracting particles, and consequently the attraction, will be greater towards K than towards H. If therefore the attraction towards K on account of the quantity of matter is just as much greater than the attraction towards H, as it is less on account of the greater distance, then it will follow that the body P will rest, because it is equally attracted both ways.

Let $PK = D$ and $PH = d$, then, by the last proposition, the force, with which the body is attracted towards K, will be to the force, with which it is attracted towards H, as the squares of PK and PH inversely, or as $\frac{1}{DD}$ to $\frac{1}{dd}$ on account of the different distances from K and H. If we imagine in the spherical surface a circle of matter whose diameter is KL, and from K through I draw KPH, and from L in the same manner draw LPI; then between these two lines KH and LI, a circle of matter, whose diameter is KL attracts the body one way, and another circle of matter whose diameter is IH opposite to the former attracts it the other. Now if L is taken infinitely near to K, and I infinitely near to H, the two triangles KPL and HPI will be similar, and their sides (Euc. book 6. prop. 4.) being proportional, it will be as $PK = D$ to $PH = d$, so KL to IH. But these opposite circles, whose diameters are KL and IH, or whose diameters are to each other as D to d, contain quantities of matter proportional to their areas, because we either consider the spherical surface as having no thickness, or else, which amounts to the same thing, suppose the thickness of it to be the same in

in every part. And since the areas of circles are to each other as the squares of their diameters, (Euc. book 12. prop. 2.) it follows that the quantity of matter and consequently the force, by which the body is attracted towards K, is to the force, by which it is attracted towards H, as DD to dd. Thus the attraction towards K, on account of the greater quantity of matter on that side, has been shewn to be as DD, but on account of the body's greater distance as $\frac{1}{DD}$, therefore on both accounts together as $DD \times \frac{1}{DD} = \frac{DD}{DD} = 1$: whilst the attraction towards H on account of the less quantity of matter on that side is as dd, on account of the body's less distance as $\frac{1}{dd}$, therefore on both accounts together as $dd \times \frac{1}{dd} = \frac{dd}{dd} = 1$. But if the attraction towards K = 1, and the attraction towards H likewise = 1, then they are equal to each other, and the body when placed at P will continue at rest.

25. *The force of gravity below the earth's surface is, at different distances from the center, directly as those distances.*

That is, the gravity of a body at P, plate I. fig. 11, is as much greater than the gravity of the same body at Q would be, as PC is greater than QC. Or, if a body was to fall through a hole bored from the earth's surface at B to its center at C, the gravity of it would constantly diminish, till at last, when it came to the center, or when its distance from the center = 0, the gravity likewise would = 0.

This perhaps will appear probable, if the reader considers what would be the case, when the stone is at the center. There is then just as much of the earth on any one side of it, as there is on the opposite side. And consequently it is equally attracted all ways, and these equal forces in contrary directions will destroy each other, and cause the stone to rest; that is, the stone will not be moved any way by the force of gravity, or will not gravitate at all. But when the same stone is at the surface, it is attracted only one way, only towards the earth's center; and when it is any where between the surface and the center, it is attracted both towards the center, and by as much of the earth as it has left behind it in falling. From whence it is evident that the gravity of the stone must be greatest of all at the surface; because when it is there, the whole earth attracts it downwards: whereas when it is any where else, a part of the earth attracts it downwards, and a part attracts it upwards. But since the weight of the stone is nothing at the center, and

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is

is greatest of all at the surface ; it follows that the force of gravity decreases, as the stone falls downwards from the surface towards the center.

Let $PC = D$ and $QC = d$; then, by proposition 23, the gravity of the body, when at P , is to the gravity of it, when at Q , as $\frac{1}{DD}$ to $\frac{1}{dd}$. But when the body is at P all that part of the earth, which is included between $ABEM$ and $IHKL$ is a hollow sphere, and such a sphere, by the last proposition, attracts it equally all ways ; therefore the body is made to gravitate towards the center only by the attraction of the solid sphere $IKHLC$. And when it is at Q , all that part, which is included between $ABEM$ and QR attracting equally all ways, it will be made to gravitate towards the center only by the attraction of the solid sphere QRC . Now the gravity of the body caused by the sphere $IKHLC$ is greater than the gravity caused by the sphere QRC , for their attractions, and consequently the weights produced by those attractions, are to each other as their solid contents, that is, (Eucl. book 12. prop. 18.) as the cubes of their semidiameters, or as PC^3 to QC^3 or as DDD to ddd . Therefore the gravity of the body, when at P , is on account of its distance from C as $\frac{1}{DD}$; on account of the quantity of matter contained in the solid sphere, which attracts it, as DDD , and on both accounts together as $DDD \times \frac{1}{DD} = \frac{DDD}{DD}$, or as the cube of the semidiameter divided by the square of it, which gives D or the semidiameter itself in the quotient. And for the same reason the gravity of it, when at Q is as $ddd \times \frac{1}{dd} = \frac{ddd}{dd} = d$. That is, the gravity of the body at P is to its gravity at Q as D to d or as PC to QC ; or as its distance from the center.

But suppose there was a hole bored from the earths surface to its center, might we not ask why a stone or any other heavy body, would fall down such a hole till it came to the center ? For since we have just now seen, that the gravity or weight of the stone is greatest of all when the stone is at the surface, and decreases, whilst it is falling, so as to be nothing at the center : ought not the stone to stop at that place where its gravity is greatest, that is, at the surface, and not fall towards the center, where its gravity is nothing ? But the answer to this is obvious. The stone gravitates only towards the center when it begins to fall. And afterwards when it is below the surface it gravitates both towards the center, which is below it, and towards that part of the earth, which is above it, and beyond which it has fallen. This makes its weight less

as

as it descends ; for when it begins to gravitate both ways, it will move one way only with the difference of these two forces. And when the stone gets to the center, this difference is nothing, for it is equally attracted all ways, and consequently will not move any way. But notwithstanding in the whole course of its fall it has gravitated towards the surface as well as towards the center ; yet it could not either stop at the surface, or ever move upwards from the center ; because of these two forces that which acted towards the center was always the greatest. So that though the stone has less weight as it gets nearer to the center, yet this weight always tends downwards ; and the stone will always fall downwards : only it falls with the greatest force, when it is at the surface, and with a less force as it approaches the center.

26. *All bodies near the earths surface, when they fall perpendicularly by their own weight or by the force of gravity, and meet with no resistance, move with equal velocities.*

In the air indeed a piece of lead will fall faster than a feather, but this is owing to the resistance which they meet with from the air. For in an exhausted receiver, if they are both let fall from the top of it at the same time, they will both come to the bottom together ; and therefore having fallen from the same height in the same time they must have moved with equal velocities.

27. *The weights of bodies, or their gravities near the surface of the earth, are as the quantities of matter contained in them.*

Any one body is just as much heavier than any other, as it contains a greater quantity of matter. For, by proposition 26, all bodies put in motion by the force of gravity move with equal velocities. Therefore, by prop. 13, because the velocities are given, the moments are as the quantities of matter. And since, by the second law of motion, the moments are as the moving forces, it follows that the moving forces are likewise as the quantities of matter. But when bodies fall by the force of gravity, the moving forces are their own weights. Therefore their weights are as their quantities of matter.

28. *The pressure of a fluid upon the constituent particles of bodies is not sufficient to account for their gravity.*

A piece of lead in the form of a bullet has the same weight, that it has, when beaten out into a thin plate. Whereas, if gravity had been owing to the pressure of a fluid, by increasing the surface, we

should find the pressure, and consequently the gravity arising from that pressure, increased at the same time. It is to no purpose to say here that the fluid is so fine as to penetrate the pores of the lead and to press only upon the solid parts of it; and therefore since there is the same number of solid parts in the bullet as there is in the plate, the pressure ought to be the same. For they, who know any thing about the pressure of fluids, understand, that more parts are exposed to this pressure in the plate than in the bullet. Suppose for instance that in any one line drawn through the bullet perpendicular to the horizon there are 2000 solid parts one under another, then the whole column of fluid, which presses upon these 2000 particles is no more than what rests upon the uppermost particle in the line. But imagine, when the lead is beaten out into a thin plate that this line is reduced to ten other shorter lines, in each of which there will be no more than 200 particles one under another: then it is plain, that each of these lines bearing the column that presses upon the highest particle will be as much pressed as the 2000 were pressed before, and consequently that the weight of these 2000 particles would be ten times greater in the plate than it was in the bullet, even though the fluid pressed upon the solid parts only. Or suppose we turn the edge of the plate of lead upwards; then a fluid which presses downright, will certainly act with less force to make the lead descend, than it would have done if the flat side of the plate had been upwards. And yet we find that the weight of the plate is the same in both positions of it.

29. *The whole world is not full of matter, but there are in nature spaces, that are quite empty, or mere vacuities.*

For if all parts of the world were full of matter, then all bodies of the same size must contain equal quantities. A cubic inch of cork for instance must contain just as much matter as a cubic inch of lead does. And since the weight of all bodies, by proposition 27, is as the quantity of matter contained in them, the cork must be as heavy as the lead. But their weights, we find, are different: from whence we must conclude that there are in some bodies interstices void of all matter, and that the lighter body must have more of these interstices than the heavier has. It cannot be replied here, that the cork would be lighter than the lead without supposing any mere vacuities in either of them; if the pores of the cork instead of being quite empty were filled with a finer and more subtil matter than those of the lead. Indeed this answer would destroy our former reasoning, if it could be made appear that the
weight

weight of matter, the quantity of it being given, could be diminished by the fineness or subtilty of its parts. But since all matter, that we can make experiments upon, whether the parts of it be fine or coarse is found to gravitate in proportion to its quantity; though the pores of the cork were filled with a finer fluid than those of the lead, yet both being full of matter they must, as their dimensions are equal, contain equal quantities of it, and therefore one must necessarily weigh just as much as the other.

In this chapter the word gravity is used in three different respects, or gravity is spoken of as capable of being greater or less in reference to three different measures. In proving the 25 proposition, the gravity of a body caused by the sphere IKHLC, Plat. I. fig. 11. is affirmed to be greater than the gravity caused by the sphere QRC. Here gravity is considered as greater or less in proportion to the cause, by which it is produced. It is referred to the same measure when we say that gravity would be greater than it is, upon supposition that the quantity of matter in the earth was double or triple what it is at present.

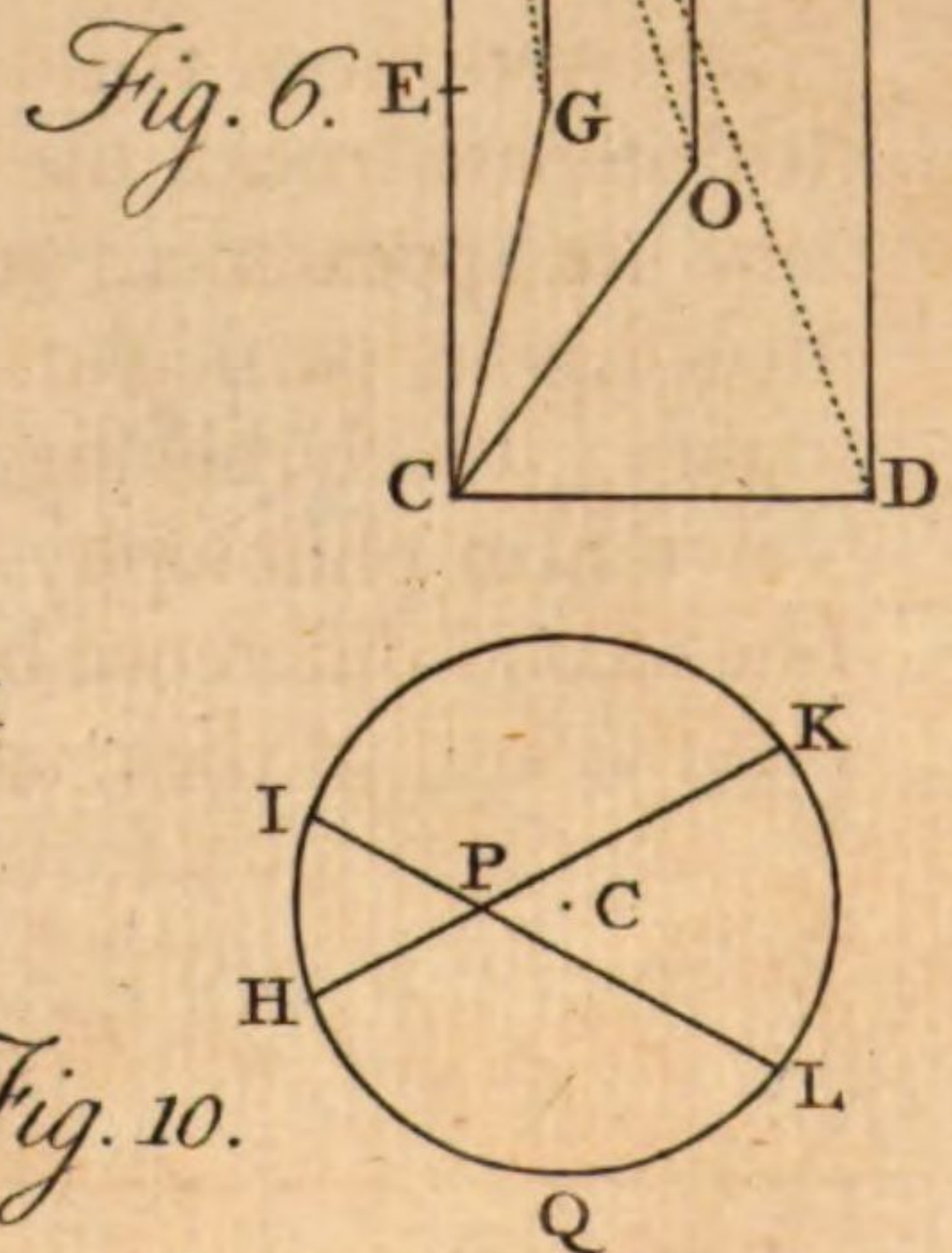
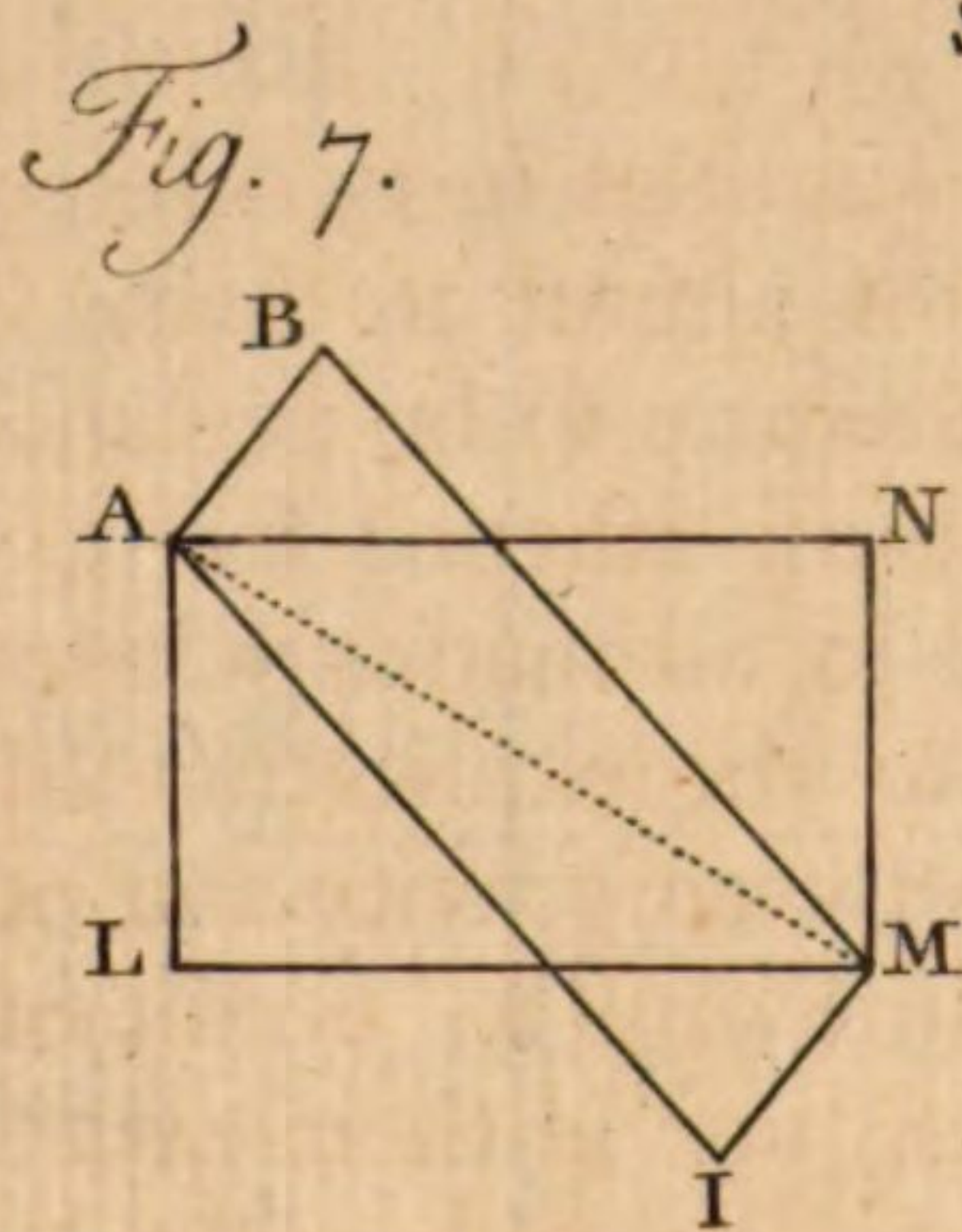
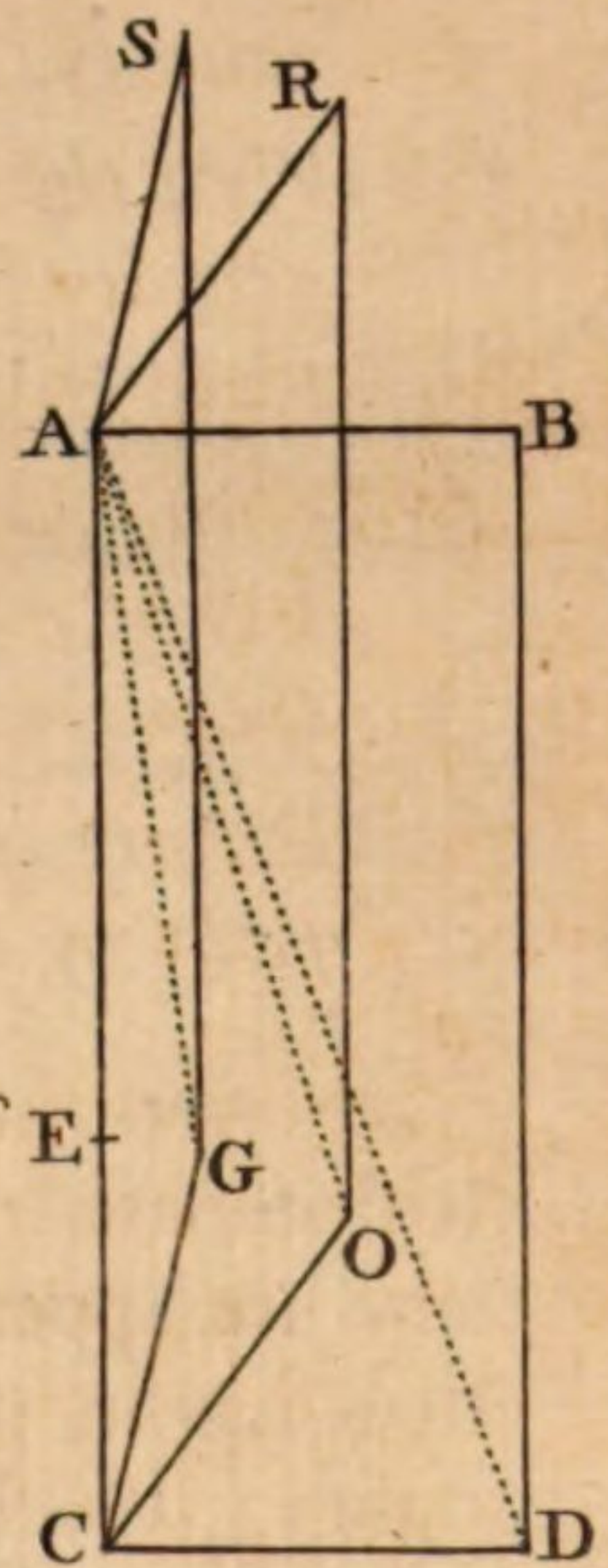
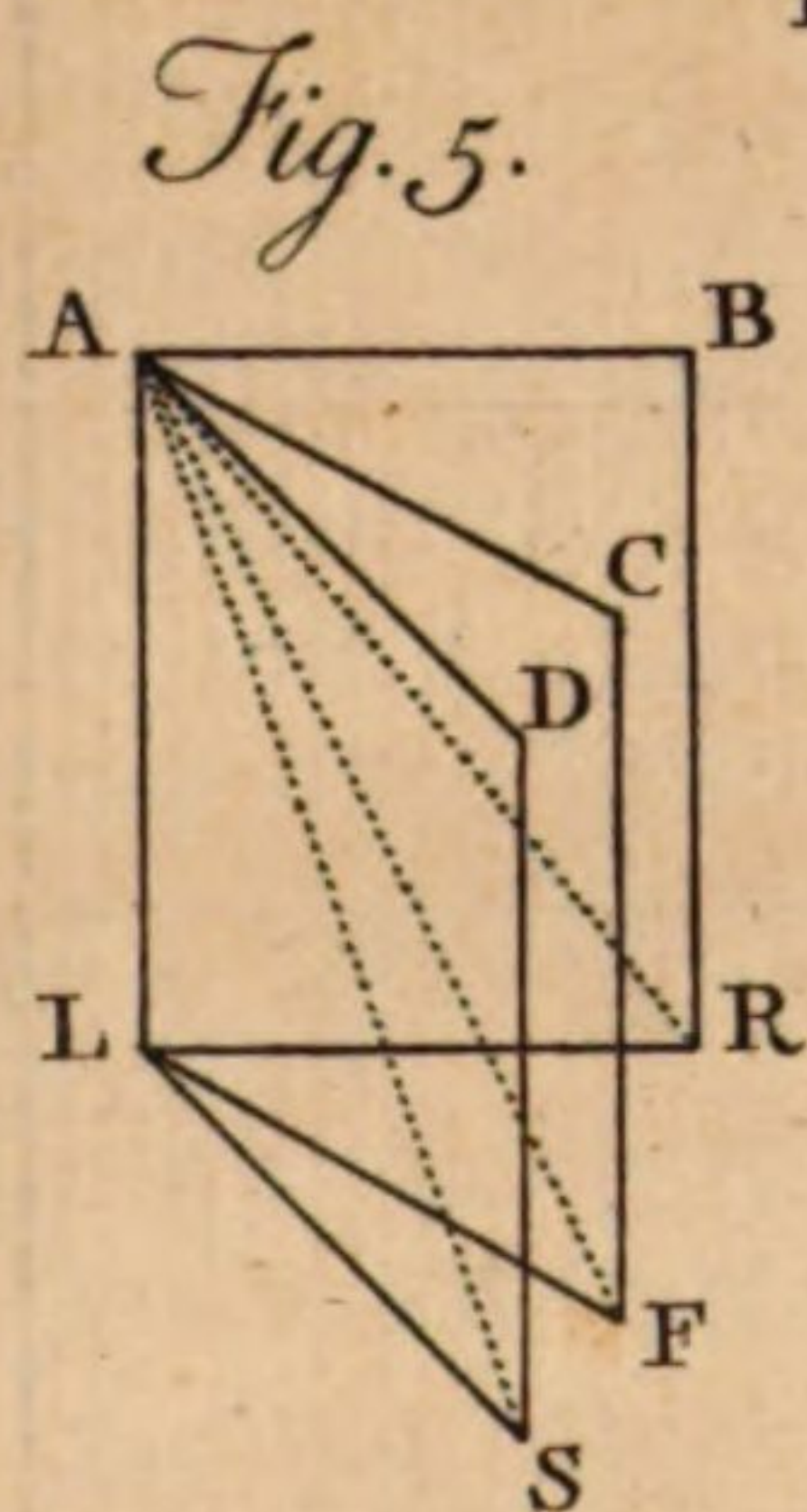
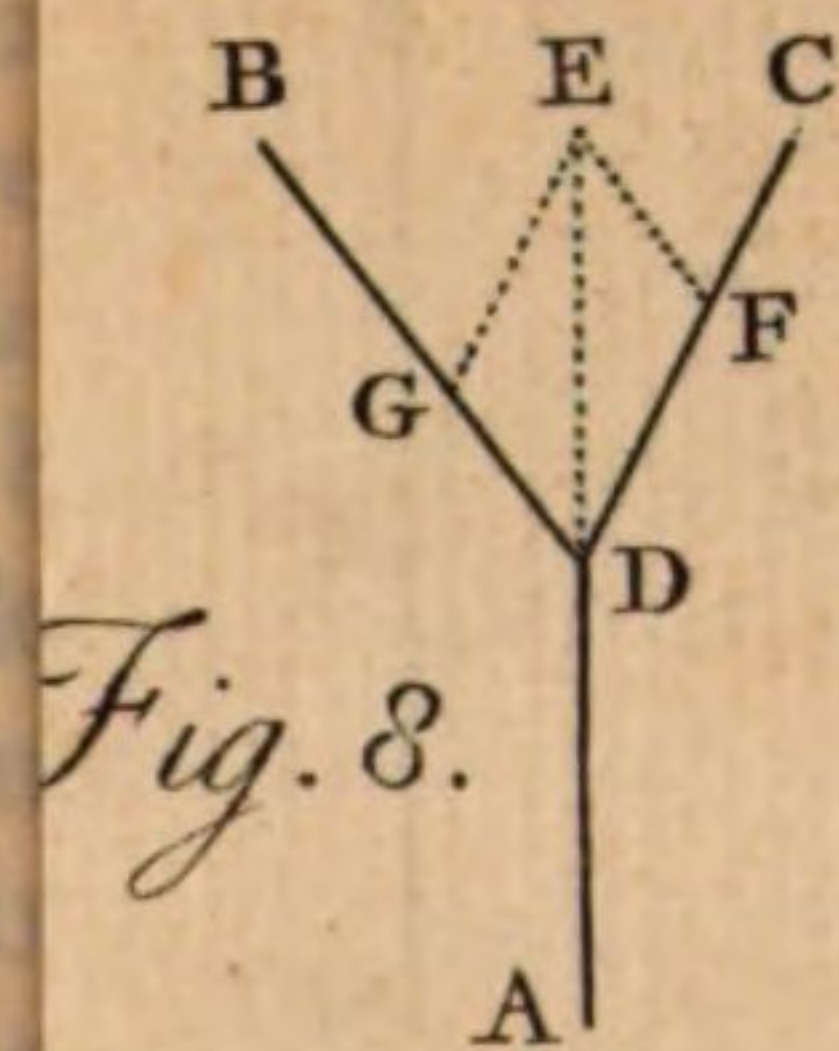
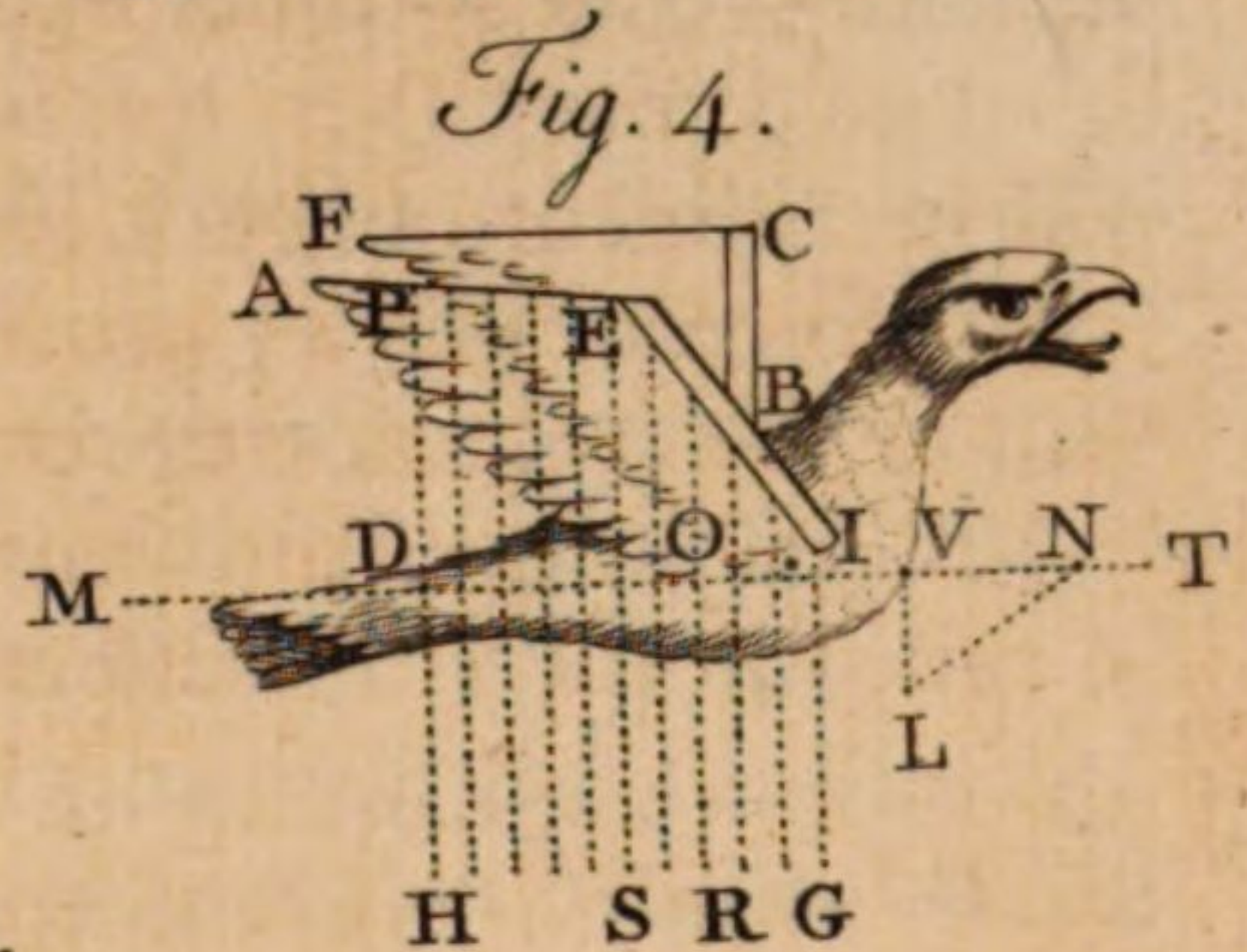
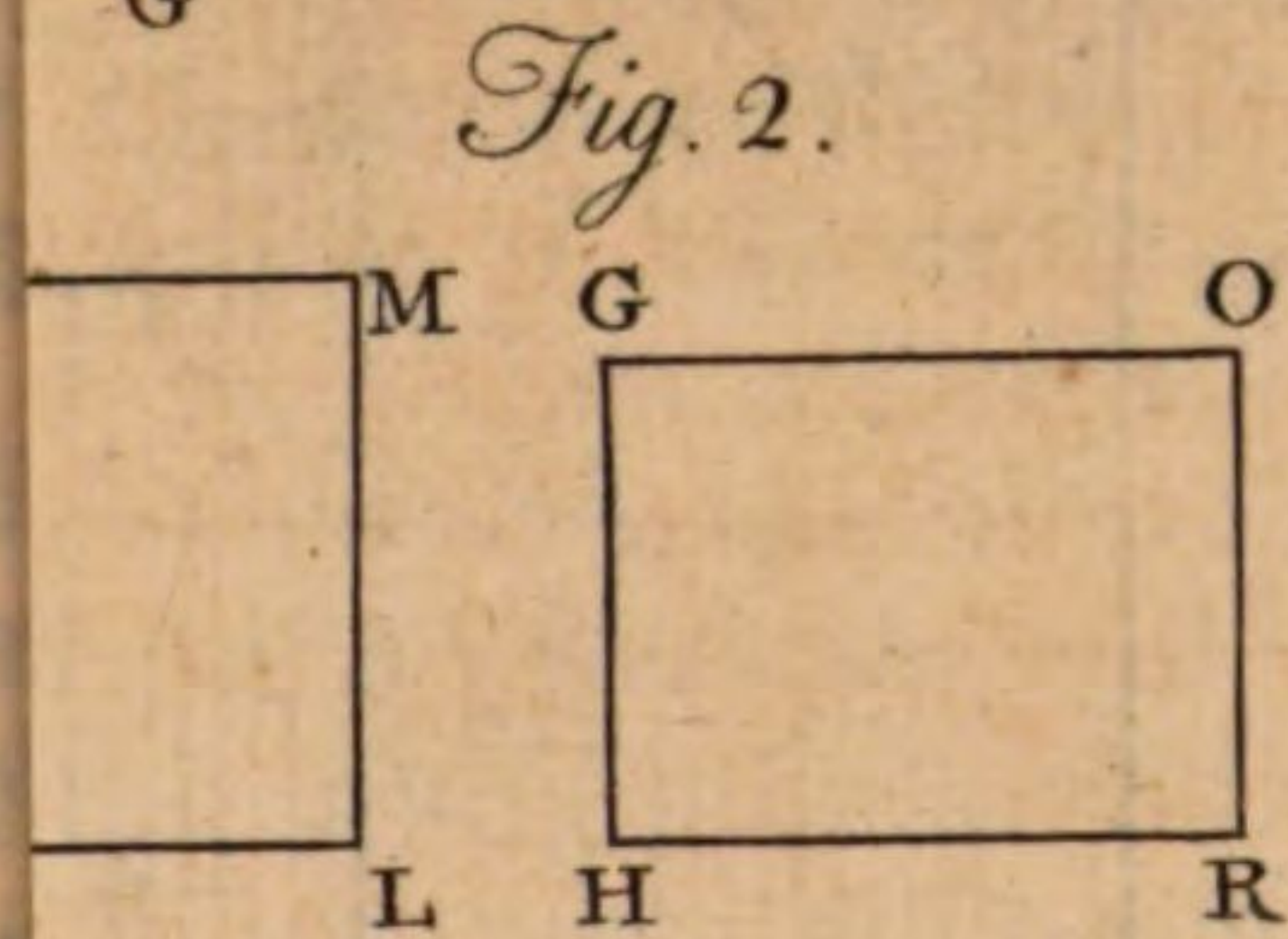
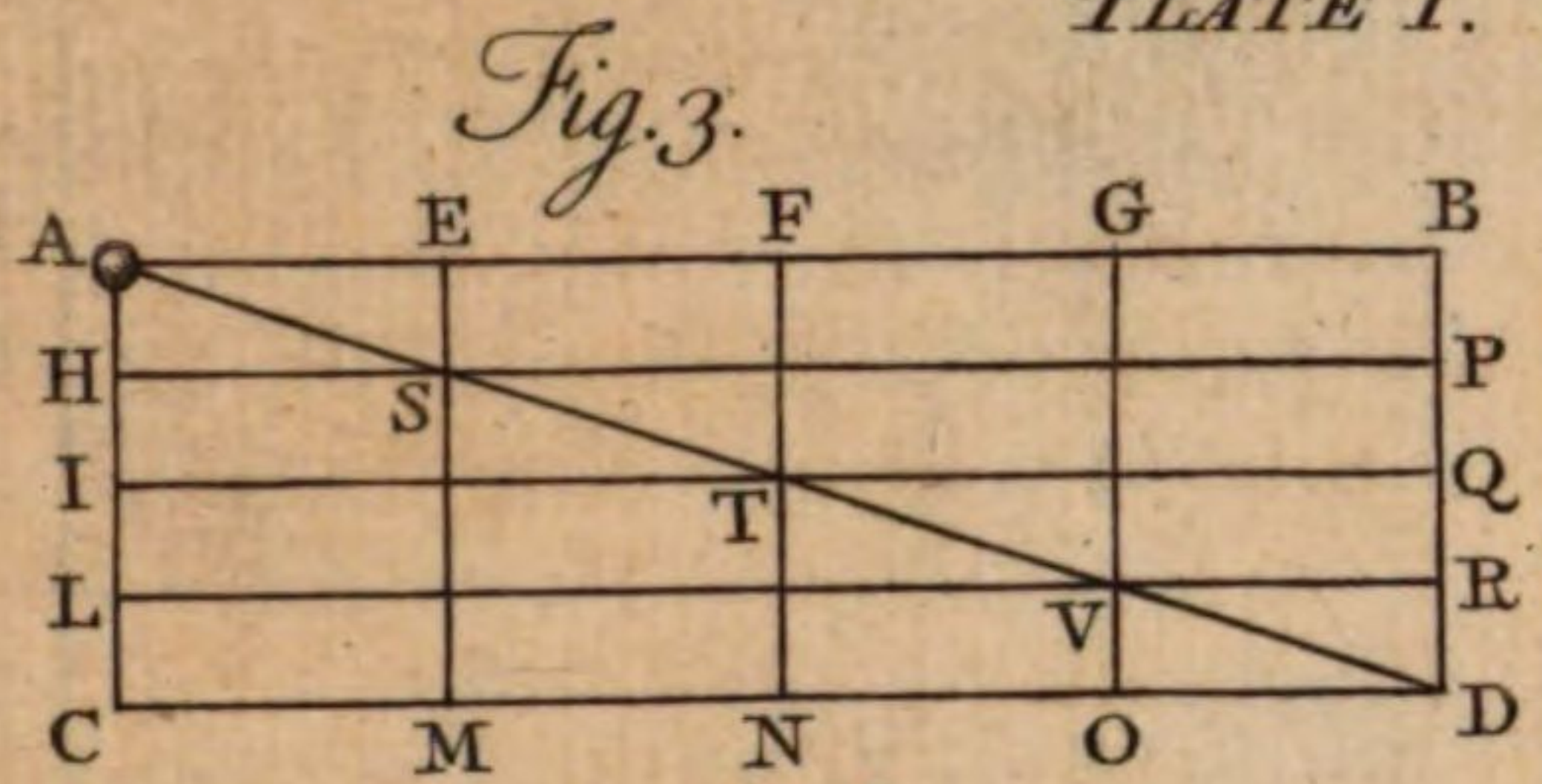
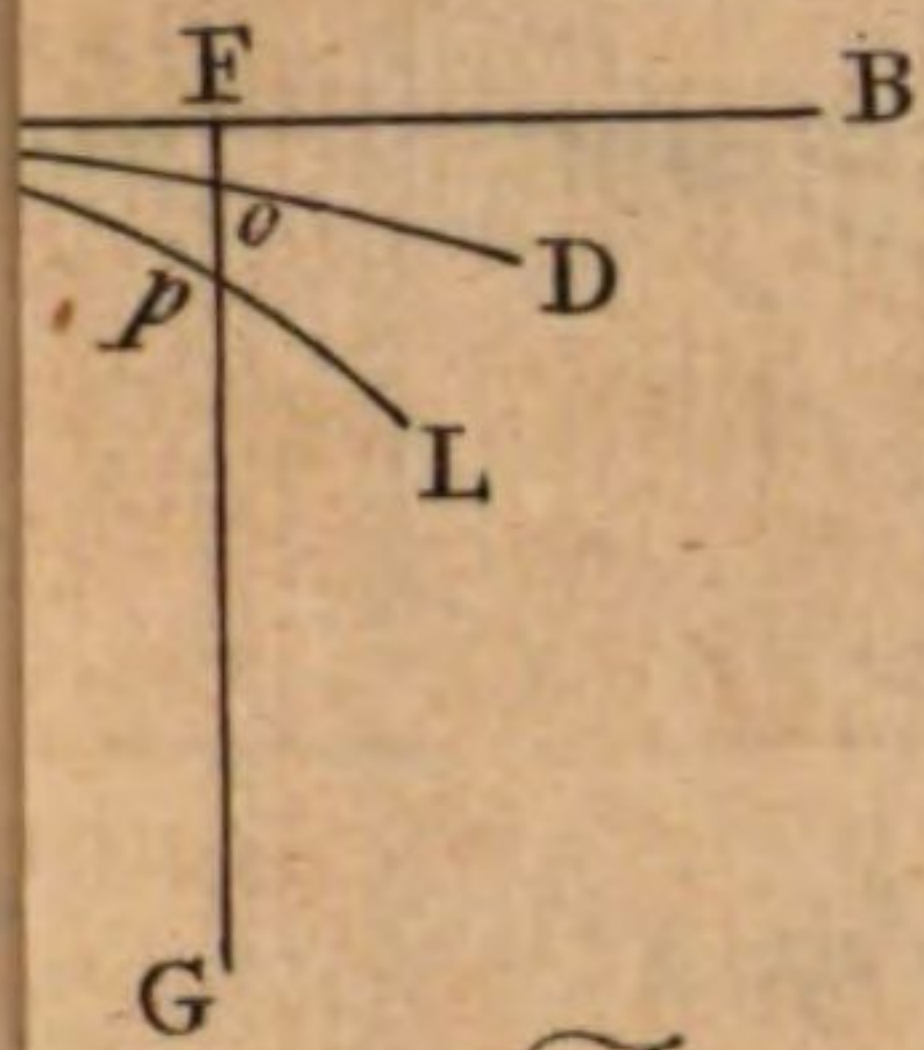
By the 23 proposition it appears, that the force of gravity decreases as the squares of the distances from the center increase; so that at distances, which from the surface upwards are in the proportion of 1, 2, 3, 4, the forces of gravity will be as 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Here the cause producing gravity, that is, the earth and the quantity of matter contained in it, continues the same, and yet the gravity propagated from it is so varied as to be greater at some distances from the center and less at others. Therefore it is here referred to some measure different from the former, and that is the velocity, which the force of it would produce in a given time. For when we say that gravity at the distance of one semidiameter from the center is four times greater than at double that distance, or as 1 to $\frac{1}{4}$ we mean that in any certain time, as a second for instance, the velocity, which the force of gravity would produce in a body at the distance of one semidiameter, is four times what it would produce in the same body at double that distance. Gravity is referred to the same standard, when the weight of a body falling perpendicularly is said to be greater than its weight if it falls down an inclined plane. For though it may begin to fall from the same height in both cases, yet a greater velocity is produced in it, in a given time, when it falls perpendicularly, than when it falls upon an inclined plane: and in reference to this velocity the gravity in one case is said to be greater than the gravity in the other.

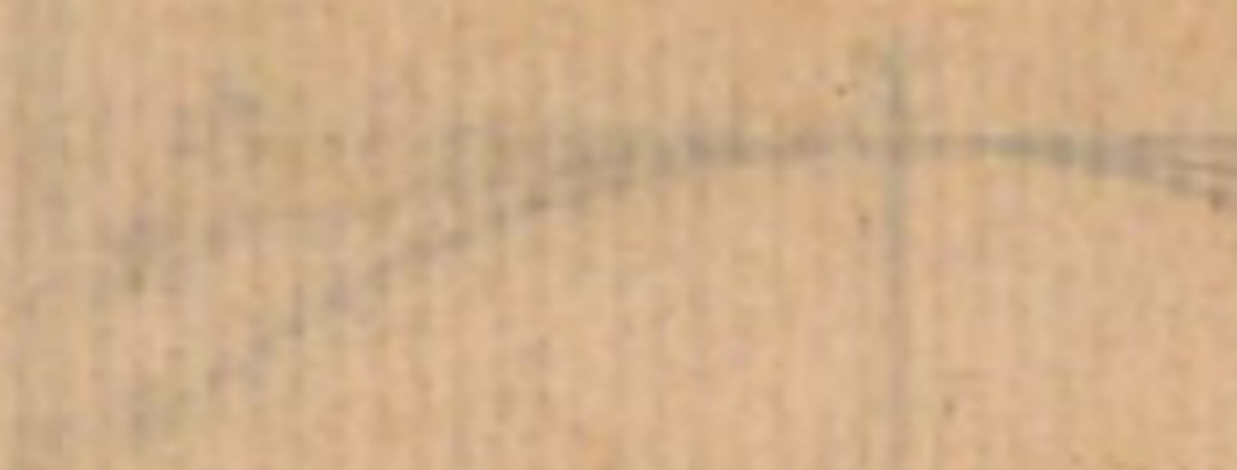
According to proposition 27, the gravity of bodies is proportional to the quantities of matter contained in them: so that, according to proposition 29, a cubic inch of lead has more gravity than a cubic inch of cork, because the former contains a greater quantity of matter than the latter. The word gravity is here referred to a third measure different from both the others. For the quantity of matter in the earth or force propagating gravity is the same: and, by 26 proposition, the lead and the cork would both fall with equal velocities. But because they fall with equal velocities, their quantities of motion produced in the same time will be different, since their quantities of matter are unequal. And, as the force of gravity produces a greater quantity of motion in the lead than in the cork, in this respect it is said to be greater.

The following definitions will inform us by what name gravity is to be called in each of these references.

30. *When gravity is considered as greater or less in reference to the efficacy of the cause, which produces it; then it is called the absolute force of gravity.*
31. *When the velocity, produced in a given time, is the measure by which gravity is said to be greater or less, then it is called the accelerating force of gravity.*
32. *When gravity is considered as greater or less in proportion to the quantity of motion, which it produces in a given time, then it is called the moving force of gravity.*
33. *Gravity is not such a quality, as the occult ones made use of in philosophy by the Aristotelians;*

It differs from those qualities in two respects. The first is that the Aristotelians make use of almost as many different occult qualities as there are appearances in nature to be explained; and so every particular effect has its particular cause assigned for it. But gravity is an universal quality, which belongs to all matter whatever, and is applied by the Newtonian philosophy to the explication of innumerable appearances. The second difference between gravity and occult qualities is, that these qualities are, as their name implies, occult ones; they cannot be made the object of our senses, nor is their existence proved by any experiments. But gravity is a well-known quality, which we may perceive by our senses, and whose existence in nature is confirmed to us by constant experience. The Newtonians therefore in the use which they make of gravity





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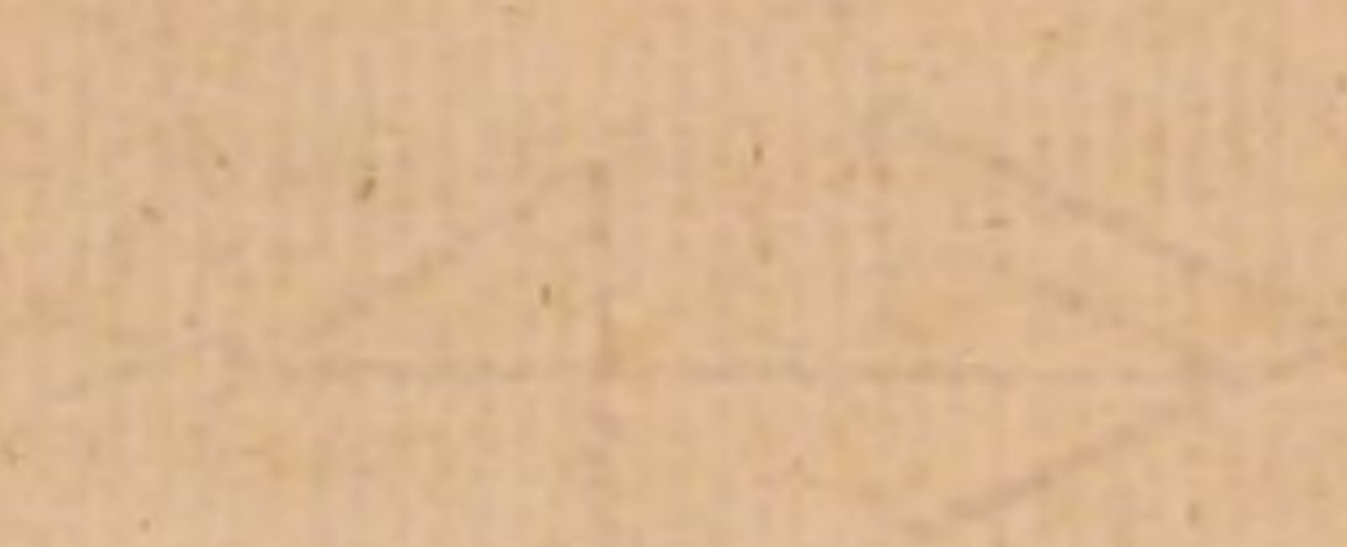
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gravity do not admit of more causes of natural things, than are both true and sufficient for explaining their appearances; and thus they observe a principal rule of true philosophy, which the Aristotelians neglect.

CHAP. V.

The center of gravity.

34. *The center of gravity in any body is a point on each side of which all the parts of the body are in equilibrio.*
 35. *The center of motion is that point round which a body does or may move.*
 36. *The line of direction is that, in which a weight or power acts.*
 37. *A body is moved by its own weight in the same manner that it would be, if the whole weight of it acted in the center of gravity.*

FOR if the whole weight acted in that center only and was not diffused through the whole mass, then the parts on one side of the center could never by their weight put the parts on the other side of it into motion, since by the supposition, there would be no weight on either side. But by the definition of the center of gravity the parts of a body on each side its center of gravity balance each other, so that those on one side can never by their gravity put those on the other side into motion. Therefore, as to producing motion in a body, the gravity of it acts in the same manner, when diffused through every part of the body, as if it were all accumulated in one point so as to act in its center of gravity only. From hence it follows that the line of direction of a weight is drawn from its center of gravity to the earths center.

38. *The weight of a body will make it fall, unless its center of gravity is supported.*

This would certainly be true if all its weight acted in the center of gravity. For then supporting this center and supporting the weight of the body, or, keeping it from falling, would be the same thing. And it is no less true though the weight is diffused through every part of the body. Since by the last proposition the motion produced in a body by its weight will be the same upon either supposition.

39. *The weight of a body can never move it in such a manner as to make its center of gravity ascend.*

Because a body is acted upon by its weight just as it would be if the whole weight was united in the center of gravity. And if it was all
 united

united there, we then may be sure that gravity could never make it ascend, unless an impulse downwards, as gravity is, could make that point, upon which such an impulse acts, move upwards.

40. *Bodies will fall by their weight, when they are set upon an horizontal plane, unless the line of direction, in which that weight acts, passes through their base.*

For when the line of direction passes through the base, then the weight of the body, because it acts in that line, presses upon the base, and will keep the body firm upon it. Whereas, if the line of direction is without the base, then the weight, will draw the body off from the base, or will make it fall. Thus the body ABDE Plat. II. fig. 2. whose center of gravity is C will stand firm; because the weight of it acts in the line CO, as if it was all united in the single point C. But a force, which presses the base DE downwards cannot possibly make it rise, as it must do before the body can fall either towards K or any other way. But the body *abde*, whose center of gravity is *c* will be thrown down by its weight. For the weight acting in the line *co* on one side of the base will make the body turn round upon the point *e*, that is, will make the base *de* rise and the body fall towards *k*.

Set one leg of the compasses at E and with the opening CE strike the arc CFK. As the body cannot fall towards K without turning round on the point E, the center of gravity C must in the fall describe the arc CFK, but that it cannot do unless it rises, for in that arc the point F is higher than C. Therefore since the weight of a body cannot make its center of gravity move upwards, neither can it make this center describe the arc CFK, that is, it cannot throw the body down. If in the same manner with the opening of the compasses *ce* we strike the arc *cfk*; where the line of direction falls without the base: then the point *f* in this arc and every other point will be lower than *c*; so that the center of gravity can descend, whilst it describes the arc *cfk*. And there is nothing in the line of direction to support it; therefore it will descend, so as to turn the body round upon the point *e*, that is, so as to make it fall towards *k*.

A wall or any other building may lean a great way before it falls. For its gravity will not make it fall; till it leans so much as to carry the line of direction beyond the foundation. For this reason it will stand the longer and may lean so much the more for having a wide foundation.

For

For a man to stand upright the line of direction of his weight must pass through his base, which is a parallelogram, or some other quadrilateral figure, contained between his two feet and two lines, of which the first is drawn across from one foot to the other at his toes and the second at his heels. It is therefore easier to stand upon both feet than it is upon one: because the center of gravity must be brought directly over the one foot that we stand upon, and the line of direction must be kept within the compass of this smaller base. When a man holds any considerable weight in his left hand, his center of gravity, that is, the center of gravity of himself and the weight considered as one mass, will be carried too far towards the left, unless by leaning the contrary way he reduces it again and brings it to such a position as will make the line of direction pass through his base.

When we are sitting on a chair, our center of gravity is upon the seat, and the line of direction falls behind our base. Therefore in order to rise, we lean forwards, that we may bring the line of direction towards our feet; and we draw our feet backwards at the same time, that we may carry our base towards the line of direction: and when by both these motions the center of gravity is reduced so as to be directly over our feet, we are able to raise ourselves upright.

In walking, the foot, from which we set off, is our base at first; till by turning the ball of it round we have thrown the center of gravity forwards beyond it, by which means we should throw ourselves down, if we did not take up the other leg from the ground, and set it before us so as to catch ourselves upon it, as upon a prop or support to prevent us, from falling. This is taking one step. And in order to take a second, the center of gravity must be brought directly over this prop or support; that is, the foot, which we put out before us, must be made our base; that so in taking a second step we may set off from it. We do this by turning the ball of the other foot still farther round so as to push the ground that we stand on back with our toe, and the ground by its reaction pushes our center of gravity forwards; till the line of direction is got to the place whither we wanted to reduce it. Therefore in walking the line of direction passes through each foot alternately, and if we set one foot directly before the other in every step that we take, then this line will move straight forwards: but if we straddle as we walk; then the line of direction does not go on evenly, but is carried out first towards the right side, and then towards the left, at the same time that it goes forwards: which is the reason, why those who straddle in their walk, as fat people commonly do, are observed to waddle. Indeed every
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body waddles either more or less, only they who walk pretty close do it so little as not to be perceived.

In skating the line of direction passes through the foot on which the skater moves; and as he raises his other foot from the ice, he scrapes it in a direction contrary to that, in which he designs to move; and the reaction of the ice upon it pushes him forwards.

In jumping the center of gravity is both made to rise from the ground and is likewise carried forwards at the same time. The manner in which the center of gravity is made to rise will be best understood if we first explain something of the same sort in a more simple instance. If you set one end of a bow-stick on the ground and with your hand bend down the other end of it; upon letting this end go, the bow flies out at once, and jumps from the ground. For the sticks elasticity unbends it equally both ways, and as the action of one end is downwards and presses upon the ground, the reaction will be upwards and the center of gravity will have a motion communicated to it in that direction, and therefore by the first law of motion, will continue to rise: and the stick will by this means be carried off from the ground, till its weight puts a stop to the farther ascent of it, and makes it fall back again. The force of the muscles is what in jumping supplies the place of elasticity. For by bending the joints of the two heels, the two knees, and the two thighs there are six bows bent; and if all of these are suddenly straitened by the force of the muscles the center of gravity of the body will have a motion upwards communicated to it, and the body itself will by this means be made to rise from the ground. If we are to jump directly upwards, then we push the ground, on which we stand, directly downwards. But if we would move obliquely so as to rise and to jump forwards at the same time, then we push the ground obliquely backwards with both feet together, in the same manner as we push it with one foot in walking.

The reader may here consider, what it is that makes the difference between walking and running. If he imagines that these two modes of motion differ only in the degree of swiftness, and that as walking is a slow motion, so running is a quicker motion of the same sort, he is much mistaken. For these two motions differ in sort rather than in degree. Indeed they do not always differ in degree, as any ones experience, will inform him: for there is no man who is not able to go one pace which he would call walking, that will carry him on faster than another pace, which he would call running. We can walk quick and run slow; and, if we please, we can rid ground faster by walking quick, than we do

do by running flow. This makes it evident that walking is not a flow fort of running, nor running a quick fort of walking. They are two entirely different modes of motion, and though we were to run or walk, with just the same degree of swiftness, yet every man would be able to tell us, when we walk and when we run. Walking is moving forwards by steps, so as never to have both legs off the ground at the same time. Running is moving forwards by jumps. Indeed in running we do not take our jumps with both legs close together, but with one of them before the other: but I call it jumping because in throwing ourselves forwards both legs are off the ground at once.

41. *Some heavy bodies slide down an inclined plane; others roll down it.*

A body is said to slide down, if it descends upon the same base constantly, or with the same side towards the plane. It is said to roll down, if in its descent the base is constantly changing. Therefore, when a heavy body is upon an inclined plane, if the line of direction falls within the base of the body, it will slide down. For, by proposition 40, the weight of the body will not throw it off the base. But if the line of direction falls beyond the base, then the body in its descent cannot continue long on the same base; but the side which is towards the plane will be constantly changing; that is, the body will roll down. Thus both the bodies A and B Plat. II. fig. 3. will move down the plane HI. Because their centers of gravity can descend. But they will not move down in the same manner. For *cr* the line of direction passes through the base *de* of the body A; therefore, in moving down, the body will always keep upon the same base or will slide down. But the line of direction *cs* passes beyond the base *fg* of the body B; therefore it will be thrown off this base and will change it for *gt*; and as this sort of motion will continue through the whole descent, the body will roll down.

42. *A heavy body, which is suspended on a center of motion, will rest, if the center of gravity is directly under the center of motion, otherwise the weight of the body will make it move.*

If a heavy body P Plat. II. fig. 1. hangs by a string on the center of motion C, the string, by which it hangs, acts in the direction PC, and the force of it must be equal to the weight of the body, otherwise it would break. And when the body is at P, these two equal forces act in contrary directions, the string in the line PC, and the weight in the line PL: therefore they will destroy each other, and the body will rest between them. But if the body is at *p*, then one of the forces acts in the line *p*C, and the other in the line *p*L. And though these forces are

equal, yet, because they act obliquely, the body by, proposition 16, will begin to move in the diagonal of a parallelogram, one of whose sides is in the direction pC and the other in the direction pL . This proof may easily be made universal. For in all cases, where a body is suspended, the force, which sustains it, must be equal to its weight. And, when the center of gravity is directly under the center of motion, these two equal forces act in contrary directions, for which reason the body will rest. But in any other situation, these forces are oblique to each other: and therefore, though they are equal, the body will be made to move in the diagonal of a parallelogram, of which the two forces are the sides.

CHAP. VI.

Of the mechanical powers.

THE subject of this chapter has given the name of *mechanics* to one branch of natural philosophy. Mechanics strictly speaking should treat of nothing but such machines or instruments as assist us in lifting weights or overcoming resistances. But because the effects of these instruments cannot be understood unless the laws of motion are first explained; the laws of motion and, in consequence, the application of these laws to the motion of bodies in general is made the subject of mechanics.

There are five simple machines. A lever, a wheel and axle, a pulley, a screw, and a wedge. Some writers reckon up more, and others reduce them to fewer. But this division will answer our purpose as well or better than any other.

43. *A lever is a bar, which turns round upon a prop or center of motion, and is applied either to raise weights or to overcome resistances.*

A lever is commonly called a hand spike, when it is made of wood; and a crow, when of iron. In theory we suppose, that the bar itself has no weight at all, and that no force whatever can bend it.

44. *By the intensity of a power is meant the absolute force of it, or the force it will have, when applied immediately to a weight, so that the power and weight move with the same velocity.*

45. *The moment of a power, like that of a weight, is found by multiplying its intensity into its velocity.*

46. *A lever is said to be of the first kind, when the prop or center of motion C Plat. II. fig. 4. is between the weight A and the power B. Of the second kind, when the weight W Plat. II. fig. 5. is between the center of motion C and the power P. Of the third kind, when the power P Plat. II. fig. 6. is between the center of motion C and the weight W.*

47. *In*

47. *In turning a lever upon the center of motion, the velocity of each point in the lever, or of any weight or power applied to that point, will be as the distance of it from the center of motion.*

In turning the lever ACB Plat. II. fig. 4. upon the center till it comes into the position aCb ; the point A describes the arc Aa , and the point B describes the arc Bb in the same time. Now it is evident that Bb is longer than Aa . Therefore the point B must move faster or must have a greater velocity than the point A, because B runs over the longer line Bb in the same time that A runs over the shorter line Aa . And the velocity of B must be just as much greater than that of A, as Bb is longer than Aa : for otherwise B could not describe Bb in the same time that A describes Aa . Now Bb is just as much longer than Aa , as BC is longer than aC : for as Bb contains Aa six times, so BC contains AC six times. And B moves just as much faster than A, as Bb is longer than Aa , that is, as BC the distance of B from the center is greater than AC the distance of A from the center; or the velocity of B bears the same proportion to that of A, that BC bears to AC.

Or otherwise: the velocities of these two points A and B, being, by proposition 10, as the spaces, which they describe in the same time, that is, as Aa to Bb , are likewise as AC to BC or as their respective distances from the center. For Aa bears the same proportion to Bb that AC bears to BC. Because the arcs Aa and Bb subtend the angles ACa and BCb , which are vertical ones and therefore equal, Eucl. b. I. p. 15. from whence it follows that the arcs being similar, Eucl. b. VI. p. 33. corol. 2. are to each other as the semidiameters of their respective circles; that is, Aa the space described in a given time by the point A or the velocity of A is to Bb the space described in the same time by B or to the velocity of B, as AC the distance of A from the center is to BC the distance of B from the center.

Let a power, such as the force of a man be applied at the point B and move along with it, whilst a weight, that is fastened at the other end of the lever, moves along with the point A. Then, since the power moves as fast as B does, and the weight no faster than A, the velocity of the power is to that of the weight as BC to AC, or as their respective distances from the center of motion.

We have here supposed the power and weight to be on the opposite sides of the center of motion, which is the case only in a lever of the first kind. For in levers both of the second and third kind, the power and weight are always on the same side the center. Thus let

CB

CB Plat. II. fig. 4. with the prop at one end of it be the lever. Then, if the power acts at B, and the weight at F, it is a lever of the second kind: but if the power acts at F and the weight at B, it is a lever of the third kind. And in either case the velocity of B is to that of F, as BC to FC, the respective distances of those two points from the center. For, as before, the velocities of B and F are as the arcs Bb and Ff , which are described by them in the same time: and, as these arcs subtend the same angle at C, they are similar. Therefore Bb , or the velocity of B, is to Ff , or the velocity of F, as BC to FC.

48. *The comparative velocity of a power or weight is to be estimated by the perpendicular distance of their line of direction from the center of motion.*

If BCD Plat. II. fig. 7. is a bent lever; and the power by means of a rope fastened at D acts in the direction DF, then the velocity of the power is as CG a line drawn from the center of motion C perpendicular to AF the line of direction, in which the power acts. For since, by the second law of motion, the power acts only in the line AF, the perpendicular distance of this line from the center, or CG must likewise be the distance of the powers action from it.

In the foregoing proposition, I estimated the comparative velocity of a power by the distance between the center and that point, where the power is applied; and the comparative velocity of a weight by the distance between the center and that point, where the weight is suspended. Thus Plat. II. fig. 4. the velocity of a power applied at B was said to be to that of a weight suspended at A, as BC to AC. But if the power pulls downright, then BE is its line of direction: and though the weight describes Aa yet the distance, to which it rises from the earths center in rising from A to a , is only AD and in such a line as AD, or in the line of its direction, the weights velocity is to be estimated. Therefore the perpendicular distances of these lines of direction from the center of motion being CE and CD, the velocity of the power, by this proposition, is to that of the weight as CE to CD. But whenever the points of application and suspension A and B are not altered by altering the position of the lever from ACB an inclined one to aCb an horizontal one, the comparative velocities are the same in respect of each other, whether they are estimated by this proportion, or by that laid down in the last proposition, because CB is to CA as CE to CD. For, since CE and CD are the perpendicular distances of BE and AD from the center, in the triangle BEC the angle at E is a right one, and so is the angle at D in the triangle ADC. The angle at C likewise in one of these triangles is
equal

equal to the angle at C in the other. And, because there are two angles in one equal to two angles in the other, therefore the remaining angle at B is equal to the remaining angle at A. Eucl. b. I. prop. 32. corol. 2. From hence it follows, Eucl. b. VI. p. 4. that these two triangles being equiangular, their sides about the equal angles will be proportional, CB to CA as CE to CD.

49. *A power and a weight, which act upon a lever, are in equilibrio, when the power is to the weight, as the distance of the weight from the center to the distance of the power from the center.*

For since their velocities by prop. 47. are to each other as their respective distances from the center; the power by what is here supposed is to the weight as the weights velocity to the powers velocity; therefore, Eucl. b. VI. prop. 16. the power multiplied into its own velocity, or the moment of the power, is equal to the weight multiplied into its own velocity, or to the moment of the weight: that is, the power and weight are in equilibrio. Or otherwise, when the power is to the weight, as the weights distance from the center to the powers distance from the same, then the power is to the weight as the weights velocity to the powers velocity; or the power and weight are to each other as their respective velocities inverted, that is, as their velocities reciprocally, therefore, by prop. 14. 45. their moments are equal.

This may perhaps be made more intelligible, if we apply it in some particular instances to a lever of each kind. A power which is equivalent to 100 pounds in weight, can lift no more than such a weight, when applied immediately to it. Because in raising the weight to any height the power and weight both move through equal spaces in the same time, that is, they both move with the same velocity. But if the lever AB is divided by the prop C Plat. II. fig. 4. in such a manner that the arm CB is to the arm CA as 6 to 1: then the same power applied at B will be in equilibrio with a weight of 600 pounds suspended at A. And, if the power is ever so little increased, it will raise the weight. For the moment at one end of the lever is equal to the powers intensity multiplied into its own velocity or distance from the center $= 100 \times 6 = 600$, and the moment at the other end is the same being equal to the weight multiplied into its own velocity or distance from the center $= 600 \times 1 = 600$.

Upon a lever of the second kind Plat. II. fig. 5. if CP the distance of the power is to CW the distance of the weight from the center as 5 to 1, then a power equivalent to 100 pounds, if applied at P, will be in equilibrio with a weight of 500 pounds suspended at W. For the power mul-

multiplied into its own velocity or distance $= 100 \times 5 = 500$, and the weight multiplied into its own velocity or distance $= 500 \times 1 = 500$. Therefore their moments are equal. And if the power is ever so little increased it will raise the weight by turning the lever round upon the center of motion.

Upon a lever of the third kind, Plat. II. fig. 6. if CP the distance of the power is to CW the distance of the weight from the center as 3 to 5 then a power equivalent to 500 pounds, if applied at P, will be in equilibrio with a weight of 300 pounds suspended at W. For the power multiplied into its own velocity or distance $= 500 \times 3 = 1500$, and the weight multiplied into its own velocity or distance $= 300 \times 5 = 1500$. Therefore their moments are equal.

I must desire the reader to take notice, that though a power and weight will rest, when they are in equilibrio, yet it is not absurd to reason concerning their velocity, and to say that they are in equilibrio, because they are to each other as their respective velocities reciprocally. For here we do not mean the velocities with which the power and weight move; that indeed might be absurd as neither of them move at all: but we mean the velocities, with which they must begin to move, if they were to move. Thus, if the center of motion divides a lever of the first kind in the proportion of 1 to 6; a power, which is equivalent to 100 pounds, applied at the end of the levers longer arm, will be in equilibrio with a weight of 600 pounds suspended at the end of the shorter: and therefore the power and weight will rest. And when, in giving a reason for the equilibrium, we say that the power and weight are to each other as their respective velocities reciprocally: we do not mean that the power moves with six times the velocity of the weight; since neither of them are supposed to move at all: but that the weight cannot begin to move without making the power move at the same time with six times its own velocity. From whence it follows, that the weight cannot move in one direction, unless it can make the power move in the contrary direction with a moment equal to its own. And since equal forces in contrary directions destroy each other, neither the weight can move the power nor the power move the weight: therefore both of them will rest, that is, they will be in equilibrio.

50. *When two weights suspended upon a lever are in equilibrio; their common center of gravity is supported by the prop.*

When instead of applying a power at one end of a lever and hanging a weight upon the other, we hang a weight upon both ends; there will

will be an equilibrium, if the weights, or quantities of matter, are to each other inversely as their distances from the prop, that is, inversely as their velocities. For it will be easy to understand, from what has been said already, that in this position their moments will be equal. Now as the center of gravity in a single body is the point on each side of which all the parts of the body are in equilibrio; so the common center of gravity of two bodies is that point on each side of which the two bodies are in equilibrio; or on each side of which the quantities of motion are equal. But, by the last proposition, the quantities of motion in the lever are equal on each side of the prop, when the two bodies are at distances inversely as their weights. Therefore if a line be drawn from the center of gravity of one body to the center of gravity of the other, the point of this line, which is over the prop, will be their common center of gravity. Or; since the distance of each weight from the prop is inversely as its quantity of matter, the common center of gravity of the two bodies is a point which divides the distance between them in the inverse proportion of their quantities of matter.

Some of those, who object to our considering the velocity of weights or powers, when these weights and powers do not move, if they are called upon to prove that two unequal weights will rest, when they are suspended upon a lever at distances from the prop, which are as the weights inversely, say, that in these circumstances the common center of gravity is supported, and therefore the weights ought to rest. But by examining this reason we shall find it proves (as the logicians say) in a circle. Suppose I ask why two unequal weights, one of which is suspended at *a*, Plat. II. fig. 4. and the other at *b*, will rest, if the distances *Ca* and *Cb* are as the weights inversely: it is just the same as if I had asked why the two unequal weights in these circumstances balance each other or are in equilibrio. And can it be a satisfactory answer to tell me, that *C*, which is the common center of gravity of the two weights is supported? for might not I ask my instructors, how they prove that *C* is the common center of gravity? If they answer, that on each side of *C* the two weights are in equilibrio; this is what I charged them with, it is proving in a circle. For first they tell me, that the two weights are in equilibrio, because *C* is the center of gravity, and is supported; and then they tell me that *C* is the center of gravity, because the two weights are in equilibrio. But if they give me the true reason why *C* is the common center of gravity of the two bodies at *a* and *b*, and say that the quantities of motion on each side of *C* are equal; then they do the same thing themselves which they blame in others: for whilst they

call it absurd in us to talk of the velocity of bodies, which are at rest, they themselves talk of the motion of bodies, which are at rest. And, unless they will consider the velocities of bodies in equilibrio, I know not by what means they would be able to prove their assertion and shew that the quantities of motion on each side of C are equal, when the quantities of matter at *a* and *b* are unequal.

51. *A common balance will not be in equilibrio, unless the weight in one scale is equal to that in the other.*

The beam of a common balance is a lever of the first kind, which is divided into two equal parts or arms by the center of motion. And, since the distance of each point of suspension, or of each scale from the center of motion is equal, therefore the velocities on each side are equal. From whence it follows, by proposition 13, that the moment of each scale will be as the quantity of matter contained in it; and consequently that the moments will not be equal, or there will be no equilibrium, unless the quantity of matter or weight in one scale is equal to the quantity of matter or weight in the other.

For this reason it is, that when you want to weigh different quantities of any thing, you must use a different standard. Thus to weigh a pound, you use a pound stone, or standard weight of a pound; to weigh 14 pounds, you use a standard of that weight, and to weigh 100 pounds you use a third standard, the weight of which is 100 pounds. It is otherwise in steelyards, where one and the same standard may be applied to estimate, and, by the construction of the instrument, may be made to balance very different weights. The reason of this will appear under the following proposition.

52. *In the steelyard different weights may be in equilibrio with the same standard: and the weight to be estimated is as the distances of the standard from the center of motion, when there is an equilibrium.*

The standard 1 Plat. II. fig. 8. may be considered as a power acting on the arm CP of the lever WP. And as the strength of a man applied at P and a weight at W, so likewise two weights one at P instead of a power, and the other at W are in equilibrio, when they are to each other inversely as their distances from the center of motion

The weight of 1 pound at the distance of 4 has 1 for the quantity of matter and 4 for the velocity, and consequently has a moment $= 1 \times 4 = 4$, and the weight of 4 pounds at the distance 1 has 4 for the quantity of matter and 1 for the velocity, and consequently has

has a moment $= 4 \times 1 = 4$, therefore their moments are equal. But the same standard of one pound at the distance 3 will have a moment $= 1 \times 3 = 3$; and therefore will be in equilibrio with a weight of only 3 pounds suspended at W or at the distance 1 on the other arm of the steelyard. For the same reason if the standards distance from the center of motion $= 2$ it would be in equilibrio with no greater weight than 2 pounds at W or at the distance 1 on the other arm. And if the standards distance was 1, or equal to the weights distance, then to make an equilibrium their quantities of matter must be equal.

The moment of the standard, which is suspended on one arm of the steelyard, is as its quantity of matter multiplied into its distance from the center of motion; and so likewise is the moment of the weight, which is suspended on the other arm. Now on the side of the standard the quantity of matter is given; therefore the moment is as the distance from the center, or as the velocity, by propositions 47. 13. And on the side of the steelyard where the thing to be weighed hangs, the distance of the weight and consequently its velocity is given. Therefore the moment is as the quantity of matter by proposition 13. From whence it follows, that when these two moments are equal, that is, when there is an equilibrium, the quantity of matter in the weight on one side is as the distance of the standard from the center of motion on the other side.

53. *The beam of a balance will not rest in any position but a horizontal one, when the weights in the opposite scales are in equilibrio with each other.*

The center of motion m of the balance Plat. II. fig. 9. does not coincide with the common center of gravity g . But, when the beam is in the horizontal position AB, the center of gravity g is directly under the center of motion m . For this reason, according to proposition 42, the whole will rest. But, if the beam is inclined in the position HI, then the center of gravity g is moved from under the center of motion towards the side I, that is towards the end of the beam which is raised. Therefore, by the same proposition, the end I will descend till the center of gravity comes under the center of motion: and, by the velocity acquired in descending, it will sink still farther; so that the end H will be raised above the level; and the center of gravity will be moved from under the center of motion towards the side H: for which reason H will descend as I did before. And the beam will keep vibrating in this manner, each end of it alternately ascending and descending; till by the resistance of the air and the friction on the center, all its motion is lost.

54. *A man standing in one scale, and in equilibrio with a weight in the opposite scale, cannot increase his own weight by pulling the ropes, to which his scale hangs; nor can he diminish it by thrusting directly upwards against the point of suspension.*

By pulling downwards he hangs either a part of his own weight, or else the whole of it to the ropes: for by, the third law of motion, the reaction of the ropes upwards, is just equal to the force with which he pulls downwards: for instance, if he weighs 160 pounds and pulls with a force equal to this weight, then the reaction of the ropes will sustain his whole weight, and no part of it will press upon the scale. If he pulls with a force equal to 80 pounds, then the ropes sustain half his weight, and the scale is pressed by the other half. And since in all cases a part of his weight equal to the force, with which he pulls, is taken out of the scale and is hung to the rope; the point of suspension will be always pressed by the same quantity; by his whole weight in the scale, when he does not pull at all; but by the remainder of his weight in the scale and by the part which he hangs to the rope, when he does pull; and these two together are equal to his whole weight.

It can scarce be asked why he may not pull with a force greater than his own weight. Because, I think, it is obvious, that if he weighs only 160 pounds and should pull downwards with a force greater than this, as suppose with a force equal to 170 pounds; then, the reaction of the ropes upwards being likewise equal to 170 pounds, he would raise his feet from the scale. Because he is drawn upwards by this reaction, or by a force greater than his own weight. And when his feet are raised from the scale and he hangs at the rope by his hands, it is plain that the rope, and consequently the point of suspension bears just his weight and no more.

If he pushes directly against the point of suspension, then his reaction downwards upon the scale will, by the third law of motion, be equal to his action upwards. And, these two equal and contrary forces destroying each other, the scale will be pressed more downwards than it is upwards only by his own weight, just as it is, when he does not push at all.

55. *A man standing in one scale and in equilibrio with a weight in the opposite scale, by pushing any part of the beam, except his own point of suspension, upwards, will make himself heavier, and by pulling any part of it downwards will make himself lighter.*

Let

Let C Plat. II. fig. 10. be the center of motion, AB the beam and D the scale in which the man stands. First by pushing upwards with a cane or by any other means against the point H, which is on the same side of the center with himself, he will make his own scale preponderate, and the end A of the beam will descend. For by pushing against H his action upwards and his reaction downwards are equal, according to the third law of motion. But his action upwards is applied at the distance HC from the center, and his reaction downwards being against the scale, where he stands, is applied as the point of suspension, or at the distance AC. And, since the moments of these two counter-actions are as their respective distances from the center, by pushing against the point H he produces a greater moment downwards, than upwards; and upon that account the scale, where he stands, will preponderate, in the same manner, as if he had been made heavier. In this case the two unequal moments of action and reaction act on the same side of the center of motion. And therefore the scale D preponderates with a force equal to the difference of these moments.

But secondly, suppose the man to push upwards against the point K on the contrary side of the center. Then his action upwards is applied at the distance KC from the center, and his reaction downwards at the distance AC as before. But both his action upwards will raise the end CB, that is, will make the scale D, where he stands, preponderate, and so likewise will his reaction downwards. In this case the scale D will descend with a force equal to the sum of the two moments of his action and reaction.

If the man has a rope fastened at H and pulls it downwards; then the reaction of the rope upwards takes a part of his weight out of the scale D, and applies it to the beam at the point H, or at the distance HC from the center of motion; that is, at a distance less than AC, by which means, the moment of this part of his weight will be diminished; and the opposite scale will preponderate in the same manner, as if he was made lighter.

Or secondly, if his rope was fastened at K on the contrary side of the center, then by pulling at it the reaction of it takes a part of his weight out of the scale at D and applies it at K on the other side of the center. Therefore, the moment on his own side being diminished and that on the contrary side increased, the scale E will preponderate, and the scale D where he stands will rise.

56. *A balance may be a deceitful one, though the beam will hang even; that is, will be in equilibrio either without the scales, or when the scales are empty.*

Let the center of motion divide the beam unequally, suppose in the proportion of 10 to 9. Or, let one arm of the balance be 10 inches long and the other 9. Then if the quantity of matter in each arm is as the length of it inversely, by prop. 49. they will be in equilibrio. And, if the weights of the empty scales or dishes are likewise made inversely as their distances from the center, there will be an equilibrium, when they are suspended upon the beam. Thus suppose one of the scales or dishes to weigh 9 ounces, and let that hang at the arm which is 10 inches long: the moment of this scale, when it is empty, will be as $9 \times 10 = 90$. Suppose the other scale to weigh 10 ounces, and let that hang at the end of the arm which is 9 inches long: the moment of it, when empty, will be as $10 \times 9 = 90$. Therefore, as their moments are equal, they will be in equilibrio.

But a balance made in this manner will be a deceitful one. And if the seller puts the standard weight into the scale, which hangs nearer to the center, the buyer will be cheated one part in ten of the weight, which he ought to have. Thus, if he ought to have a pound, he would instead of it have only $\frac{9}{10}$ of a pound. For, as what he buys is put into the scale, which hangs farther from the center; the greater velocity of it will make up for the defect in quantity, and the moment of it will be equal to that of the standard, though the quantity of matter is less. A pound standard or $\frac{10}{10}$ of a pound, if put into the scale, which is at the distance of 9 inches from the center, produces a moment as $10 \times 9 = 90$. And $\frac{9}{10}$ of a pound, if put into the other scale at the distance of 10 inches from the center, produces a moment equal to the former, since it is as $9 \times 10 = 90$.

For the sake of making this the more intelligible I have supposed the difference of the two arms to be so great in proportion to their whole length that the eye would discover the cheat. But the difference may be so small in proportion, that it could not be discovered any other way than by changing scales. And if the balance is a false one, this method will certainly destroy the equilibrium. For, when the pound standard or $\frac{10}{10}$ of a pound is put into the scale, whose distance from the center is 10 inches, the moment will be as $10 \times 10 = 100$. And when $\frac{9}{10}$ of a pound, that is, the thing to be weighed, are put into the scale whose distance is 9 inches, the moment will be only as $9 \times 9 = 81$.

57. *When*

57. *When two men carry a weight, which hangs between them upon a pole; this pole is a lever of the second kind. And the moment which each man sustains will be inversely as the distance of the weight from him.*

If we support any weight upon a pole, which we hold out at arms length, and no person has hold of the other end of the pole; then the nearer the weight is to us, so much less is the moment, which we sustain. For this is a lever of the third kind. Our own shoulder is the center of motion, and to raise the weight we must turn the lever, which is the pole and our own arm together, round upon this center, and must raise the farther end of the pole. For this reason by bringing the weight nearer to us, we bring it at the same time nearer to the center of motion, and consequently diminish its moment. Therefore in this case the moment is directly as the distance of the weight from us.

But, when another person has hold of the other end of the pole, it is changed into a lever of the second kind. For the center of motion is his hand, if he holds the pole in his hand, or his shoulder, if he rests it there. And to raise the weight we must raise the end, which is next to us, by turning it round upon that center. The weight therefore is between the power and the prop. And, since the center of motion is in this case beyond the weight, the farther the weight is from us, the nearer it is to the center, and therefore the moment of it is the less. That is, the moment sustained diminishes as the distance from us increases, and consequently is as the distance inversely.

From hence it appears, that when two men of unequal strength support a weight upon a pole; each of them will bear a moment in proportion to his strength; if the distance of the weight from each of them is inversely as his strength.

58. *Several mechanical instruments are levers of the first or of the second kind: but scarce any instance can be found, where a lever of the third kind is used.*

Levers are used to overcome resistances as well as to raise weights. For a pair of scissors is a double lever of the first sort; the power is applied at the bow or handle of each lever, the rivet is the common center of motion, and the thing to be cut, or resistance to be overcome, is placed on the other side of this center. In a pair of scissors both the levers are moveable, but in a brasiers sheers one of them is fastened to a couple of strong standards in a block, and a man by pressing upon the long handle of the other sheer or lever, which is moveable upon the center,

center, is able to overcome the resistance of a plate of copper a quarter of an inch thick so as to cut it.

A cutting knife used by patten-makers to cut their wood is moveable on a joynt or center at one end, whereby it is fastened to a plank. The wood to be cut is placed on this plank between the joynt and the handle where the power acts. Therefore it is a lever of the second kind applied to overcome the resistance of the wood. The oar of a boat is a lever of the second kind: for the water by its resistance makes a sort of prop for one end of the oar, whilst the power acts at the other end, and the boat or weight to be moved, is between them.

A lever of the third kind is seldom used but from necessity. For the power in this lever is always nearer the center of motion than the weight is: and consequently in moving the instrument the power will always have a less velocity than the weight has. This is a mechanical disadvantage, which makes it necessary, when such a lever is used, to apply a power that would be more than sufficient to lift the weight or to overcome the resistance without any lever at all. In raising a ladder, the weight of it is lifted at first by a lever of the second kind, and afterwards by one of the third kind. The end of the ladder, which rests upon the ground, is the prop or center of motion; the man, who is to raise it, applies the power at the other end; and its center of gravity, or the weight to be raised, is between the power and the prop. But as the ladder rises the man walks under it towards the prop. And when he has got nearer to the prop than the center of gravity is; he acts upon a lever of the third kind, and accordingly finds the difficulty of raising the ladder increase as he gets nearer to the prop. A pair of sheep-sheers are two levers of the third kind. The wool or resistance is applied at one end; the bow-spring at the other end is the common center of motion; and the hand or power is applied between them.

59. *When a weight instead of being suspended by a rope, is fastened to the lever; the point, upon which the center of gravity acts, may be considered, as the point of suspension. And, if the center of gravity is above the lever, this point by inclining the lever is made to approach the lower end, but if below it, then it is made to approach the higher end.*

When the lever, Plat. II. fig. 11. with the weight c fastened above it is in an horizontal position; the line of direction is cp : and p is the point, on which the center of gravity acts. When the lever is inclined in the position SM , e the point, through which the line of direction passes,

passes, is nearer to the lower end of the lever S than p is. And when it is inclined the other way, in the position TR , a the point, through which the line of direction passes, is likewise nearer to the lower end R than p is. From hence by a bare inspection of the figure it appears, that when a weight, which has its center of gravity fastened above a lever of the first kind, is below the horizontal level, as the point of action by approaching the lower end of the lever gets farther from the center of motion, the velocity and consequently the moment of the weight will be increased. But when the weight is above the level, as the point of action by approaching the lower end of the lever gets nearer to the center of motion, the velocity and consequently the moment of the weight will be diminished.

But if the center of gravity c is fastened below the lever, and p is the point of action, whilst the lever *Plat. II. fig. 12.* is in the horizontal position AL ; then by inclining it into the position KR , ic will be the line of direction, and i the point of action will be nearer to R , the higher end of the lever, than p is. Also by inclining it into the contrary situation DF , ce will become the line of direction, and e the point of action will be nearer to the higher end of the lever than p is. From hence again it will appear by a bare inspection of the figure, that when a weight, which has its center of gravity fastened below a lever of the first kind, is below the horizontal level, the point of action, by approaching the higher end of the lever, gets nearer to the center of motion, by which means the moment of the weight is diminished. But when the weight is above the level, the point of action by approaching the higher end of the lever, gets farther from the center of motion and the moment of the weight is increased.

Nothing of this sort will happen, when the weight is suspended by a rope: because the point of suspension or point of action is not altered by inclining the lever; since, by proposition 42, as the weight swings, the center of gravity will always keep directly under the same point. Neither would the point of action be altered by inclining the lever, though the weight was fastened to it, if the center of gravity was neither above the level nor below it, but in the same horizontal plane with it.

The reader will easily perceive that what I have proved in a lever of the first kind is equally true in both the other kinds of lever. I shall apply it in the instance of two men carrying a weight between them. When two dray-men carry a barrell on a coul-staff, to which it is suspended by a chain, the point on which the weight acts is not altered by inclining the staff, and therefore in going either up or down hill, there will be no

inconvenience suffered, nor advantage gained, either by the first man or by the last. But if they carry the barrell upon two dogs, one of which is fastened at each end, then the weight does not swing, and the center of gravity is below the lever. Therefore the point, on which the weight acts, will by inclining the lever be made to approach the higher end. And in going down hill, the first man by having this point removed from him, will, according to proposition 57, be eased of a part of his burden, and the last man will have an equal part added to his. When two chair-men are going down hill, if the center of gravity of the weight, which they carry, is above the plane of their poles, the point of action will approach the lower ends of the poles, and the man that walks first will have the disadvantage. If the center of gravity is below that plane, the point of action will approach the higher ends, and the man that walks last will have the disadvantage. But, if the center of gravity is in the plane of the poles, neither of them will suffer any disadvantage by inclining the poles or by going down hill. Because the point of action, notwithstanding the poles or levers are inclined, will be the same, as if they lay horizontally, or as if the men walked upon level ground.

60. *The quantity of motion produced by a power, which acts upon a lever, is always proportional to the moving force.*

We say indeed that the weight, which a man is able to lift by the help of a lever is greater than his strength. But by this we only mean that the weight is greater than he could raise with his strength directly applied to it, in such a manner as to make his power and the weight that is to be lifted, move with the same velocity. And what enables him to raise it with the help of a lever is, that by suspending the weight at the shorter arm of the lever and applying his strength at the longer arm, the weight is made to move slower than the power. And though the moment of the weight, when it moves with the same velocity as the power does, might be greater than his strength; yet the proportion, which the velocity of the weight bears to that of the power, may, by means of the lever, be diminished, till the moving force, which he exerts, shall be equal to the moment of the weight.

No motion therefore is produced by the lever itself: the only use of it is to make the weight move slower than the power. So that by means of this mechanical instrument we can indeed lift a greater weight than we could without it: but at the same time we produce only the same quantity of motion. For just as much as the moment we can raise with

a lever exceeds in weight, so much it falls short in velocity of the moment we could raise without one. What has been here said concerning the lever, I must desire the reader to apply, as he goes along, to all the other mechanical powers.

61. *In a wheel and axle, the velocity of the power is to the velocity of the weight, as the circumference of the wheel is to the circumference of the axle.*

This machine is much used and is applied in very different shapes. Let the axle A Plat. II. fig. 13. and the wheel CD be fastened together, so that one cannot turn round but the other must turn round with it. Suppose a power or a smaller weight to be applied at a rope P, which winds about the circumference of the wheel, and a weight to be suspended at W by another rope, which winds about the circumference of the axle the contrary way. To turn the wheel once round, by pulling at the rope P, as much rope must be drawn off as winds once about the wheel, that is, as much as is equal to the circumference of the wheel. Therefore, when it is turned once round, the power, which acts at P, must have run through a space equal to the length of rope, that it has drawn off the wheel; or through a space equal to the wheels circumference. But whilst the wheel is turning once round, the axle A will turn once round along with it, and the rope, by which the weight is suspended, will wind once about the axle. Upon this account the weight W will be raised through a space equal to as much rope as will go once round the axle; or through a space equal to the circumference of the axle. Since therefore in the time that the machine is turned once round, the space, which the power describes, is equal to the circumference of the wheel; and the space, which the weight describes, is equal to the circumference of the axle; and since, by proposition 10, the velocities are as the spaces described in the same time; it follows, that the velocity of the power is to the velocity of the weight, as the circumference of the wheel to the circumference of the axle.

In all the variety of shapes, in which this machine is used, there is the same proportion between the velocities of the power and weight. Suppose that the power does not act by a rope which winds round the wheel, but that it is a mans strength applied immediately to the handles K, E, F, G, H, I. If the man first lays hold of the handle H, and pushes it down to the place G, his hand will have run through the space HG or the sixth part of a circle, and the handle I will be brought down to the place of H. He then lays hold of I and pushes it down to G as

before, by which means his hand will have gone through another sixth part of a circle. And thus by laying hold of every one of these six handles in its turn, and pushing it down to G he will turn the wheel once round, and his hand or the power, will have described $\frac{6}{8}$ or the whole circumference of a circle, whose diameter is IF. This circle is to be looked upon as the circumference of the wheel; though the wheel is then an imperfect one, for the handles are only the spokes of it. And since, whilst the power is describing this space, the weight has risen, as before, through a space equal to as much rope as winds once round the axle, it follows that, in this instance likewise, the velocity of the power is to the velocity of the weight as the circumference of the wheel to the circumference of the axle.

62. *In a wheel and axle, the power and weight are in equilibrio; if the power is to the weight, as the circumference of the axle to the circumference of the wheel.*

For then, by the last proposition, the power is to the weight, as the weights velocity, to the powers velocity. And if there be four quantities, of which the first is to the second as the third to the fourth, then the first multiplied into the fourth is equal to the second multiplied into the third. Eucl. book VI. prop. 16. Therefore the power multiplied into its own velocity, or the moment of the power, is equal to the weight multiplied into its own velocity, or to the moment of the weight.

Or otherwise, since the power is to the weight as the velocity of the weight to the velocity of the power; therefore the power and weight are to each other as their velocities inverted, and consequently, by proposition 13, their moments are equal.

We may apply this in the following instance. Let the circumference of the wheel be to that of the axle as 12 to 1, and suppose a weight of 1 pound suspended at P, Plat. II. fig. 13. and another weight of 12 pounds suspended at W, then P or 1 pound, is to W, or 12 pounds, as the circumference of the axle 1, or W's velocity, to the circumference of the wheel 12, or P's velocity. Therefore $1 \times 12 = 12$, which is the quantity of matter in P multiplied into its velocity, or P's moment, is equal to $12 \times 1 = 12$ which is the quantity of matter in W multiplied into its own velocity, or to W's moment.

In using this machine the weight is not always suspended to the axle by a rope, but is sometimes fastened to the axle itself. This is the case of a bell, the weight of which is moved, in ringing it, by a wheel and axle.

And,

And, in once turning the wheel round, the velocity of the bell is as the circumference described by its center of gravity.

In a crane wheel on the other hand the power is not applied to the wheel by means of a rope; nor does it act upon any handles or spokes. But as the man or the horse within the wheel steps forwards, the part, upon which he treads, by this means becomes the heaviest part of the wheel, and descends till it is the lowest. Thus he keeps going on, and by treading upon every part of the wheels circumference in its turn he walks over that whole circumference, and brings the wheel once round.

A capstan is a cylindrical axle perpendicular to the horizon, which is turned by four levers or bars that cross it. These are the spokes of an imperfect wheel. And the velocity of the power applied to them is as the circumference, which it describes in one revolution of the capstan.

63. *The motion of a watch is made uniform by means of the fusee; though the force of the spring is not always the same, but is greatest of all when the watch is just wound up, and keeps decreasing as the watch goes down.*

From proposition 61 it follows, that the velocity of a weight or of any other force applied to the axle of a wheel, will increase as the circumference of the axle increases. Thus Plat. II. fig. 13. a weight, or any other force, acting upon the axle B will have a greater velocity than one which acts upon the axle A. If the force was applied to the circumference TV or RS as to an axle, its velocity would be farther increased. Now the fusee of a watch is an axle in the shape of a cone. And the spring by means of the chain acts upon the smallest part of this axle, when the watch is just wound up. As the watch goes down the chain is applied to different parts of this cone, and to the thickest part of all when the watch is just down. Therefore as the elasticity or absolute force of the spring decreases, the velocity, with which it acts to move the watch increases by being applied to a greater part of the axle in proportion: and consequently, if the fusee is regulated as it ought to be, the moment produced in the wheels and in the hands of the watch will be always the same.

65. *An upper pulley, that only turns round upon its center and does not move with the weight, gives no mechanical advantage to a power.*

If a weight F Plat. II. fig. 14. is suspended at one end of a rope running across the pulley ED, which turns on the center C, but is fixed to the beam A in such a manner as neither to move upwards nor downwards; and

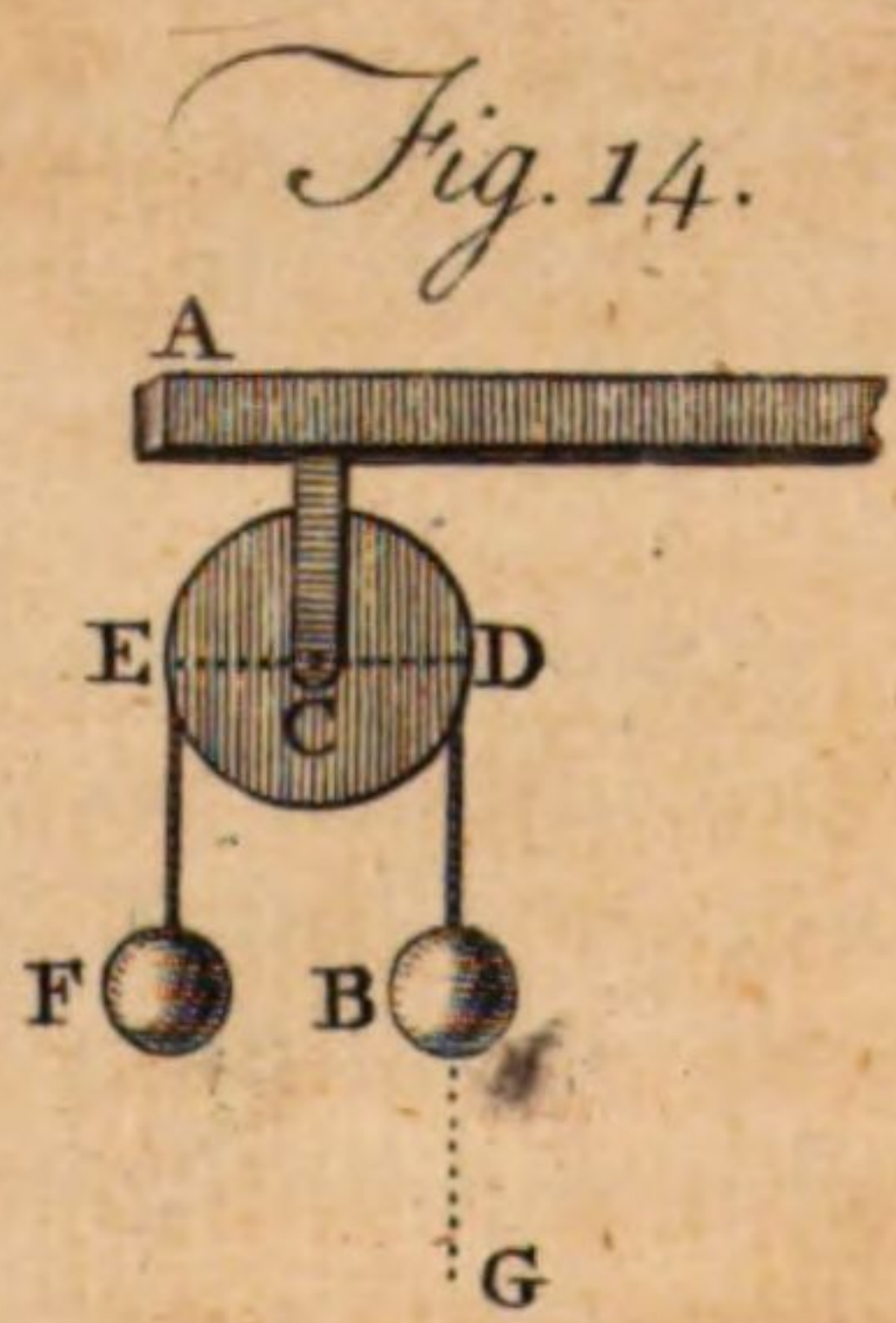
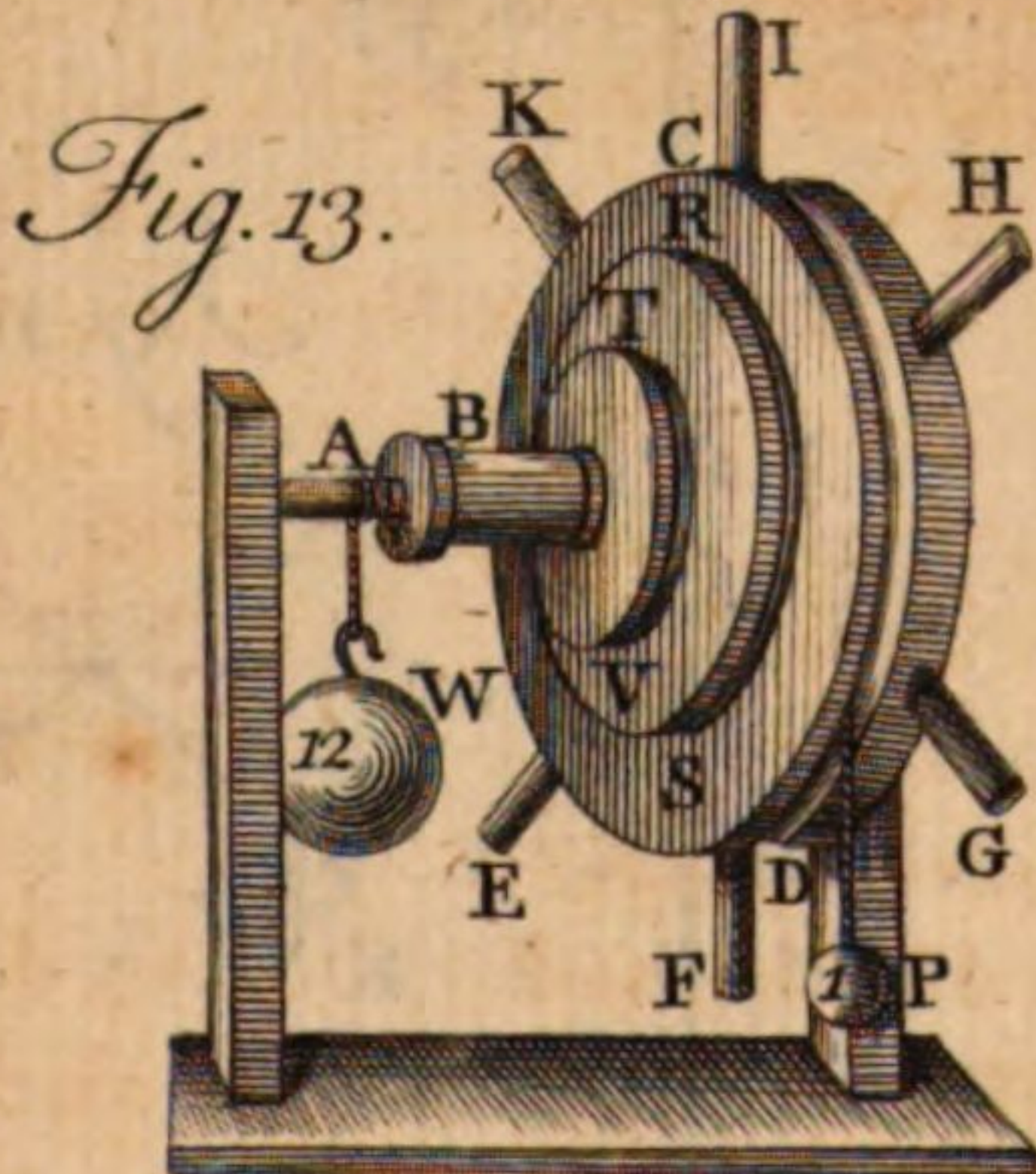
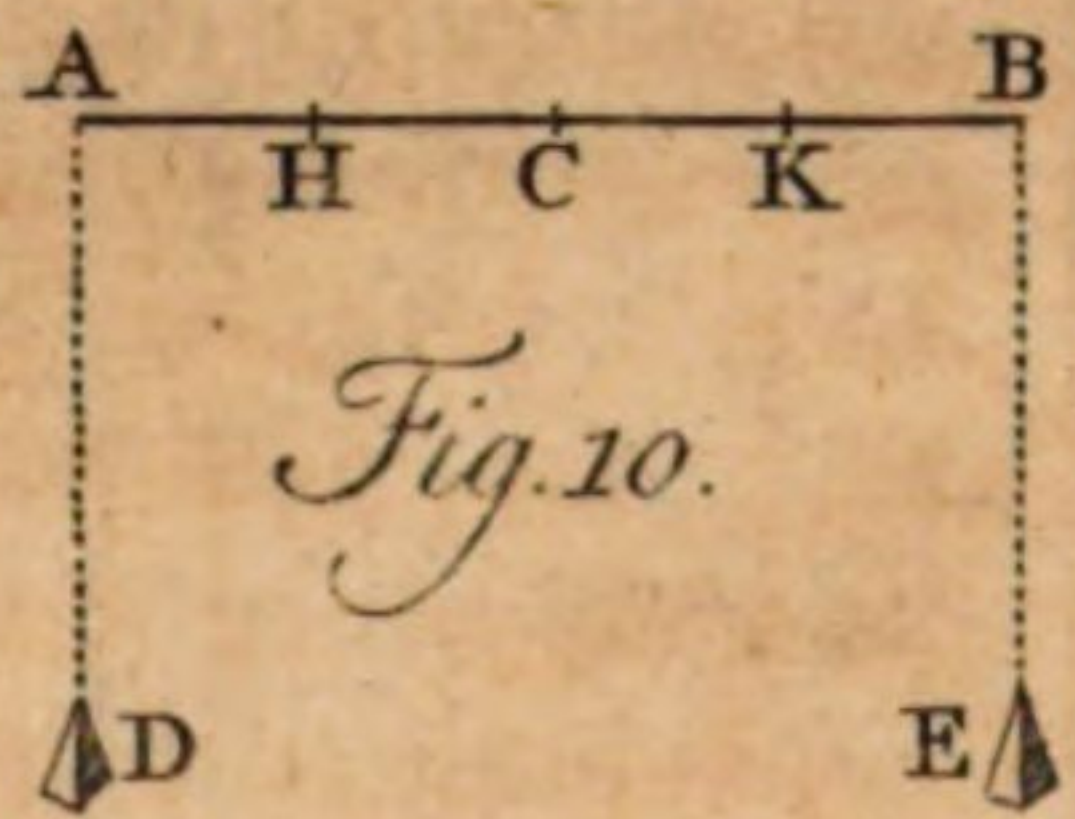
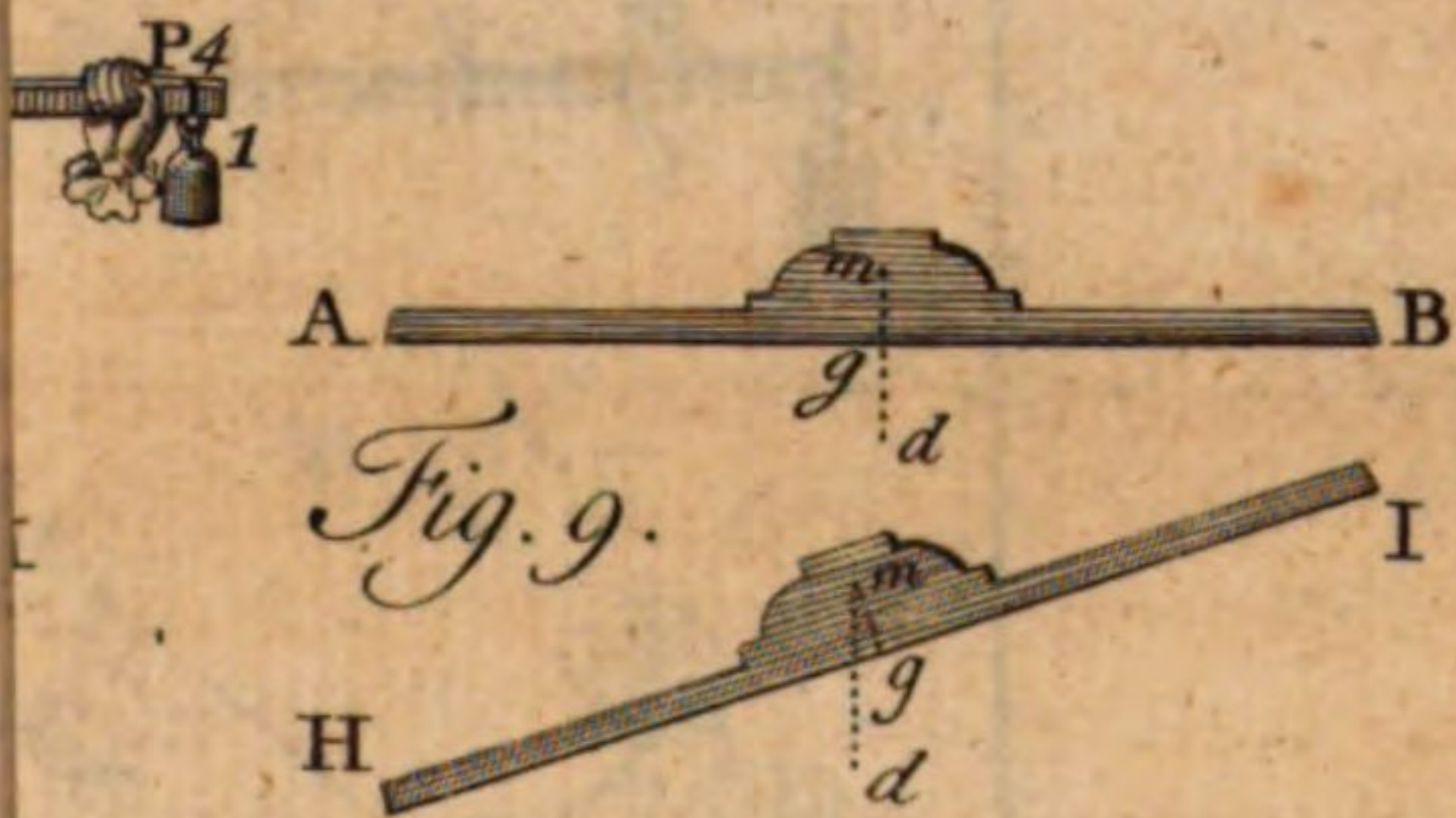
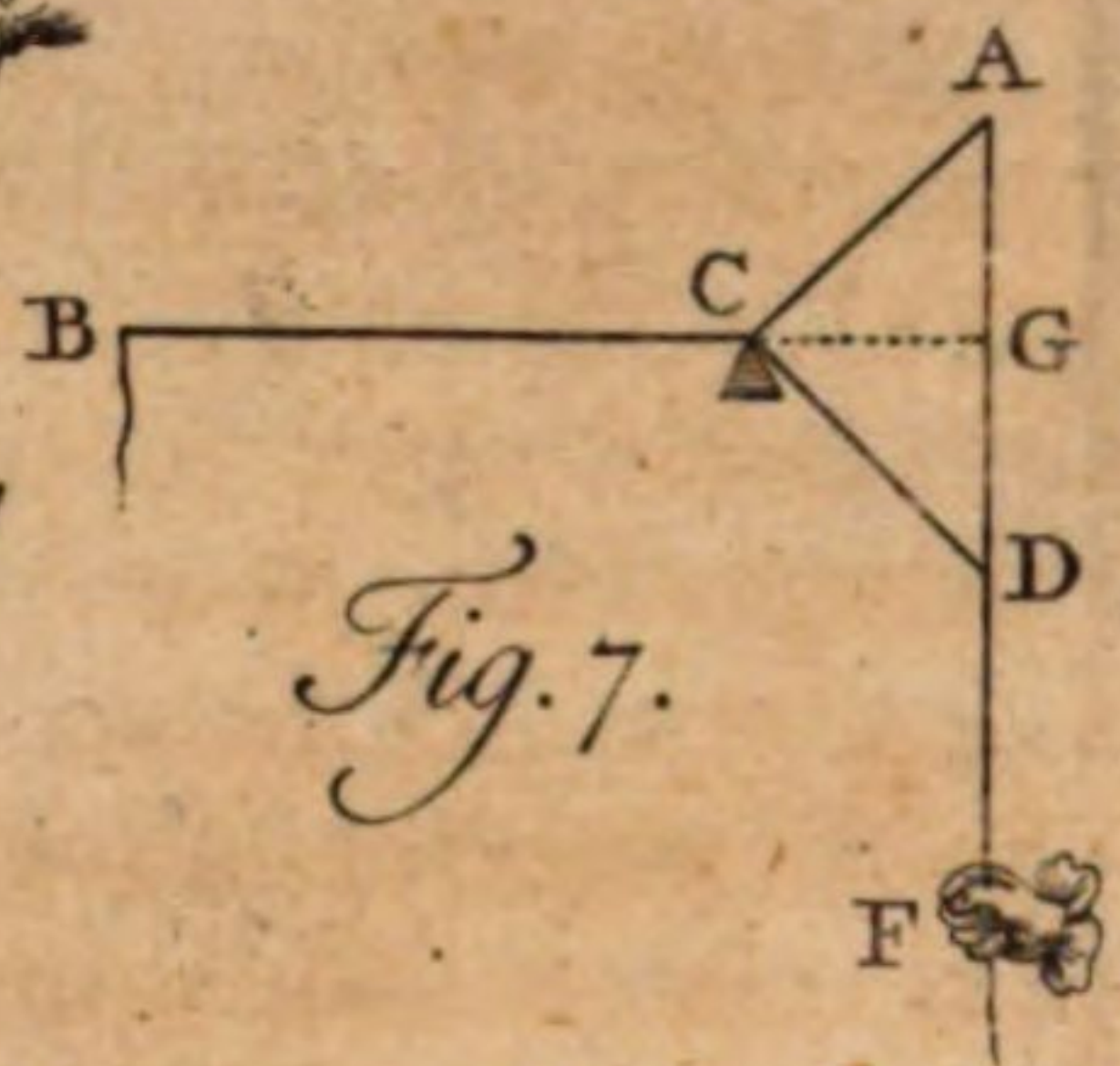
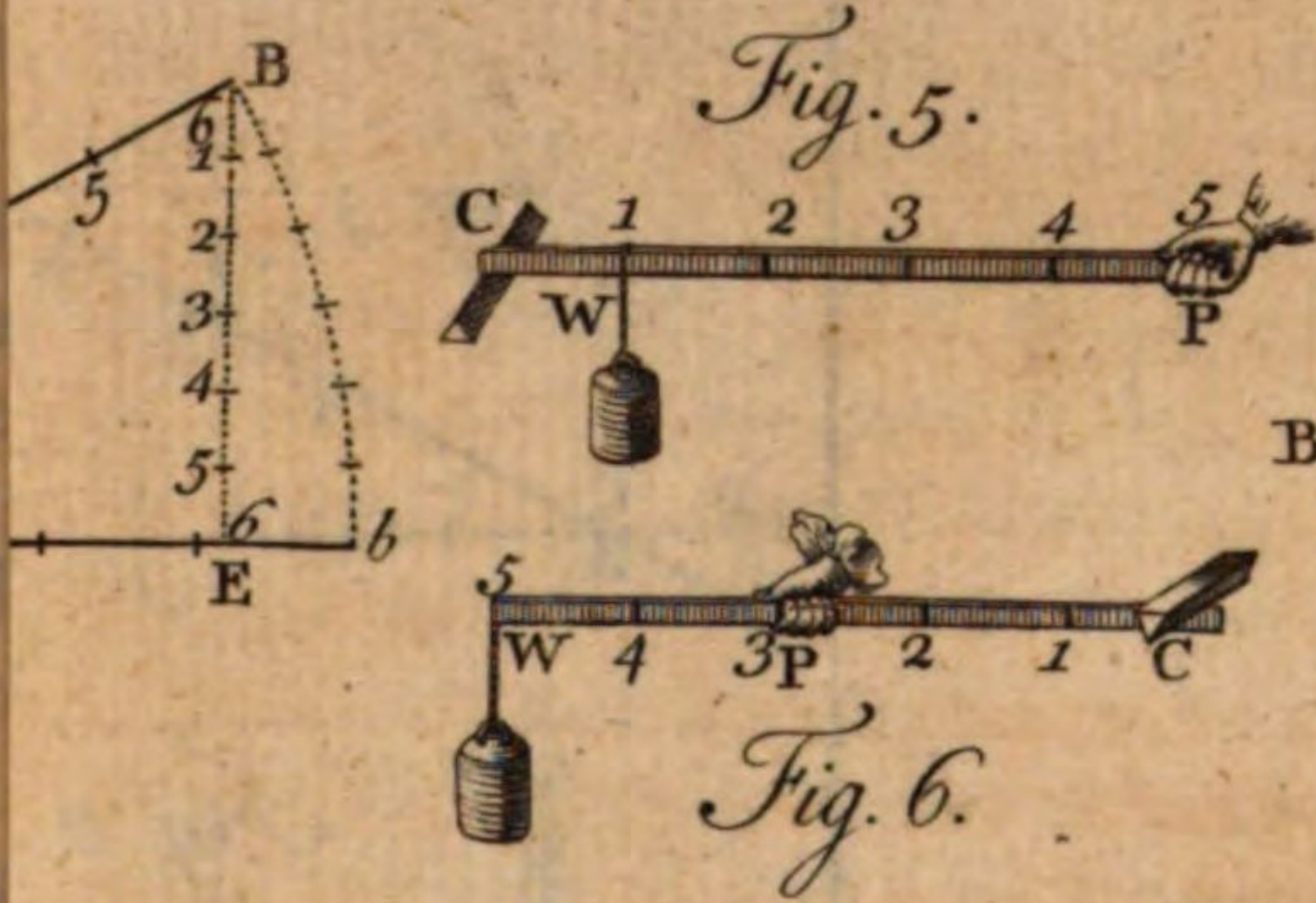
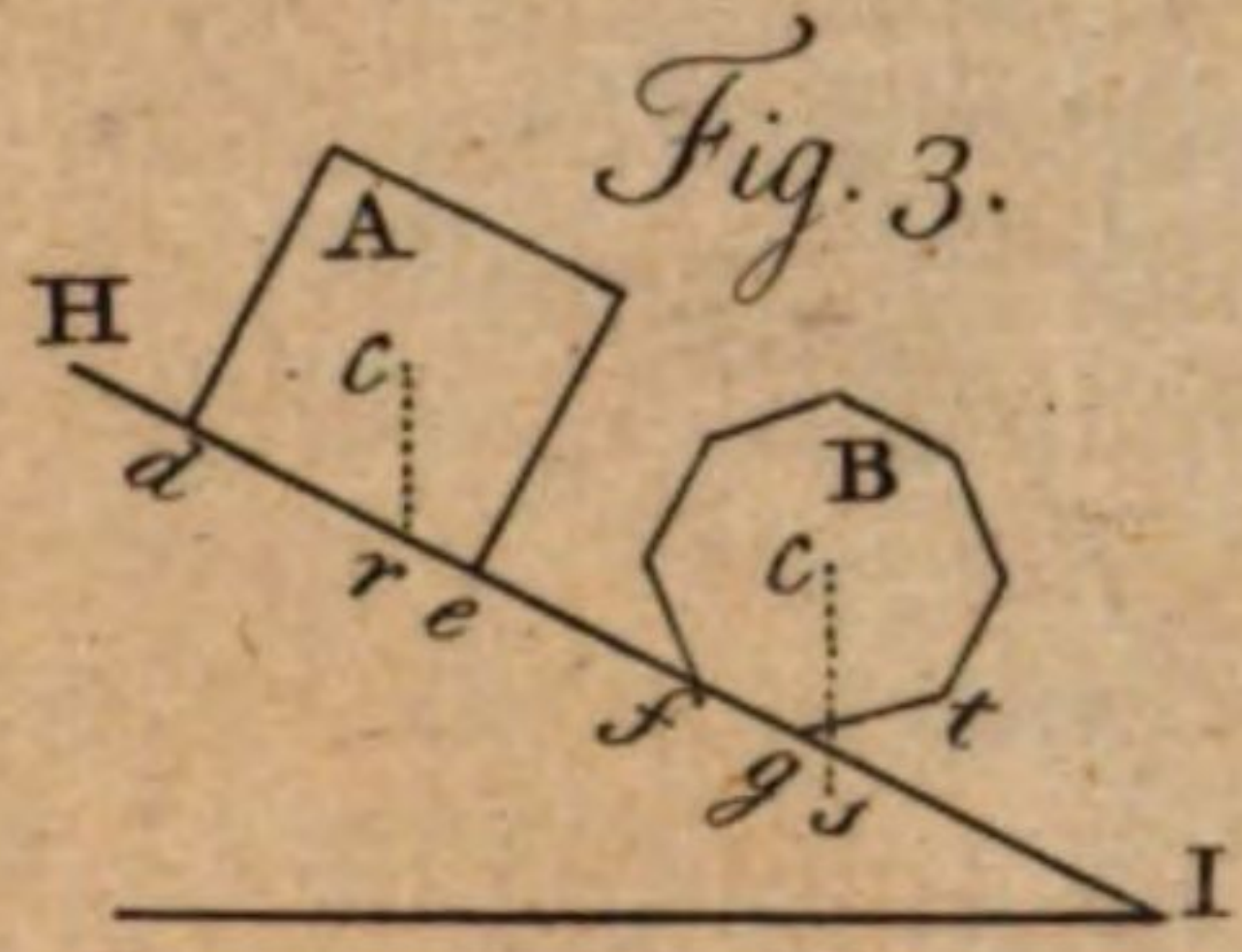
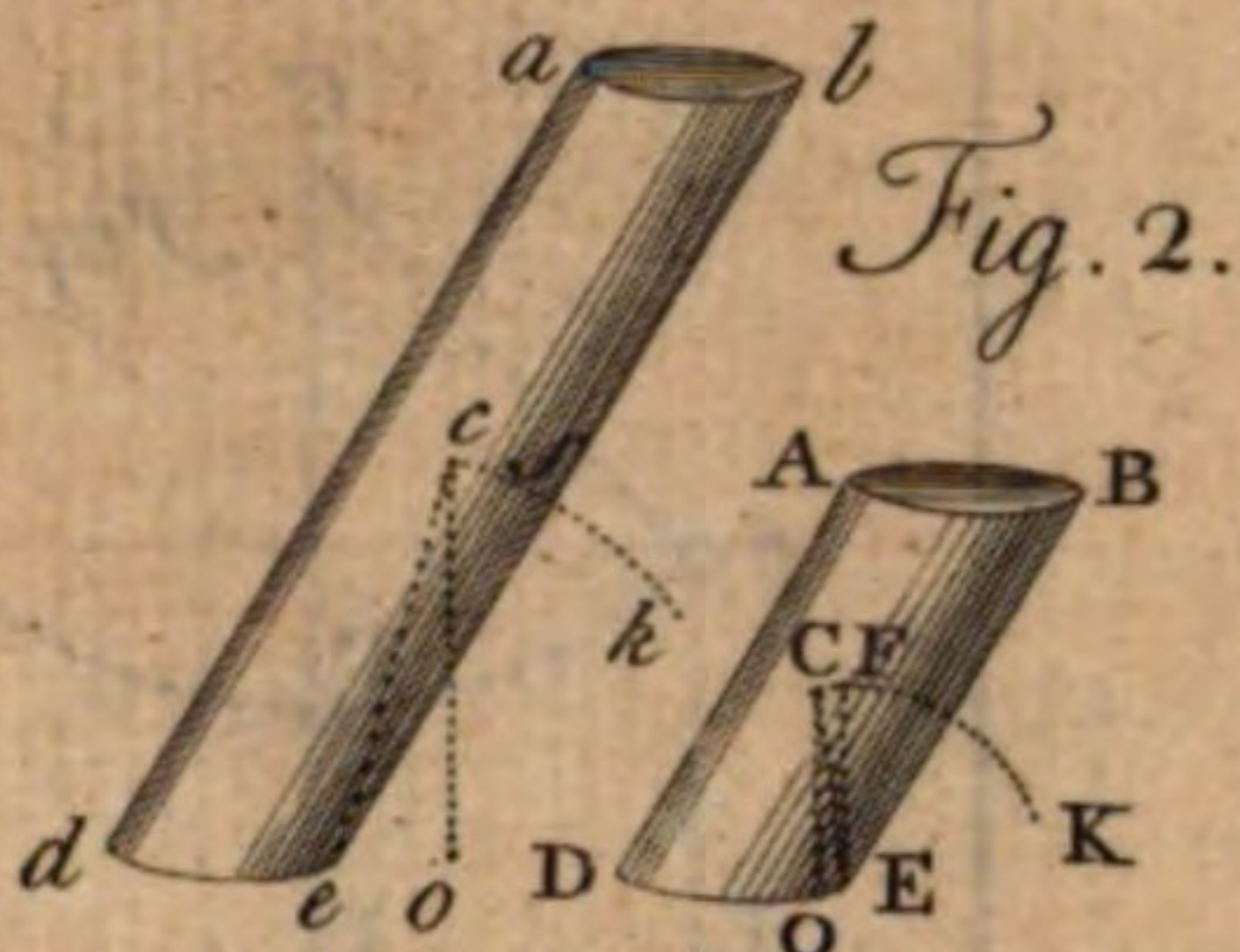
and if another weight B is suspended at the other end of the rope : then it is evident that B cannot descend through any space BG but F must rise through an equal space, because all the rope, which is drawn to the side of B in descending, is taken from the other side where F hangs; that is, as much as the side DB is lengthened, so much the side EF is shortened. Therefore the two weights move with equal velocities, and consequently, by proposition 13, will not be in equilibrio, or have equal moments, unless their quantities of matter are equal. In the same manner, if instead of a weight at B a power had been applied there, we might have proved that the velocity of the power would be equal to that of the weight, and that a power, which could lift the weight F with such a pulley as this, might lift it without any pulley at all.

If a rope is thrown across a pulley fastened at the top of a room, and a mass of lead is hung at one end of the rope ; a man, who pulls at the other end, cannot raise the mass, if it weighs more than he does himself. For supposing him to weigh 160 pounds; and the lead to weigh 200 pounds. Then if he pulls with a force less than his own weight, it is evident that he cannot raise the lead. Because a force less than 160 pounds cannot raise a weight of 200. It is likewise evident that he cannot raise it, if he pulls with a force equal to his own weight. And if he pulls with a force greater than his own weight, instead of raising the lead, he will draw himself up from the ground. For, by the third law of motion, the reaction of the rope upwards is equal to his action downwards. Therefore if he pulls downwards with a force equal to 170 pounds weight, which is 10 pounds more than he weighs himself, the reaction of the rope upwards is equal to 170 pounds or 10 pounds more than his weight. And, since the rope reacts upon him with a force greater than his own weight, he will rise with the difference.

If the mass of lead was to weigh just as much as the man himself weighs, or 160 pounds, he could not raise it. Because, if he pulls with a force less than his own weight he cannot raise a weight equal to his own, or 160 pounds. And if he pulls with a force exactly equal to his own weight, he will hang by his hands at one end of the rope, and the mass of lead, which hangs at the other end, will be in equilibrio with him, and consequently will not rise. But if he pulls with a force greater than his own weight, then, instead of raising the lead, he will draw himself up from the ground as before.

But suppose instead of the mass of lead that there was a noose at that end of the rope, and that the man was to put his foot into this noose, whilst

PLATE II.



Gul. Stephens calavit

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whilst he pulls at the other end. In this case his own weight is hung at that end of the rope, where his foot is; and his own strength is applied at the other end to raise this weight. Let us therefore enquire whether he could in these circumstances pull himself up to the top of the room. We have seen that he could not raise a weight of 160 pounds. But though he weighs 160 pounds yet he will be able to raise himself. For here it must be observed that his whole weight cannot hang upon both ends of the rope at the same time. But whatever part of his weight he hangs at that end of the rope, where his hand is, by pulling at it; just so much he takes off from the other end, where his foot is in the noose. Thus for instance, suppose him to pull with a force equal to 80 pounds or half his own weight; he then hangs half his weight to that end of the rope where he pulls, and consequently leaves but 80 pounds or half his weight in the noose at the other end. In this case his weight at one end, and his strength, with which he pulls at the other, would just be equally balanced. But suppose him to pull with any force greater than half his weight he would raise himself from the ground. Thus if he pulls with a force equal to 90 pounds, which is 10 pounds more than half his weight; then he hangs 90 pounds of his weight at the end, where his hands are, by pulling at it; and consequently leaves only 70 pounds in the other end, where his foot is. Therefore his strength, or the weight he hangs at one end by pulling at it, will overbalance the weight, which he leaves at the other end: and upon this account the end, where his foot is, will rise; and part of the rope will come over the pulley to that side, where his hand is. And if he keeps gathering up this rope as fast as it comes over, he may raise himself up to the top of the room. From this instance the reader will easily understand what was before affirmed, that this will always be the case, if the man pulls with any force which is greater than half his own weight.

65. *When, besides the upper pullies, there is another lower block of pullies, which will move as the weight moves: the velocity of the weight is to the velocity of the power as 1 to the number of ropes, that support the lower block.*

The weight W Plat. III. fig. 1. is fastened to a lower block with one pulley in it; and the block is supported by two ropes *fe*, and *dg*: that is, by two parts of the same rope, which has one end fastened upon the beam A, and running across the lower pulley is divided by it into two parts; and then running across the upper pulley at *d* comes over to the side BP, where we suppose the power to act. We have reckoned only

only two ropes: for the third BP to which the power is applied, does not support the lower block.

Now in this instance the velocity of the weight is to that of the power as 1 to 2. For to draw the weight through any space, as HW, each of the ropes *fe* and *dg* must be shortened by a quantity equal to HW; and what rope is drawn from this side of the fixed pulley at *d* will run over to the side BP, therefore the space, which the power describes, whilst the weight is ascending through HW, will be equal to BP or to twice HW. And, since the velocities, by proposition 10, are as the spaces described in the same time, the velocity of the weight is to that of the power as 1 to 2, or as 1 to the number of ropes.

Universally; whatever is the number of ropes, by which the lower block is supported; when the weight is to be raised through any space, each rope must be shortened by a quantity equal to that space. And, since the sum of all the parts taken from each rope runs over to the side of the power; it follows that, whilst the weight is rising through any given space as 1, the power will describe a space which exceeds it as much as the sum of all the parts exceeds one part, or as much as the number of ropes exceeds 1. Therefore the space described by the weight is to the space described by the power in the same time, or the velocity of the weight is to the velocity of the power, as 1 to the number of ropes.

The weight W, Plat. III. fig. 2. which hangs to the lower block, is supported by 5 ropes: the rope, to which P the power or the lesser weight is applied, is not to be reckoned into the number. Now to raise the lower block and the weight along with it 1 foot higher than it is, or one foot nearer to the upper block, these 5 ropes must be shortened 1 foot each. And, since the sum of all these parts, or 5 feet of rope, runs over to the side where P is applied; it follows that, whilst the weight is rising through one foot, the power describes 5 feet. Therefore the velocity of the weight is to that of the power as the spaces described in an equal time, or is as 1 to 5, or as 1 to the number of ropes.

These two blocks are made in an inconvenient form. For by placing the pulleys one below another the blocks are made so long, that they will touch each other, before the weight comes near the hook, which the upper block hangs upon. To prevent this, the pulleys are usually placed so as to turn upon the same axis: and then a block for 3 pulleys Plat. III. fig. 3. is no longer than a block for one. The weight W is here supported by 6 ropes. And since each must be shortened 1 foot in order to raise the weight 1 foot, and the power will in the mean time have

have 6 feet of rope run over to the side, where it is applied, and consequently will describe 6 feet, whilst the weight is describing 1. Therefore the velocity of the weight is to that of the power as 1 to 6 or as 1 to the number of ropes.

66. *In pullies the power and weight will be in equilibrio, if the power is to the weight as 1 to the number of ropes that support the lower block.*

For then, by proposition 65, the power and weight are to each other as their velocities inversely; therefore, by proposition 13, their moments are equal.

Let W Plat. III. fig. 3. be 6 hundred weight, and P 1 hundred; then P is to W as 1 to the number of ropes. But P's velocity is to W's velocity as the number of ropes to 1, or as 6 to 1. Therefore P 1 hundred weight multiplied into its own velocity or $100 \times 6 = 600$, and W 6 hundred weight multiplied into its own velocity or $600 \times 1 = 600$.

From hence we may see that a weight acts with less velocity, and consequently with less moment in proportion to the number of ropes that support it. Therefore 1 hundred weight suspended at a single rope, which runs across an upper pulley, will turn a jack as well as 6 hundred weight suspended at a lower block supported by 6 ropes. But there is a good reason for preferring 6 hundred weight with tackle of six, (as it is called) or with a set of pullies having six ropes, to 1 hundred weight with a single rope. For in the former case there is 6 times as much rope as there is in the latter; and consequently the weight will be 6 times as long in going down; or the jack will go 6 times longer without winding up.

67. *The velocity of two equal resistances, of which one acts perpendicularly upon one side of an isosceles wedge and the other perpendicularly upon the other side, is to the velocity of a power, which acts perpendicularly upon the back of it, as the base of the wedge is to the sum of its sides.*

Let ABC Plat. III. fig. 4. be an isosceles wedge, that is, a wedge, whose sides AC and BC are equal to each other. And let PP be two equal weights, which are to be separated from each other by driving the wedge between them. When the wedge is quite driven in by a power, such as the stroke of a hammer, applied perpendicularly to the back AB; the space described by the resistance, which these two weights make, will be to the space described by the power in the same time, or, by proposition 10, the velocity of the resistance will be to the velocity of the power, as AB the base of the wedge is to AC added to CB the sum of the sides.

H

When

When the wedge is driven between the weights into the situation MON ; one weight will have been removed from P to Q on one side, and the other from P to Q on the other side : so that CK perpendicular to one side is the space described by one of them, and CL perpendicular to the other side is the space described by the other of them. But since CK is equal to CL, the space described by the whole resistance is the same, whether one weight describes CK, and the other CL, or both together describe CL. Or, since one weight is only half the resistance, and the other weight is the other half, CK added to CL is described by half the resistance, CK by one weight which is only half the resistance made by both of them and CL by the other. But CK is equal to CL. Therefore CK and CL together are equal to twice CL ; and consequently as it is only half the resistance which in driving the wedge in describes a space equal to twice CL ; the whole resistance describes but half this space or once CL. In the mean time the power, which is applied at D, having driven the wedge into the position MON, has moved from D to C or has described the space DC. Therefore, taking DG perpendicular to BC and equal to CL, the space described by the resistance is to the space described by the power in the same time, and consequently, by proposition 10, the velocity of the resistance is to the velocity of the power, as DG to DC. But since DC the direction of the power is perpendicular to AB the back of the wedge, it follows that BDC is a rightangled triangle. And, since DG the direction in which the resistance moves is perpendicular to BC, the triangle DGB is similar to the triangle BDC, Eucl. b. VI. pr. 8. and DG is to DC as DB is to BC. Therefore the velocity of the resistance is to the velocity of the power either as DG to DC, or which has just been proved to be the same thing, as DB half the base of the wedge to BC one of its sides. But as DB half the base to BC one side, so is twice DB or AB the whole base to twice BC, or (since CB is equal to AC) to BC added to AC or the sum of the sides. Therefore the velocity of the resistance is to the velocity of the power as the base of the wedge to the sum of the sides.

68. *If a power acts perpendicularly to the back of an isosceles wedge, and a resistance is so divided that half of it may act perpendicularly to one side of the wedge and half of it perpendicularly to the other ; they will be in equilibrio, if the power is to the resistance, as the base of the wedge to the sum of its sides.*

For, by the last proposition, the velocity of the resistance is to the velocity of the power as the base of the wedge to the sum of its sides.

There-

Therefore, if the power is to the resistance as the base to the sum of the sides, the power and resistance are to each other as their velocities inversely; and, by proposition 13, their moments are equal.

This proposition may be proved independently of the foregoing one. Let ABC Plat. III. fig. 5. be an isosceles wedge, that is, a wedge, whose sides AC and BC are equal to each other. A power applied perpendicularly to the back of it acts in the direction GC. Let ED half the resistance act in the direction EG perpendicular to the side AC; and *ed* the other half resistance act in the direction *eG* perpendicular to the side BC. Now since EG and *eG* are oblique to GC, that is, since the resistance acts obliquely to the power, it follows that the whole resistance does not oppose itself to the power: for only that part of the resistance, which is directly contrary to the power's direction, or to GC, opposes itself to the power, and prevents the motion of the wedge in that direction. Therefore the power will be in equilibrio with the resistance, though it is not equal to the whole resistance; provided it be equal to that part of it, which acts in a direction opposite to GC. To find out what part of the half resistance ED is opposed to the power and prevents the wedge from moving in the direction GC, resolve the oblique force ED into two others, of which EF is parallel to AB the back of the wedge and FD perpendicular to it. This is done, according to proposition 20, by considering ED as the diagonal of a parallelogram, of which EF and FD are the two sides. In the same manner the other half resistance *ed* may be resolved into *ef* parallel to AB, and *fd* perpendicular to it. Therefore the part of the resistance, which by acting perpendicularly to the back of the wedge opposes itself to the power, is to the whole resistance as $FD + fd$ to $ED + ed$; or, since FD is equal to *fd*, and ED equal to *ed*, as twice FD to twice ED. But since the power will be in equilibrio with the resistance, when it is equal to that part of the resistance, which is opposed to it, there will be an equilibrium, when the power is to the whole resistance as twice FD to twice ED, that is, as I shall prove presently, as the base of the wedge to the sum of its sides. In the mean time it will be necessary to say something concerning the two other parts of the half resistance on each side EF and *ef*. Now these two parts being equal and in contrary directions destroy each other, and so neither oppose the power nor hinder the motion of the wedge in the direction GC.

But I am to prove that twice FD is to twice ED as the base AB to the two sides AC and CB taken together. EFD and GDA are similar triangles. For the angle EFD is a right one, because FD is perpendi-

cular to AB and EF is parallel to it. Eucl. b. I. pr. 29. And GDA is likewise a right angle, and therefore equal to EFD; because EDG is perpendicular to AC by the supposition. AGE is likewise equal to GEF, because AB and EF are parallel, and these two are alternate angles. Euc. b. I. pr. 29. From whence it follows, that, since the three angles in every triangle are only equal to two right ones, and two angles in the triangle EFD are respectively equal to two angles in the triangle ADG, the third angle EDF in one is equal to the third angle GAD in the other. Eucl. b. I. pr. 32. corol. 2. and the two triangles are similar. But AGC is a rightangled triangle, and the line GD drawn from the right angle is perpendicular to AC: therefore the triangle ADG and consequently the triangle DFE is similar to the triangle AGC. Eucl. b. VI. pr. 8. And FD is to ED as AG to AC, or twice FD to twice ED as twice AG or AB the base of the wedge to twice AC or AC and BC together, which is the sum of the sides. Eucl. b. VI. pr. 4.

From this proposition, when it is thus demonstrated, the foregoing proposition may be deduced as a corollary. For since the power and resistance are in equilibrio, when the power is to the resistance as the base of the wedge to the sum of its sides. And since, by proposition 13, when they are in equilibrio, they are to each other as their velocities inversely. It follows that the velocity of the resistance is to the velocity of the power, as the power is to the resistance, or as the base of the wedge to the sum of its sides.

69. *If in cleaving wood the power is to the resistance of the wood, as the base of the wedge to the sum of its sides, they will be in equilibrio; and the power, if it be ever so little increased, will drive in the wedge.*

I find that one writer upon mechanics makes the proportion of the power to the resistance of the wood to be less than this, and maintains that there will be an equilibrio, when the power is to the resistance, as a quarter of the base to one side, or as half the base to the sum of the sides. He supports his opinion in this manner. If the wedge ACB Plat. III. fig. 6, is driven into the wood VCF, which is split no farther than the point of the wedge; it is evident that the velocity of the wood at F and V must be greater than the velocity of it any where else between F and C or V and C: since the two pieces CF and CV turn round upon C as upon a joynt or center of motion. And indeed the velocity of each piece, or the velocity of the whole resistance will in this case be only half what it would have been, if the lower parts had moved with the

the same velocity as the upper ones. For let us reckon from C to F, and imagine four points taken in CF at equal distances from each other C, M, R, and D; then let the velocity of C=0, the velocity of M=1, that of R=2, and that of D=3. Therefore the sum of all the velocities, or the velocity of the whole piece = $0+1+2+3=6$. But if all these points had moved with the same velocity as D, that is, if each point C, M, R, D, had had 3 degrees of velocity, then the sum of all these velocities or the velocity of the whole piece would have been = $3+3+3+3=12$. And universally, the velocity of a resistance turning round upon a joynt as C may be represented by the sum of a number of terms in arithmetical progression beginning from 0 and ending with the greatest velocity; whereas the velocity of a resistance, which does not turn round in this manner, but has the same velocity at the bottom as at the top, may be represented by the sum of an equal number of terms, each of which is equal to the last in the other progression. But the former sum will only be half the latter; as in the instance just produced, 6 is the half of 12. Therefore the velocity and consequently the moment of the whole resistance, when each of the pieces turns round upon a joynt, is but half what it would have been if the lower parts had moved with the same degree of velocity as the upper parts. Now were all the parts of the wood to have the same degree of velocity, the power, when there is an equilibrium, would be to the resistance as AB to AC+BC that is as the base to the sum of the sides, or as GB to BC half the base to one side, by the last proposition. Therefore when the moment of the resistance is but the half of this, or when the resistance turns round on a joynt, then half the power will be sufficient to produce an equilibrium: that is, a power which is to the resistance, as half the base of the wedge to the sum of the sides, or as a quarter of the base to one side.

In answer to this reasoning, I shall first establish the truth of the proposition, and then shew what it was which misled this writer.

First, I am to shew that, if there is an equilibrium on a wedge, the power must be to the resistance, as the base to the sum of the sides, or as half the base to one side, not only when the lower parts of the pieces FC and VC move with the same velocity, as the upper parts, but likewise, when they turn round on a joynt C at the bottom. The parts of the wood I, and Q, are not separated by the whole wedge ACB, but by a small wedge ICQ which is a part of the whole one. And at M the velocity of the resistance, which the wood makes there, is to the velocity of the power, by which this smaller wedge is driven in, as NM to

NC

NC or as NQ to QC, that is, since GCB and NCQ are similar triangles, as GB to BC, or as half the base of the whole wedge to one side of it. Therefore to make an equilibrium at M, the power must be to the resistance as NQ to QC or as half the base to one side. So likewise the parts of the wood E and O are not separated by the whole wedge ACB, but by a part of it ECO, which may be considered as a small wedge by itself. And at R the velocity of the resistance made there by the wood is to the velocity of the power considered as driving in the wedge ECO in the proportion of HR to HC or of HO to OC, that is, because HCO and GCB are similar triangles, as GB half the base of the whole wedge to BC one of its sides. And lastly the velocity of the resistance at D is to the velocity of the power, which separates that resistance by driving in the whole wedge as GD to GC, or as GB to BC. But since the velocity of each part of the resistance is to the velocity of the power by which that part is separated as half the base of the wedge to one of its sides; it follows, Eucl. b.V. pr. 12. that the velocity of the whole resistance is to the velocity of the whole power in the same proportion, that is, as GB to BC half the base to one side, or as AB the base to AC and BC together, which is the sum of the sides. But there will be no equilibrium unless the power and the resistance are to each other as their velocities inversely, or unless the power is to the resistance as half the base to one side or as the whole base to the sum of the sides.

The writers mistake was owing to this. He had proved that the resistance made by the two pieces FC and VC if they turn round upon a joynt at C has but half the moment of the same resistance, supposing the two pieces were separated as far at the bottom as they are at the top, and therefore he concludes that half the power will be sufficient to make an equilibrium. Whereas the same power would be requisite in both cases. Because the same power has but half the moment when the two pieces turn round upon a joynt at C that it would have, if the lower parts were to move as far asunder as the upper parts do. For the velocity of the power, which separates the wood at C = 0, because the point of the wedge is only contiguous to the wood at that place and does not enter it to any depth. The space described by the power to drive in the wedge ICQ, which separates the wood at I and Q is NC call this 1 and then the velocity of the power in that place = 1. The space described by the power to drive in the wedge ECO, which separates the wood at E and O is HC this is double NC, therefore the velocity of the power in that place = 2. In like manner, the space described by the power, which drives in the whole wedge ACB and separates the wood at

at the top, is GC or triple NC, therefore the velocity of the power at the top $= 3$. From whence it follows that the sum of all the velocities or the whole velocity of the power reckoning from the bottom to the top $= 0 + 1 + 2 + 3 = 6$. But if the lower parts of the wood had been separated as far as the upper parts, then the effect of the power upon the wood at C would have been as great as if the whole wedge had been thrust quite down from VF to C, that is the power would have acted in every part of the wood with the same velocity that it acts with to drive in the whole wedge, or the same that it acts with upon V and F or with the velocity $= 3$. Therefore the sum of all the velocities at the four places C, N, H, and G would have been $3 + 3 + 3 + 3 = 12$. And universally the velocity of a power, when it acts by means of a wedge on a resistance, which turns round upon a joynt as C, may be represented by the sum of a number of terms in arithmetical progression, beginning from 0 and ending with the velocity with which the power acts to drive in the whole wedge: whereas the velocity of a power, if the two resisting bodies separate as far asunder at the bottom as at the top, may be represented by the sum of an equal number of terms, each of which is equal to the last in the other progression. But the former sum will only be half the latter, as in the instance just produced 6 is the half of 12. Therefore the same power has but half the velocity and consequently but half the moment, when the two resisting bodies turn round upon a joynt, that it would have, if the lower parts of them were to separate as far asunder as the upper parts do.

Now it is granted that to make an equilibrium, a power must be to a resistance as half the base to one side, if the lower parts of the wood were to separate as far as the upper ones. And though the moment of the resistance is but half, when instead of this, they turn round as upon a joynt at C: yet the same power will be requisite to make an equilibrium in both cases. Because, as in the latter case, the moment of the resistance is but half what it is in the former; so the moment of the same power, when the parts of the wood turn round as upon a joynt, will be but half the moment it would have, if they separated as far asunder at the bottom as they do at the top. Therefore in cleaving wood with a wedge, there is an equilibrium, when the power is to the resistance as half the base to one side of the wedge, or as the whole base to the sum of the sides.

70. *In a screw the velocity of the weight or resistance is to the velocity of the power, as the distance between two contiguous spiral rounds of the thread to the circumference, which the power describes in one turn of the screw.*

An infide screw is a cylinder of iron, brass, or wood, or any other substance, with a spiral thread winding round it; which thread is made out of the same piece with the cylinder: G is the top of the cylinder Plat. III. fig. 7. ABb or CdD are parts of the spiral thread which winds round it. The outside screw, is a hollow cylinder in the plank or plate of metal Kk , and in this hollow cylinder is another spiral thread like that of the infide screw; this is called the nut, and the thread of the infide screw turns in the cavities of the nut or outside one. The power is commonly applied to a bar or handle GF . If there is any weight to be raised, such as the beam of a house which has sunk, the piece IH with the whole machine being set under it, the screws GG are turned round by the bars GF : the ends of the screws bear against the plank ML ; and as they turn round the plank Kk , in which are the infide screws, rises and the piece HI rises with it, by which means the weight at I is raised. Now in one revolution of the screw, the plank Kk , and the piece HI , and consequently the weight at I rises through a space equal to Dd , the distance between two contiguous rounds of the spiral thread: for if the plank hung upon the lower round D at first, by turning the screw once round, it will be made to hang on the next round d . In the mean time the power has described the circumference of a circle, whose semidiameter is GF . But the velocities are as the spaces described in the same time, by proposition 10; therefore the velocity of the weight is to the velocity of the power as the distance between two contiguous rounds of the thread to the circumference, which the power describes in one revolution of the screw.

71. *A power and a weight or resistance are in equilibrio by means of a screw, if the power is to the weight or resistance, as the distance between two contiguous rounds of the thread to the circumference, which the power describes in one revolution of the screw.*

For then, by the last proposition, the power and weight are to each other as the velocity of the weight to the velocity of the power, that is, they are to each other as their velocities inversely. Therefore, by proposition 13, their moments will be equal.

72. *In*

72. *In compound machines there will be an equilibrium; if the power is to the weight, as the velocity of the weight is to that of the power.*

ABD Plat.III. fig.8. is a compound lever: for it consists of three levers. The velocity of the weight is to that of the power as 1 to 5, in the first of them, or A: as 1 to 4, in the second of them, or B: and as 1 to 6, in the third of them, or D. By proposition 49. Therefore, when all three are used together, the velocity of the weight is to that of the power as 1 to 5 and 1 to 4 and 1 to 6: that is, as $1 \times 1 \times 1 = 1$ to $5 \times 4 \times 6 = 120$. and consequently if the power is to the weight as 1 to 120, by proposition 14, their moments will be equal.

AB and FG are two wheels, CI and KH are their axles. Plat. III. fig. 9. The axle of the upper wheel has a pinion on it, the teeth of which, by taking the teeth in the circumference of the lower wheel, will turn it round. Let the circumference of the wheel AB be to the circumference of KH the axle of the other wheel FG as 8 to 1: and let the pinion have 8 teeth and the wheel FG have 40. Then the circumference AB, on which the power acts, being to the circumference of the axle KH, where the weight is suspended, as 8 to 1; the velocity of the weight is to that of the power as 1 to 8, by proposition 62. But since there are only 8 teeth in the pinion at C and 40 in the circumference FG, it follows, that whilst the wheel AB and the pinion are turning round five times, the wheel FG will turn round but once. Therefore again the velocity of the weight suspended at the axle KH will be to the velocity of the power acting at the wheel AB as 1 to 5. So that the velocity of the weight is to the velocity of the power, as 1 to 8 and as 1 to 5, or as $1 \times 1 = 1$ to $8 \times 5 = 40$. And if the power is to the weight as 1 to 40, they will be as their velocities inversely, and consequently, by proposition 14, their moments will be equal.

In a machine compounded of four pulleys with a separate string for each of them, Plat. III. fig. 10. a power M equal to 1 pound will support a weight P equal to 16. For in each pulley used alone there would be an equilibrium, if the power was to the weight as 1 to 2. Therefore when all four are used together, the power must be to the weight in a ratio compounded of four times the ratio of 1 to 2, or as $1 \times 1 \times 1 \times 1 = 1$ to $2 \times 2 \times 2 \times 2 = 16$.

CHAP. VII.

Concerning the mutual action of bodies upon each other.

73. *All attraction is mutual.*

IF the earth attracts a stone, then, by the third law of motion, proposition 15, equal quantities of motion will be produced in each of them; that is, they will approach each other with equal moments, or the stone will attract the earth just as much. For suppose the attraction on the side of the earth to be the greater, and consequently that the quantity of motion produced by this attraction is greater in the stone than in the earth: then if there was any obstacle, such as a plate of iron, placed between the stone and the earth to prevent them from coming together, this plate must be more pressed by the stone one way than it is by the earth the other way; and consequently as the plate is acted upon by two unequal forces; the stone, the plate, and the earth would all begin to move in the direction in which the stone was moving, by proposition 19, and this motion so produced would always continue, and would be constantly accelerated so long as the moment of the stone was greater than the moment of the earth. But this is absurd and contrary to the first law of motion, proposition 11: for it cannot be true unless a mass of matter, such as the stone, the plate, and the earth, can move itself, and when in motion accelerate itself. Therefore the plate could not be more pressed one way by the stone than it is the other way by the earth: but the quantities of motion, and consequently the attractions on each side, must be mutual and equal. Indeed the quantity of matter in the stone is so small in respect of the earth, that, since, by proposition 13, to make the moments equal, the velocity of the earth must be small in proportion, we may consider the whole velocity as belonging to the stone without any sensible error.

74. *The common center of gravity of two or more bodies is not made to change its state of resting or of moving uniformly in a right line by the mutual action of the bodies upon each other.*

The common center of gravity of two boats, is a point which divides the distance between them in the inverse proportion of the quantity of matter. That is; the common center of gravity is a point taken between two unequal boats, as much nearer to the great one than it is to the little one as the quantity of matter in the former is greater than the quantity of matter in the latter, by proposition 50. Now if a man in
one

one boat pulls a rope, which is fastened to the other, they will approach with velocities that are inversely as their quantities of matter, by propositions 47, 15; that is, with velocities that are to each other as the respective distances of the boats from the common center of gravity. Since therefore the velocities produced on each side of this center by this mutual action are inversely as the quantities of matter, it follows, by proposition 14, that the quantities of motion on each side of it will be equal; that is, the two boats will approach it with equal moments. Or, if the boats were very near to each other, and the man in one of them was to push with a pole, or by any other means, against the other; then likewise by his action against the boat, which he pushes, and the reaction of that against the boat in which he stands, the boats will be separated from each other with velocities that are inversely as their quantities of matter, by proposition 16; that is, with velocities, that are to each other as the respective distances of the boats from the common center of gravity. Since therefore the velocities produced on each side of this center in this case of mutual action are likewise inversely as the quantities of matter, it follows as before, that the quantities of motion on each side of it are equal; that is, the two boats will depart from it with equal moments. In all instances of impulse, and collision, of attraction, and repulsion, or of mutual action of any sort, the bodies which act upon each other will for the same reasons either mutually approach to the common center of gravity, or mutually depart from it with equal moments: but whether they mutually approach to it or mutually depart from it, these equal moments on each side of the center are produced in contrary directions: and as a single body or the center of gravity of a single body would neither be accelerated nor retarded, nor be any ways made to change its present state of motion or rest, by equal moving forces acting upon it, or by equal changes of motion produced in contrary directions, by proposition 19; so neither can these equal changes of motion produced in contrary directions on each side of the common center of gravity of two bodies accelerate or retard that center or any ways change its state of motion or rest,

Suppose there were four bodies, which mutually acted upon each other: then, if A and B were two of them, by what has been said already, the common center of gravity would not change its state of resting or of moving uniformly in a right line on account of the mutual action of these two upon each other. And for the same reason, if C and D were the other two, their common center of gravity would not be made to change its state by their mutual action. And lastly, since the distance

between these two centers is divided by the common center of gravity of all four in the inverse proportion of the quantities of matter on each side of it, and since these two centers do not change their state of resting or of moving uniformly, therefore that center will not change its state, upon account of the mutual action either of A and B or of C and D upon each other. For the same reason it would not change its state upon account of the mutual action either of A and C or of B and D upon each other. And since the mutual action of all the four bodies may be thus reduced to the action of any two and two of them upon each other, as of A and B of C and D; or of A and C of B and D; or of C and B of A and D, &c; and since the common center of gravity of all four will not be made to change its state by the mutual action of any two of these bodies upon each other: therefore neither will it change its state by the mutual action of all four upon each other. The same sort of proof may be applied, whatever is the number of bodies, that act upon each other.

75. *The quantity of motion in the same direction is not altered by the mutual action of bodies upon each other.*

If two bodies move in the same direction, then the quantity of motion in the same direction is the sum of their moments: but, if they move in contrary directions, then it is the difference of their moments. The former of these assertions is easily understood, and does not need any farther explanation. To explain the latter of them, it is necessary to observe that if motion in one direction is affirmative, motion in the contrary direction must be negative. These ideas of affirmative and negative may perhaps be made more intelligible by considering them in the case of two lines drawn from the same point A, Plat. III. fig. 11. one in the direction AB the other in the contrary direction AC. Now I say if we look upon AB as an affirmative quantity, then AC will be a negative one, or the line AC in respect of a line drawn in the direction AB is less than nothing. For imagine the point B or extremity of the line or string AB to be in motion, and that it runs in a strait course from B to A. Imagine farther that, in this motion of the point B towards A, the line or string BA runs up, that is, contracts itself, shortens, or decreases; then at the instant B gets to A and coincides with it, this string will have no length or will have decreased to nothing. Since therefore a motion of the point B in the direction BA diminishes the line or shortens the string BA; if we imagine it to move farther than A in the same direction, this farther motion of B will produce the same effect in the line BA
that

that it did before, or will still diminish it. But when B coincided with A, $AB = 0$: therefore, whilst it moves farther than A towards C, AB being still farther diminished will become less than nothing. But the point B moving from A towards C will begin to draw the string out again the other way, that is, will describe or by its motion will generate the line AC; therefore though AC considered in itself is as much a positive quantity as AB is, yet considered in respect of AB it is less than nothing; that is, if a line AB drawn in one direction is looked upon as affirmative, then a line AC drawn in the opposite direction must be looked upon as negative.

From this instance of affirmative and negative lines the reader I hope will be able to understand, that, though a motion in the direction AC is in itself as much an affirmative motion as any can be, yet in respect of a motion in the direction AB it may be looked upon as less than nothing or negative.

I do not know whether the following case of motion, changing from affirmative to negative, will make this more intelligible. Suppose a body was moving in the direction AB, and a person was to move his hand against it in the contrary direction or BA. Then, if the impulse of his hand was less than the moment of the body, he would destroy a part of the motion; and the body would continue to move in the direction AB with the remainder. If he was to increase the impulse of his hand, then he would destroy a greater part of the body's motion. And, if he made this impulse equal to the moment, then he would destroy the whole motion or would stop the body. Suppose therefore he was to increase the impulse of his hand in the direction BA still farther, what effect must it produce in the body's motion? The impulse was sufficient to stop the body before or to make its motion $= 0$: therefore, when it is farther increased, it will more than stop the body, or will make its motion less than nothing; that is, will change it from affirmative to negative. But if a body is moving in the direction AB, and any person impels it with his hand in the contrary direction or BA with a force greater than the body's moment, he will turn it back again and make it move in the direction CA. Therefore, if a motion in the direction AB is affirmative, then a motion in the contrary direction, or AC, must be considered as negative. For the same reason, if a motion from B towards A is affirmative, then a motion in the contrary direction or from C towards A must be considered as negative. The reader may see this matter farther explained in proposition 49 of Optics.

From hence we may be able to understand why the quantity of motion in two bodies, when they move in contrary directions, is estimated by

by the difference of their moments. For if one body as X moves in the direction B A, and another as Z in the direction C A: then if we look upon the moment of X as affirmative, the motion of Z will be negative. Therefore the motion of both together must be the affirmative moment of X and the negative moment of Z together; that is, the moment of X with the moment of Z subtracted from it, or the difference of their moments; for this is the method of joyning an affirmative and negative quantity together. Let the moment of $X = 20$, and the moment of $Z = 8$. Then, since the moment of Z is negative; the moment of both together is equal to the affirmative moment 20 with the negative one 8 added to it; that is, it is equal to $20 - 8 = 12$ or to the difference of their moments in contrary directions.

The reader by this time understands, that to prove the quantity of motion in the same direction not to be altered by the mutual action of two or more bodies upon each other, we need only shew, that the sum of their moments in the same direction or the difference of their moments in contrary directions is not altered by this mutual action. When bodies impell or strike upon each other, then, by the third law of motion, action and reaction being equal and in contrary direction, whatever change of motion is produced on one side, an equal change, but in a contrary direction, will be produced on the other side. Thus, if two bodies X and Z are moving the same way, and X overtakes Z, then X is said to impell Z, and whatever change it makes in the moment of Z by the action of this impulse, it will by the reaction of Z, have an equal change produced in its own moment in the contrary direction: that is, whatever moment X communicates to Z just so much it will lose itself. From whence it follows, that the moments of X and Z taken together will be the same after the impulse that they were before; since all the moment which X loses is acquired by Z. But if X and Z are moving in contrary directions and strike upon each other, then likewise equal changes will be made in the moments of both, that is, equal moments will be destroyed in both. But, if equal moments are taken from both, the difference of their moments is not altered by the stroke.

76. *If two perfectly hard bodies, which are void of elasticity, are moving in the same direction; and one impells the other: or if they are moving in contrary directions, and strike upon each other: the common velocity with which they will both move on together, after the impulse or congress, is as the sum of their moments in the same direction, or as the difference of their moments in contrary directions, divided by the sum of their quantities of matter.*

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When two bodies void of elasticity, as 2 balls of clay, are moving in the same direction, and that which moves the faster of the two overtakes the other; they will not fly off from each other, after the impulse, but will continue moving in the same direction as one mass, with a velocity, that will be the same in both of them. This, which is called the common velocity, may be thus found. The moment of each body before the impulse being supposed known, add these moments together, and, by the last proposition, this sum is not altered by their mutual action upon each other. Therefore the moment after the impulse, that is, the moment of the two bodies moving together as one mass is known, for it is the sum of their moments before it. The quantity of matter likewise is known, for it is the sum of the quantities in both bodies. And since, by proposition 13, the moment is a product arising from the quantity of matter multiplied into the velocity, it follows, that if the moment is divided by the quantity of matter, the quotient will be the velocity.

Suppose a body, which we will call A, weighs 2 ounces, and another which we will call B, weighs 4 ounces. Let A be moving in any direction with 3 degrees of velocity, and let B move after it in the same direction with 6 degrees. If neither of the bodies are elastic, then, after B has overtaken A, they will move on together; and each of them will have the same velocity. To find out what this velocity will be, we must observe, that before the impulse A containing a quantity of matter $= 2$ was moving with 3 degrees of velocity and consequently with a moment $= 2 \times 3 = 6$. For the same reason B's moment before the impulse was $4 \times 6 = 24$. Therefore the sum of their moments in the same direction $= 6 + 24 = 30$. This moment, by proposition 75, continues the same or 30 after the impulse, when both bodies A and B move on together, that is, when the quantity of matter $= A + B = 2 + 4 = 6$. But if the moment 30 is divided by the quantity of matter 6, the quotient, which is the common velocity, or velocity of both bodies together, will be 5.

When two bodies void of elasticity are moving in contrary directions and strike upon each other; they will after the congress move on together as one mass in that direction, in which the stronger of the two moved before it. The moment, when they move together, will, by the last proposition, be equal to the difference of their moments before the congress. Therefore their moments before the congress being supposed known, the moment of them, when they move on as one mass, is known likewise: and if this moment be divided by the quantity of matter

matter in motion, that is, by the sum of the quantities in both of them together, the quotient will be the common velocity after the congress.

Suppose A weighs 2 ounces and has 3 degrees of velocity, that is, suppose A's moment $= 2 \times 3 = 6$, if it meets B which weighs 4 ounces and has 6 degrees of velocity, or has a moment $= 4 \times 6 = 24$: then, as the difference of their moments in contrary directions is $24 - 6 = 18$, this, by the last proposition, will be the moment, when, after the congress, if they are void of elasticity, they both move together as one mass. The moment therefore being 18, and the quantity of matter being $A + B$ or $2 + 4 = 6$ which is the sum of the quantities of matter in both bodies; if 18 is divided by 6 the quotient or 3 will be the common velocity.

This proposition may be applied to find the common velocity after the stroke, when before it only one of the bodies is in motion and the other is at rest. For if A moves alone and strikes upon B at rest, then A's moment may be considered either as the sum of A's and B's moments in the same direction or as the difference of their moments in contrary directions.

Suppose A weighs 2 ounces and moves with 3 degrees of velocity or with a moment $= 2 \times 3 = 6$ and strikes upon B, which weighs 4 ounces: then, because B is at rest, its moment $= 0$, therefore the sum of their moments is $6 + 0 = 6$, and the difference of them is $6 - 0 = 6$. From whence it follows, by what has been said already, that if 6 or A's moment is divided by $A + B$ or $2 + 4$ which is the sum of the quantities of matter in both bodies, then the quotient, or 1, will be the common velocity, or the velocity with which A and B will move on together after the stroke.

77. *The change of velocity which is made in any case of impulse or congress, where the bodies are perfectly elastic, is double the change, which would be made, in like circumstances, where the bodies are void of elasticity.*

If an elastic body D Plat. III. fig. 12. is at rest and another elastic body C, which is moving in the direction AB strikes upon it, the sides of both of them will be pressed in by the stroke of impulse, and C, by proposition 75, will lose velocity in the direction AB and D will acquire it. But the elasticity of the bodies will make their sides fly out again with a force equal to that which pressed them inwards. This produces a second stroke, which we may call the stroke of elasticity. This stroke of elasticity is equal to the former or to the stroke of impulse, because the parts of the bodies, which are pressed in, fly out again with a force equal

equal to the impulse that pressed them inwards; and it is likewise in the same direction; for by the stroke of impulse C lost velocity in the direction AB and D acquired it; and it is evident that the side of the body D flying out towards A will diminish C's velocity in the direction AB and the side of the body C flying out towards B will increase D's velocity in the same direction. Therefore, since in all cases the stroke of elasticity is equal to that of impulse and both of them produce a change of velocity in the same direction, it follows that the change produced by both these strokes together will be double the change produced by the stroke of impulse alone. But where the striking bodies are elastic, there is always a stroke of impulse and another of elasticity, and where they are not elastic, there is only a stroke of impulse. Therefore the change of velocity produced in any case, where the striking bodies are elastic, will be double the change produced in like circumstances, where they are void of elasticity.

This proposition, though true in theory, cannot be applied with exactness to any bodies, that we can make use of in trying whether it is true by experiments. Because the demonstration of it depends upon the perfect elasticity of the striking bodies. Whereas neither glass balls, nor ivory ones, nor any other body in nature is perfectly elastic. The sides of many bodies when pressed inwards will fly out again, but not with a force equal to that, with which they were pressed inwards.

From this proposition the two following rules may be deduced for determining the velocity and direction of elastic bodies, after they have struck upon each other directly.

78. *If a body, that is void of elasticity, would acquire velocity in any case of impulse or congress; a body, that is perfectly elastic, will acquire twice as much in the same circumstances.*

For the change of velocity made in elastic bodies is always double the change made in those which are not so.

79. *If a body, that is void of elasticity, would lose velocity in any case of impulse or congress; a body, that is perfectly elastic, will lose twice as much in the same circumstances.*

This rule is proved in the same manner as the foregoing one. Only here we must observe that the velocity, which a body is to lose by this rule, will often be greater than the velocity, which it has. And in this case the difference between what it has before the stroke and what it is to lose by the stroke is negative, or is to be taken in a direction contrary

to that, in which the body was moving. Thus, if a body is moving in any direction with 3 degrees of velocity, and according to the rule ought by a stroke on another body to lose 4 degrees; by this loss it will have 1 degree of velocity less than nothing or 1 degree of negative velocity, or, by what has been said already in explaining proposition 75, it will be turned back in the contrary direction with 1 degree of velocity; that is, with the difference between 4 and 3, or between what it has and what it is to lose.

In the propositions which follow upon this subject we will apply these rules to some particular cases.

80. *If an elastic body A in motion strikes upon another equal and elastic one B at rest; then by the stroke A will communicate all its velocity to B and will rest itself.*

Since A and B are equal, the quantity of matter in each of them may be called 1. Let A's velocity be called 6, and consequently its moment will be $6 \times 1 = 6$, which is the product of its velocity multiplied into its quantity of matter. Now if neither of the bodies had been elastic, the common velocity after the stroke, according to proposition 76, might have been found by dividing 6 the moment by $A + B$ or by $1 + 1 = 2$ or by the sum of the quantities of matter in both bodies together. For the quotient of $\frac{6}{2}$, or 3, would have been this common velocity. And, as A had 6 degrees of velocity before the stroke and has only 3 after it; A, when it is a body void of elasticity, loses 3 degrees of velocity. Therefore, by proposition 79, A an elastic body will in the same circumstances lose double 3 degrees, or 6 degrees of velocity, which is all it has, and consequently must rest after the stroke. B likewise, when it is a body void of elasticity, rests before the stroke, but after the stroke moves with the common velocity 3; that is, it acquires 3 degrees of velocity. But, by proposition 78, B an elastic body will in the same circumstances acquire twice as much or 6 degrees, which is the same velocity that A had before the stroke.

81. *If there are six equal and elastic bodies, of which five as B, C, D, E, F, lie in a row contiguous to each other, and the other as A strikes directly upon B the first in the row; then A and all the other bodies in the row will rest, except F the last of them, and this will fly off with the same velocity, that A had when it struck upon B.*

By the last proposition, A will communicate all its velocity to B and will rest itself. For the same reason B will communicate all its velocity to

to C and will rest itself. And then C becomes the striking body, and therefore will communicate all its velocity to the next or D and will rest itself. The same may be said of all the other bodies in the row, though there were ever so many of them, till we come to the last of them or F. For since there is no other body beyond F to which its velocity may be communicated, it will move off with the velocity received from the next before it or E, that is, as appears from what has been said already, with the velocity, that A had when it struck upon B.

82. *If there are six equal and elastic bodies, of which four only as C, D, E, F, lie in a row contiguous to each other, and two as A and B moving along close together strike upon C the first in the row; then A and B and all the other bodies in the row will rest, except E and F the two last of them, and these will move off together with the same velocity that A and B had before the stroke.*

By the last proposition, if B alone had struck upon C then B and all the others in the row would have rested except F. But when B has lost all its velocity by the stroke, then A becomes the striking body, and will communicate all its velocity along the row to the last body, which is then the body E. Thus B makes F fly off, and A makes E fly off: and because A and B strike together, E and F will fly off together.

83. *If an elastic body A strikes upon another elastic body B, which is greater than itself: then the moment of the body B after the stroke, will be greater than the moment of A before it.*

Let A weigh 2 ounces and move with 3 degrees of velocity or a moment 6, and let B, which is at rest, weigh 4 ounces. Then if these two bodies had been void of elasticity, their common velocity after the stroke, by proposition 76, (where this case is made one of the instances) would have been 1. Thus B, when void of elasticity, acquires by the stroke 1 degree of velocity, and consequently, by proposition 78, B, if elastic, will acquire twice as much or 2 degrees. Therefore since B's quantity of matter = 4, and its velocity after the stroke = 2, its moment must = $2 \times 4 = 8$, which is the product of its velocity multiplied into its quantity of matter. Whereas the moment of A before the stroke was only 6.

But notwithstanding the moment 8, which is produced in B by the stroke, is greater than the moment 6 in A, which produced it; yet the

moment of the 2 bodies considered together is exactly the same after the stroke that it was before it: as it ought to be, by proposition 75. For, after the stroke, B moves along with the moment 8 in the same direction that A was moving before it. And let us consider what in the mean while becomes of A. Now A had 3 degrees of velocity before the stroke, and, if void of elasticity, would lose 2 degrees by it, according to what was proved in this very instance under proposition 76. But, by proposition 79, A, if elastic, will lose twice as much or 4 degrees. And, since it has but 3 degrees in all, it will be turned back again or reflected with the difference between 3 and 4 or with 1 degree of velocity: and since its quantity of matter $= 2$ its moment $= 1 \times 2 = 2$. But since after the stroke B moves in one direction, and A, which is reflected, moves in the contrary direction, A's moment is negative in respect of B's: and the moment of the 2 bodies taken together will be $8 - 2 = 6$ which is equal to the moment, that A had before the stroke.

I must therefore desire the reader to remember, that though there is motion produced by the stroke, if we consider B or the greater body alone, yet if both bodies are taken together there is no motion produced in the same direction that A was moving, but the moment in that direction is exactly the same after the stroke that it was before it.

84. *If an elastic body A overtakes another equal one B, which is moving slower than itself and in the same direction; after the stroke they will continue to move in the same direction with interchanged velocities.*

Let A have 8 degrees of velocity and B 4 degrees. Call A's quantity of matter 1; and B's, which is equal to it will be 1 likewise. Therefore A's moment $= 8 \times 1 = 8$; and B's $= 4 \times 1 = 4$. The sum of their moments in the same direction is 8 added to 4 $= 12$, this sum divided by the sum of their quantities of matter or by $A + B = 1 + 1 = 2$, gives 6 in the quotient. From whence it follows, by proposition 76, that, if neither of these bodies had been elastic, they would have moved on together after the stroke with the common velocity 6. Therefore A, whose velocity before the stroke was 8, would have lost 2 degrees; and B, whose velocity was 4 would have gained 2. But A, if it is perfectly elastic, will lose twice as much velocity or 4 degrees: and B, if it is perfectly elastic, will gain twice as much or 4 degrees: by propositions 79. 78. And since A, which had 8 degrees of velocity at first, loses

4. of them by the stroke ; its velocity afterwards will be $8 - 4 = 4$. And since B, which had 4 degrees of velocity at first, gains 4 more ; its velocity afterwards will be $4 + 4 = 8$. Thus before the stroke A's velocity was 8, and B's 4 ; and after the stroke A's is 4, and B's 8 ; that is, their velocities are interchanged.

85. *If an elastic body A meets and strikes upon another equal one B ; after the stroke they will be reflected with interchanged velocities.*

Let A have 8 degrees of velocity and B 4 degrees. Call A's quantity of matter 1, and then B's, which is equal to it will be 1. The moment of $A = 8 \times 1 = 8$: and the moment of $B = 4 \times 1 = 4$. The difference of their moments in contrary directions is $8 - 4 = 4$. Divide this difference 4 by $A + B = 1 + 1 = 2$, or by the sum of the quantities of matter, and $\frac{4}{2} = 2$, which is the quotient, would be the common velocity, by proposition 76, with which both the bodies, if neither of them had been elastic, would have moved on together after the stroke in the same direction that A was moving in before it. Thus A, if not elastic, would lose 6 degrees of velocity ; for it had 8 degrees before the stroke and would have only 2 after it. Therefore, by proposition 79, A, if elastic, will lose twice as much or 12 degrees. But it has no more than 8 degrees, and consequently the difference between 8 and 12 or 4, that is, the difference between the velocity, which it has, and the velocity, which it is to lose, is negative or is taken in the contrary direction. So that A will be reflected with the velocity 4. B moved with the velocity 4 to meet A before the stroke, and if neither body had been elastic, would have moved afterwards with the velocity 2 in the same direction with A. Therefore B, if it was not elastic, would lose 4 degrees of velocity in one direction and acquire 2 degrees in the opposite direction. From hence it follows, by proposition 78, 79, that B, if elastic, will lose twice as much velocity or 8 degrees in one direction, and acquire twice as much or 4 degrees in the opposite direction. But since it is to lose 8 degrees in the direction in which it moved before the stroke and has no more than 4 degrees to lose, the difference between 8 and 4, or $8 - 4 = 4$ is to be taken in the opposite direction : and these 4 degrees added to the other 4 which it acquires in that opposite direction, will make the velocity with which it is reflected $= 4 + 4 = 8$. Thus before the stroke A's velocity was 8 and B's 4 ; and when they are reflected A's is 4 and B's 8 : that is, the velocities are interchanged.

86. *If*

86. *If two equal and elastic bodies A and B, which are moving in contrary directions, meet and strike upon each other with equal velocities; each of them will be reflected with the same velocity that it had before the stroke.*

This may be considered as one case of the last proposition: for if the two bodies had equal velocities at first, they will likewise have equal velocities after they have exchanged. Or otherwise; the bodies, if they had not been elastic, would have lost all their velocities, because both of them strike with equal moments, and these equal and contrary forces would have destroyed each other, and the two bodies would have rested after the stroke. But A, as it is supposed elastic, will lose twice as much velocity, by proposition 79, that is, double what it has, or will gain as much negative velocity as is equal to the difference between the velocity, which it has, and the double of that velocity. This difference is plainly equal to the velocity which it has. The same may be said of the body B. Therefore the negative velocity, or velocity with which each body is reflected, is equal to that, with which it moved before the stroke.

87. *To determine the direction, in which two elastic bodies will be reflected, when they strike upon each other obliquely.*

Let two equal and elastic bodies P and Q move in the respective directions PA and Qa and strike upon each other. These oblique motions PA and Qa may each of them be resolved into two others, by proposition 20, in such a manner, that one part of PA, as PB or CA, shall be parallel to one part of Qa as Qb or ca; and the other part of PA as PC or BA shall be directly opposite to the other part of Qa, as Qc or ba. Now the two parallel motions CA and ca will not be altered by the stroke. For, if the bodies had moved in these directions only, they would never have approached each other, and consequently could never have struck upon each other. Therefore as no part of the stroke is made by either of these motions, they can neither of them be at all affected by it. And if we take AE equal to CA and in the same direction; and take ae equal to ca and in the same direction; from what has been said it follows, that after the stroke the parallel motion of P will be AE and that of Q will be ae. The opposite motions BA and ba will have their velocities interchanged, by proposition 85. Therefore taking AD equal to ba, and ad equal to BA, AD will be the motion, with

with which P will be reflected, and ad will be that, with which Q will be reflected. Since therefore after the stroke P is carried on with the motion AE and the motion AD , make these the two sides of a parallelogram, and, by proposition 16, the diagonal Ap will be the line in which the body P will be reflected. In like manner, since the two motions of Q after the stroke are ae and ad , make these the sides of a parallelogram and draw the diagonal aq , then, by proposition 16, aq is the line in which the body Q will be reflected.

From this instance the reader will understand, that, in all other cases of the oblique congress of two elastic bodies, their motions may each of them be resolved into a parallel one and an opposite one, that the parallel motion of each will be the same after the stroke that it was before it, and that the other motion of each after the stroke may be found by the foregoing rules relating to the direct congress of bodies. And when the two motions of each body after the stroke are thus found, then by making these the sides of a parallelogram and drawing the diagonal, the direction in which the bodies will be reflected is determined: for they will always be reflected in that diagonal.

88. *If an elastic body strikes perpendicularly upon an immoveable obstacle, it will be reflected in the same line that it described, and with the same velocity that it had, before the stroke.*

If an elastic body moves in the direction EB Plat. III. fig. 14. and strikes perpendicularly on the immoveable elastic obstacle DBF ; the body will be reflected in the line BE with the same velocity that it had before the stroke. For if the body and obstacle had been void of elasticity then by the stroke it would have lost all its velocity. Therefore, by proposition 79, because they are elastic, it will lose twice the velocity that it has, or will lose its own velocity, and acquire a negative one that is equal to it, that is, will be reflected or be made to move in a contrary direction with the same velocity that it had before the stroke.

89. *When a body strikes obliquely on an immoveable obstacle, the angle contained between the line, which the body describes and a line, that is perpendicular to the obstacle at the point where the body strikes it, is called the angle of incidence.*

Thus Plat. III. fig. 14. if AB is the line, which the body describes, and BE a perpendicular to the obstacle DBF at the point B where the

body strikes it, then the angle of incidence is the angle ABE . The reader will observe that ABD is the complement of the angle of incidence to a right one.

90. *When a body, after striking obliquely on an immoveable obstacle is reflected, the angle contained between the line, in which it is reflected, and a line, which is perpendicular to the obstacle at the point of reflection, is called the angle of reflection.*

Thus Plat. III. fig. 14. the angle of reflection is EBC contained between BC the line which the reflected body describes and EB a perpendicular to the obstacle at B the point of reflection. CBF is the complement of reflection.

In these two definitions of the angles of incidence and reflection, I have copied from Sir Isaac Newtons definitions of them in his optics. The mechanical writers commonly call ABD the angle of incidence, and CBF the angle of reflection.

91. *When an elastic body strikes obliquely on an immoveable and elastic obstacle, and is reflected from it: the angle of reflection is always equal to the angle of incidence.*

In order to prove this, supposing the body to strike the obstacle in the direction AB , Plat. III. fig. 14. let us first trace out the line in which it will be reflected. The oblique motion AB of the incident body may be resolved into two others, by proposition 20, one of them parallel to the obstacle as AE equal to DB , the other perpendicular to it, as AD equal to EB . Of these two motions the parallel one DB will not be altered by the stroke. For if the body had moved in this parallel direction only, it would never have approached the obstacle, and consequently could never have struck upon it. Therefore as no part of the stroke is made by the parallel motion AE or DB , it is not affected by the stroke but will be the same after it that it was before. And, if BF is taken equal to DB , from what has been said it follows that BF will be the direction and velocity of the body after the stroke. The perpendicular motion, which before the stroke is AD or EB , will, by proposition 88, be the same after it, only in a contrary direction as BE . Therefore after the stroke the body being carryed on by the two motions BF and BE will, by proposition 16, describe BC the diagonal of a parallelogram, of which BF and BE are the sides. Thus BC is proved to be the line, in which the body will be reflected.

And

And from what has been said in tracing out this line BC, in which the body will be reflected, we may shew that the angle of reflection EBC is equal to the angle of incidence ABE. For AE and DB are equal; because they are opposite sides of a parallelogram. Euc. book I. prop. 34. And for the same reason EC and BF are equal. But DB and BF are equal by the construction, because one is the parallel motion before the stroke and the other is the same motion after it. Therefore AE because it is equal to DB and EC because it is equal to BF are likewise equal to each other. The side EB is common to the two triangles AEB and CEB. And, since AC is parallel to DBF and EB is perpendicular to it by the construction, therefore EB is perpendicular to AC, and the angles AEB and CEB are equal. But, because AE, EB, in one of these triangles are equal to CE, EB, in the other and the angles AEB and CEB contained between these equal sides are likewise equal; it follows, Euc. b.I. prop. 4. that the other two angles in one triangle are each of them respectively equal to the other two angles in the other triangle, which are subtended by equal sides. Therefore EBC the angle of reflection is equal to ABE the angle of incidence.

92. *If a body, which is in motion, strikes upon another body either in motion or at rest; the quantity of the stroke is proportional to the moment, which is lost by the striking body,*

If one of the bodies has a greater moment than the other, then that which has the greater moment, or if their moments are equal, then either of them, may be considered as the striking body.

Now the quantity of the stroke is the force impressed by the striking body upon the other. But since, by the third law of motion, proposition 15, reaction is equal to action, whatever force is impressed on the body, which is struck, will by its reaction be destroyed in the striking body. Therefore the motion lost by the striking body is proportional to the force impressed by it on the other, or to the quantity of the stroke. From hence it follows, that when a moving body strikes perpendicularly upon an immoveable obstacle; the quantity of the stroke will be as the striking body's moment; because its whole moment is lost by the stroke.

93. *If a given body strikes upon another given body at rest, the quantity of the stroke is always proportional to the velocity of the striking body.*

This follows from the last proposition, because the moment lost is always proportional to this velocity. Call one of the bodies A and let

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it

it be supposed to weigh 2 ounces : call the other B, and let its weight be 4 ounces : we may consider this proposition in two cases.

First, if B is the body at rest, then the quantity of the stroke or moment lost will always be as A's velocity. If A's velocity before the stroke $= 3$, and consequently its moment $= 3 \times 2 = 6$; then supposing it void of elasticity, it appears by proposition 76, that its velocity after the stroke $= 1$ and consequently its moment $= 1 \times 2 = 2$. Therefore the moment lost by the stroke $= 6 - 2 = 4$. By the same 76 proposition it may be proved that, if A's velocity before the stroke had been 6 and its moment $= 6 \times 2 = 12$, then its velocity after the stroke would $= 2$ and its moment $= 2 \times 2 = 4$. Therefore the moment lost by the stroke $= 12 - 4 = 8$. But since as 4 is to 8 so is 3 to 6, that is, since the moment lost is as A's velocity, it follows by the last proposition, that the quantity of the stroke is as A's velocity.

The second case of this proposition is, that when the velocity is given the quantity of the stroke will be the same whether A strikes upon B or B upon A. Thus if the given velocity is called 3, the quantity of the stroke will be the same whether A with the velocity 3 and consequently with the moment $3 \times 2 = 6$ strikes B at rest, or B with the same velocity 3 and consequently with the moment $3 \times 4 = 12$ strikes A at rest. It has just been proved that if A strikes B the moment lost $= 4$. And, by proposition 76, it may be shewn that, if B weighing 4 ounces, and moving with 3 degrees of velocity, that is, with a moment $= 12$, strikes A at rest, the common velocity after the stroke, supposing both of them void of elasticity, will be 2. Therefore B's velocity being this common velocity or 2 and its quantity of matter 4, its moment after the stroke will be $2 \times 4 = 8$. So that the moment lost by the stroke $= 12 - 8 = 4$. And since the moment lost is 4 whether A strikes with a given velocity upon B at rest, or B strikes with the same velocity upon A at rest : it follows, by the last proposition, that the quantity of the stroke will be the same.

We may by assigning a reason for this last case, which seems a paradox, give a general proof of the proposition. When A is the moving body the moment is 6, and when B is the moving body the moment is 12 : and yet B will not hit double the blow that A does, but the quantity of the stroke would in either case be the same. For we must remember that the quantity of the stroke does not depend upon the moment of the striking body only, but likewise upon the resistance of the body that is struck. Thus if we hold our hand steady against any blow, we shall feel more of it, than if we had made no such resistance, but had

had let our hand yield to it. And in all other cases the more the body, which is hit, bears up against the blow, so much the greater stroke it will sustain; and if it makes no resistance at all, the blow will be nothing. Now, where the resistance is given, the quantity of the stroke is as the moment of the striking body, and where this moment is given, it is as the resistance of the body that is struck. Therefore in all cases it is in a ratio compounded of them both or as the moment multiplied into the resistance. The resistance of the body, which is struck, being, where other circumstances are alike, as the quantity of matter contained in it, when A strikes B with 3 degrees of velocity the quantity of the stroke is as $6 \times 4 = 24$, or as A's moment into B's quantity of matter. And when B strikes A it is as $12 \times 2 = 24$ or as B's moment into A's quantity of matter.

Therefore universally, where the quantity of matter in two bodies, of which one moves and the other rests, is given, the quantity of the stroke will be as a product arising from one body multiplied into the other and then multiplied again into the velocity with which the striking body moves. But this product will always be as this velocity, because the quantity of matter in both is given. Therefore the quantity of the stroke will likewise be as the velocity.

Upon this principle it is that those, who call themselves strong men, will let a smith cut a bar of iron asunder upon an anvil, which is laid upon their breast. The only difficulty in this is to bear the weight of the anvil: for the force with which the hammer strikes it, though it is sufficient to drive a chissel through the iron, will make the anvil give a very inconsiderable blow to their breast, who support it. They lay their back hollow from the ground, that they may have room to yield to the blow; and then the quantity of the stroke will be proportional to the velocity, with which the anvil is driven down upon their breast. One blow of the hammer upon their naked breast without the anvil would kill them. But though the hammer strikes the anvil with an equal force, and this force communicated to the anvil strikes their breast; they will feel little or no inconvenience from it. Indeed there are two reasons why a blow of the hammer immediately upon their breast should hurt them more, than the same blow would when communicated first to the anvil. The first reason is because the whole force of the hammer strikes upon a very small part of the breast; whereas if by striking the anvil it was to make the anvil hit as great a blow, yet this would reach over the whole breast and consequently would be less upon each part, than if all the force had struck one single part. But the other reason, why a blow

immediately from the hammer would hurt them so much more than the same blow upon the anvil, is, that the anvil moves with a much less velocity. For suppose the hammer to weigh 1 pound and the anvil to weigh 100 pounds. The hammer by striking the anvil communicates its motion to the anvil; so that the moment of one is equal to that of the other; but the anvil will have only a hundredth part of the hammers velocity, and consequently will strike a less blow. From hence it is evident, that the heavier the anvil is in proportion to the hammer, the less will be the velocity of the anvil and therefore so much less the stroke upon the breast of a man, which supports it. The greater quantity of matter there is in the anvil, so much the more difficult it will be to support it, but when it is supported, this will make the blow the easier: and if the quantity of matter in the anvil was infinite, the velocity communicated by the stroke of the hammer would be nothing; and if a man could bear the weight of such an anvil as that, he would feel no blow at all, whilst the iron was cut asunder upon it.

94. *If a given body in motion strikes directly upon another given body, which is moving slower and in the same direction; the quantity of the stroke is the same, as if the body, which moves slower, had rested, and the other had struck upon it with the difference of their velocities.*

The quantity of the stroke is, by the last proposition, as the striking velocity, that is, as the velocity with which the two bodies approach each other. But when both of them are moving in the same direction they approach only with the difference of their velocities; and so they do when one of them rests and the other moves with this difference. Therefore the quantity of the stroke will be the same in both cases.

Call one body A and let it be supposed to weigh 2 ounces; call the other B and let its weight be 4 ounces. Then, if A's velocity is 3, its moment will be $3 \times 2 = 6$, and if B moves after it in the same direction with the velocity 6, its moment will be $6 \times 4 = 24$. And if they are both void of elasticity, the case will be so much the more simple and, by proposition 76, their common velocity after the stroke will be 5. Therefore B's moment after the stroke will be $5 \times 4 = 20$: and as its moment before was 24, it must have lost a moment $= 24 - 20 = 4$. But suppose A had been at rest and B had struck upon it with the difference of their velocities, which is in the case proposed $6 - 3 = 3$. Then B's moment before the stroke is $3 \times 4 = 12$. And when this moment is divided by $A + B$, or $2 + 4 = 6$, or by the sum of the quantity of matter in the

two.

two bodies, the quotient is 2, which, by proposition 76, is the common velocity after the stroke. Therefore B's moment after the stroke will be $2 \times 4 = 8$, and, as its moment before was 12, it must have lost a moment $= 12 - 8 = 4$. And since 4 is the moment lost whether A and B are both in motion in the same direction, or A rests and B strikes upon it with the difference of their velocities; it follows, by proposition 92, that the quantity of the stroke is the same in both cases.

95. *If two bodies, which are moving in contrary directions, strike upon each other; the quantity of the stroke is the same, as if one of them had rested and the other had struck upon it with the sum of their velocities.*

The quantity of the stroke is, by proposition 93, as the striking velocity, that is, as the velocity with which the two bodies approach each other. But when both of them move in contrary directions they approach with the sum of their velocities; and so they do, when one of them rests and the other moves with this sum. Therefore the quantity of the stroke will be the same in both cases.

Call one body A, and let it be supposed to weigh 2 ounces; call the other B, and let its weight be 4 ounces. Then if A's velocity is 3, its moment will be $3 \times 2 = 6$. If B moves in a contrary direction with the velocity 6, its moment will be $6 \times 4 = 24$. And, if they are both void of elasticity, their common velocity after the stroke, by proposition 76, will be 3. Therefore B's moment after the stroke will be $3 \times 4 = 12$; and, as its moment before the stroke was 24, it must have lost a moment $= 24 - 12 = 12$. But suppose A had been at rest and B had struck upon it with the sum of their velocities which in the case proposed is $6 + 3 = 9$. Then B's moment before the stroke is $9 \times 4 = 36$. And when this moment is divided by $A + B$ or $2 + 4 = 6$, or by the sum of the quantities of matter in the two bodies, the quotient is 6, which, by proposition 76, is the common velocity after the stroke. Therefore B's moment after the stroke will be $6 \times 4 = 24$, and as its moment before was 36 it must have lost a moment $= 36 - 24 = 12$. And since 12 is the moment lost whether A and B are both moving in contrary directions, or A rests and B strikes upon it with the sum of their velocities, it follows, by proposition 92, that the quantity of the stroke is the same in both cases.

96. *The motion of bodies amongst themselves is the same whether the place, in which they are all included, rests or moves uniformly in a right line.*

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The motion of two or more balls on a billiard table on board a ship, is the same in all respects and particularly as to the strokes upon each other, whether the ship and the table in it are at rest, or whether they move on uniformly in a right line. For, by the two last propositions, the quantity of the stroke is always as the difference of the striking velocities in the same direction or as the sum of them in contrary directions. And since the uniform motion of the table, or place, in which all of the balls are included, is common to all of them, it will not alter the difference of their velocities in the same direction or the sum of them in contrary directions. Therefore the uniform motion of the table will not make their strokes upon each other different from what they would have been if the table had rested.

We may apply this proposition more particularly to another case; where it is of more importance. If the earth turns round its axis from west to east once in every day; then if you discharge a bullet from west to east against a wall, the bullet and wall are both moving in the same direction; but if you discharge it from east to west they are moving in contrary directions. It is therefore enquired, why the stroke of the bullet with the sum of the velocities, as in the latter case, should not be greater than it is in the former, when it is made only with their difference. To this I answer that the difference of the bullets and walls velocities in the former case is exactly the same as what is here called the sum of their velocities in the latter; and either of them is the same as the velocity with which the bullet would have struck the wall supposing the earth and the wall had been at rest. Therefore, by the three foregoing propositions, the quantity of the stroke will be the same in all the three cases. Suppose that the velocity with which the earth turns round its axis is to the velocity, with which the bullet is discharged as 10 to 1: for though this is far from being the true proportion, it will illustrate the case as well as any other would. Then if the wall had been at rest the striking velocity of the bullet upon it would be $=1$. But if the wall and earth are supposed to move from west to east with the velocity 10; then it must be remembered that the bullet before it is discharged is moving along with the earth in the same direction and with the same velocity; and consequently when it is discharged from west to east there is 1 degree of velocity added to it in this direction so that it moves with a velocity $=10+1=11$. Now as the wall moves with the velocity 10, and the bullet in the same direction with the velocity 11; the striking velocity will be the difference between them or $11-10=1$. But if the bullet is discharged from east to west, then because, before it was dis-

discharged it was moving along with the earth from west to east with 10 degrees of velocity, and by being discharged has only 1 degree of velocity given to it in the contrary direction, though in respect of the earths surface and the things upon the surface it appears to move from east to west, yet in absolute space it will still continue to move from west to east with the difference between the velocities 10 and 1 or with the velocity 9. So that what is called in the enquiry the sum of the bullets and walls velocity is properly the difference. For after the bullet is discharged it moves from west to east with the velocity 9 and the wall moves in the same direction with the velocity 10. Therefore the striking velocity will be $10 - 9 = 1$. And since the striking velocity is 1, whether the wall is at rest and the bullet is discharged against it with the velocity 1; or the wall is in motion and the bullet is discharged against it either from west to east or from east to west with the same velocity; it follows, by propositions 93. 94. 95. that the blow, which the bullet hits upon the wall, will be the same in all the three cases.

97. *The quantity of an oblique stroke, where the moment is given, is to the quantity of a direct one, as the cosine of incidence is to the radius.*

Thus Plat. III. fig. 14. if a given body strikes the bar DBF obliquely in the direction AB with a velocity represented by the same line AB, the blow will be less than if the body had struck it directly or perpendicularly with the same velocity, as much less as AD the cosine, or sine of the complement, of incidence is less than AB, which is called the radius. For, by a consequence deduced from proposition 92, the stroke, when a body strikes perpendicularly with the velocity AB, is as the moment, that is, if the quantity of matter in the body is given, as the velocity. But, when the same body strikes with the same velocity obliquely or in the direction AB, then this oblique motion may be resolved into two others, of which one as AE will be parallel to the bar and the other as AD perpendicular to it. Now it is evident that the parallel part of this motion would never bring the body to the bar or make it strike upon it. Therefore the whole stroke is made with the perpendicular part of the motion or AD. And, because the oblique stroke is given only with the moment AD, but the direct or perpendicular stroke would be given with the whole moment AB, it follows that the oblique stroke is to the direct one as AD to AB, or as the cosine of incidence to the radius.

98. *The friction of the parts of a machine against each other will in time destroy any motion, that has been communicated to it.*

Amongst the other mutual actions of bodies upon each other we reckon friction. No machine can have the surface of its parts made exactly smooth by polishing, though their roughness may be made too small for our eyes to discover it. And as a body by striking directly upon another, that is either at rest or in motion, will lose velocity, so, as the parts of a machine rub against each other, when the prominence of one part, though ever so small, strikes upon the prominence of another part, the motion of the parts will by this means be diminished, and by frequent strokes of the same sort will in time be entirely destroyed.

From hence it follows, that no machine can be so contrived as to have a perpetual motion, or always to preserve the motion once impressed upon it. For the motion communicated will by the friction of the machine be constantly decreasing, and must end at last.

To prevent this, in all the contrivances for a perpetual motion, the usual attempt has been to find out a way of repairing the motion which is lost by friction. And as bodies move themselves by the force of gravity; and as motion is likewise generated by the spring of such bodies as are elastic; the way of repairing the motion, as it decays by friction must be by the application of one or other of these two properties of matter. The motion of some clocks is kept up by the force of gravity: but then this motion is not a perpetual one; for the clock stops as soon as the weights are down. Other clocks are kept in motion by elastic springs, but this motion ceases likewise, when the spring by having expanded itself is grown too weak to repair it any longer. And indeed neither of these properties can be so applied as to make a perpetual motion. This I will endeavour to shew, from one or two of the principal contrivances, in the two following propositions.

99. *The force of gravity cannot produce a perpetual motion.*

The general reason for this has been already given. For since gravity produces motion in bodies only whilst they can descend; the motion of a machine, which is lost by the friction of its parts, can be repaired by a weight no longer than till the weight is down, or is come to the ground and is prevented by it from descending any longer.

In order to remove this difficulty machines have been contrived with more weights than one, of which some are to be constantly ascending, whilst the others are descending. Where we may observe that the de-
scending

PLATE III.

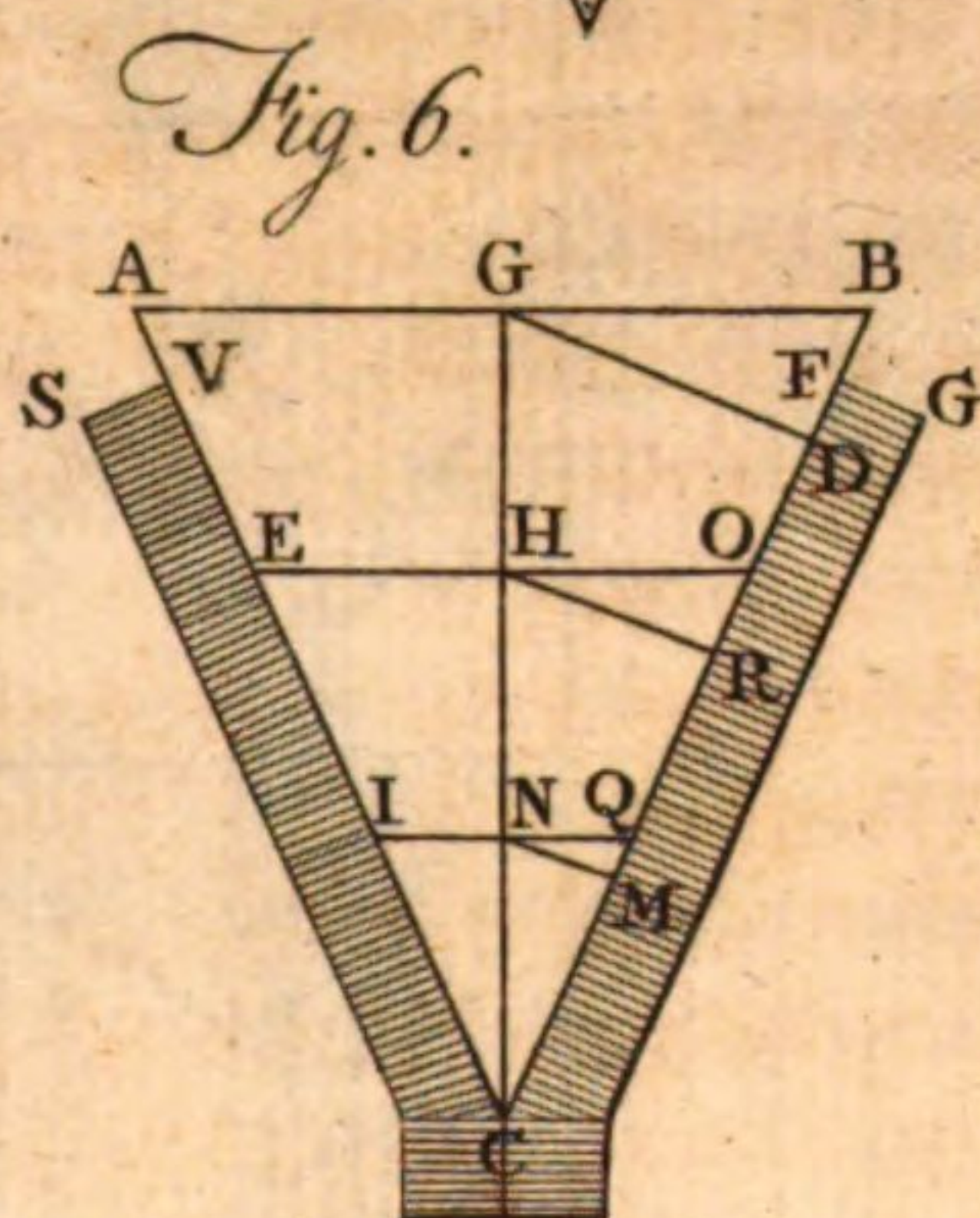
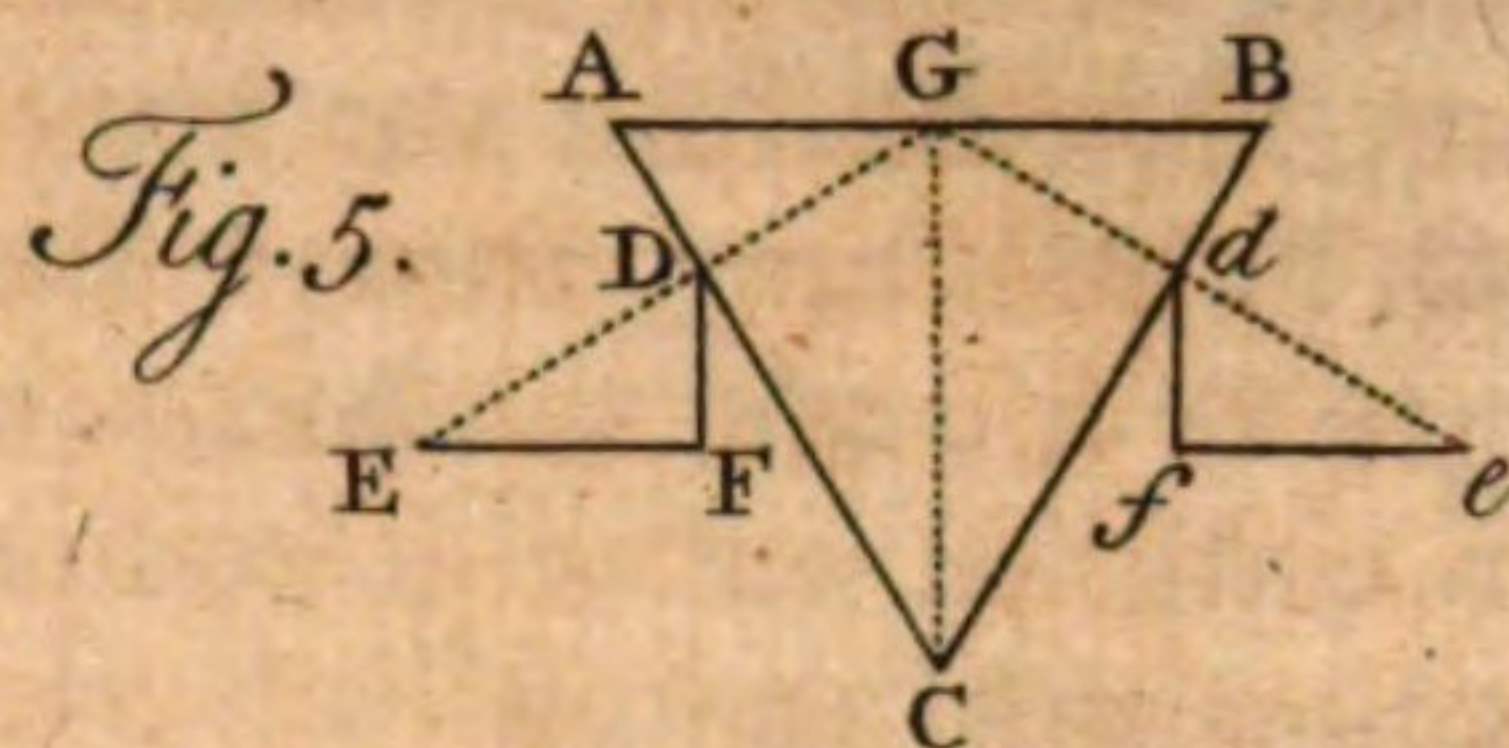
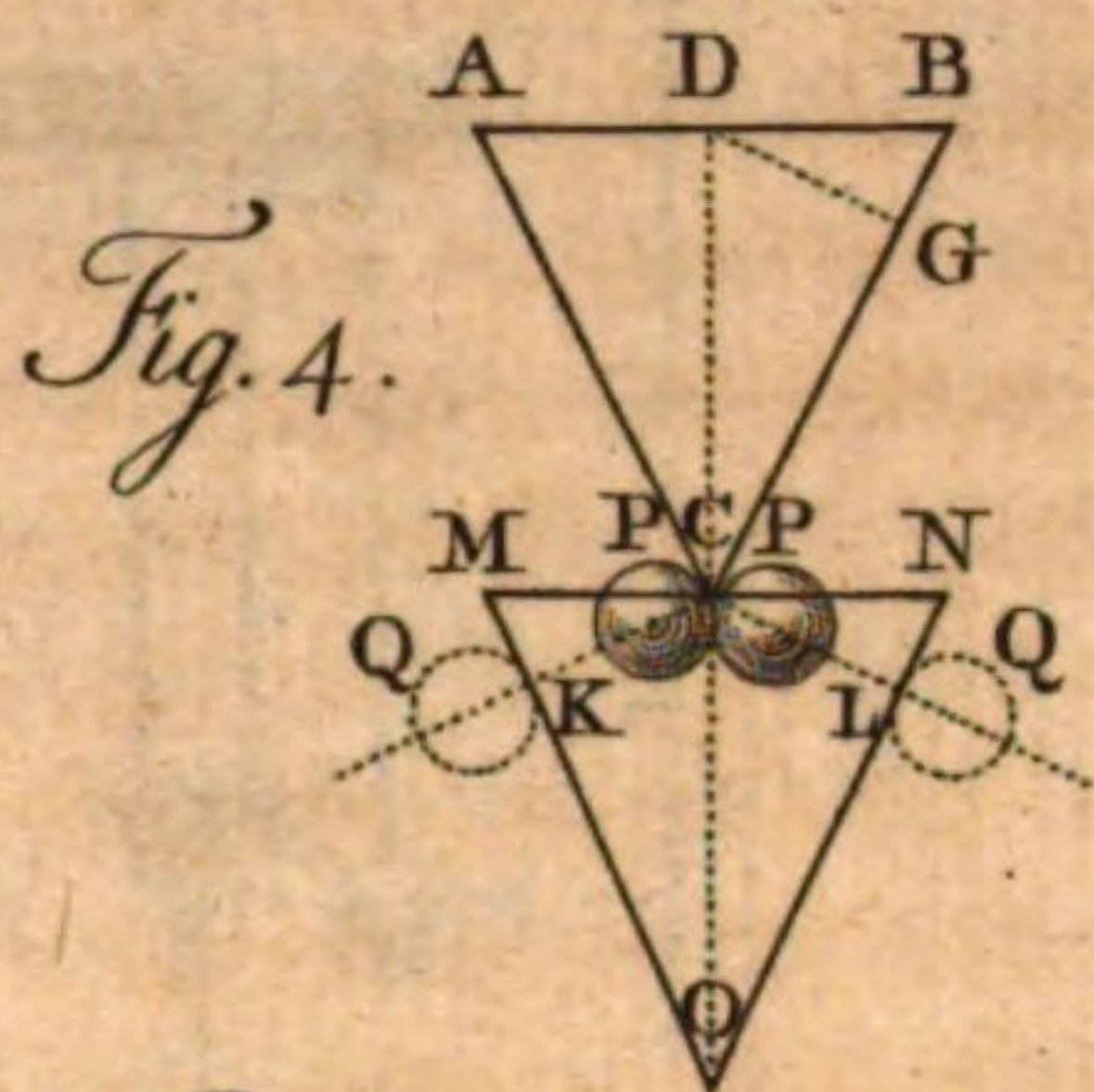


Fig. 7.

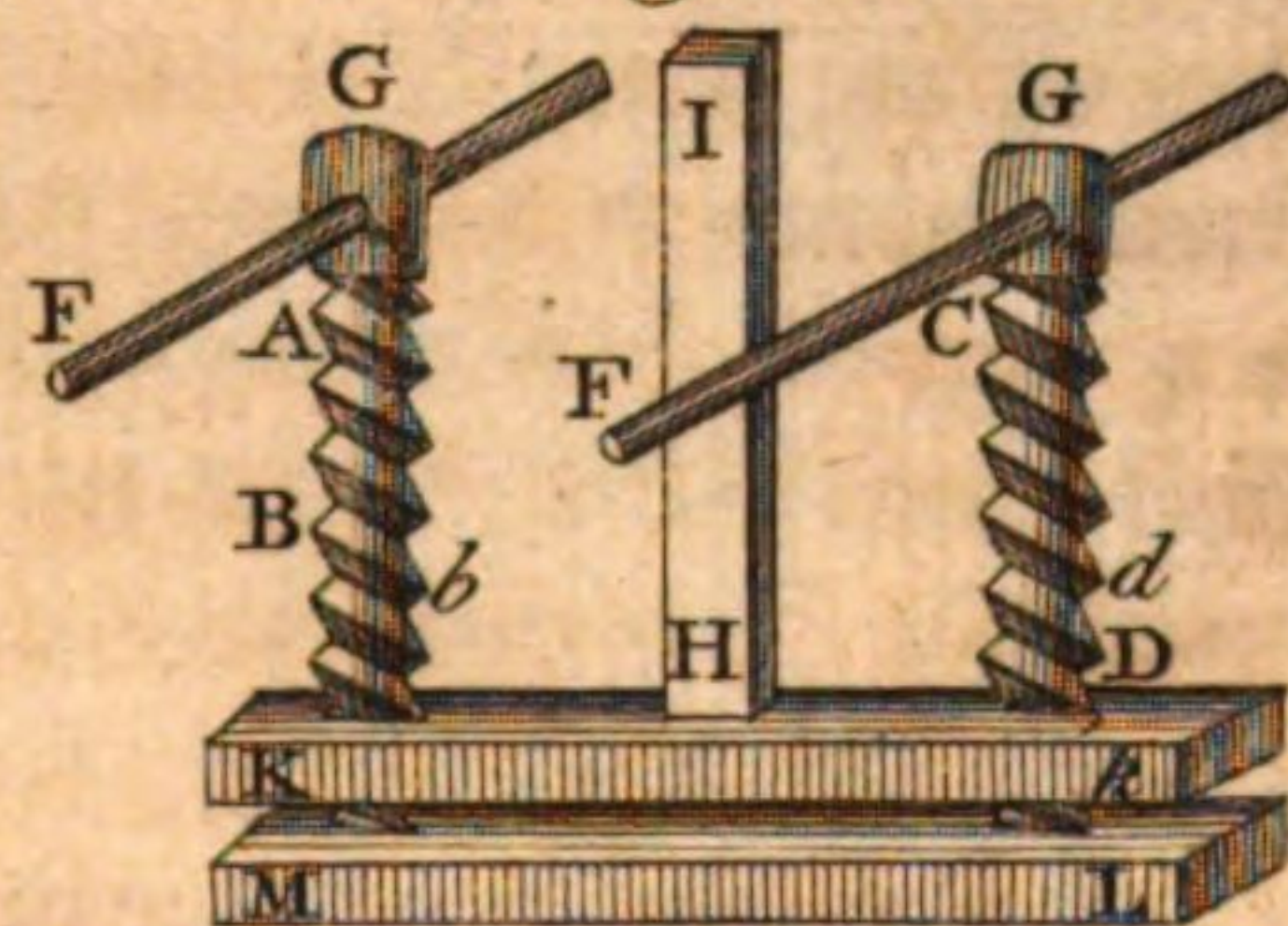


Fig. 8.

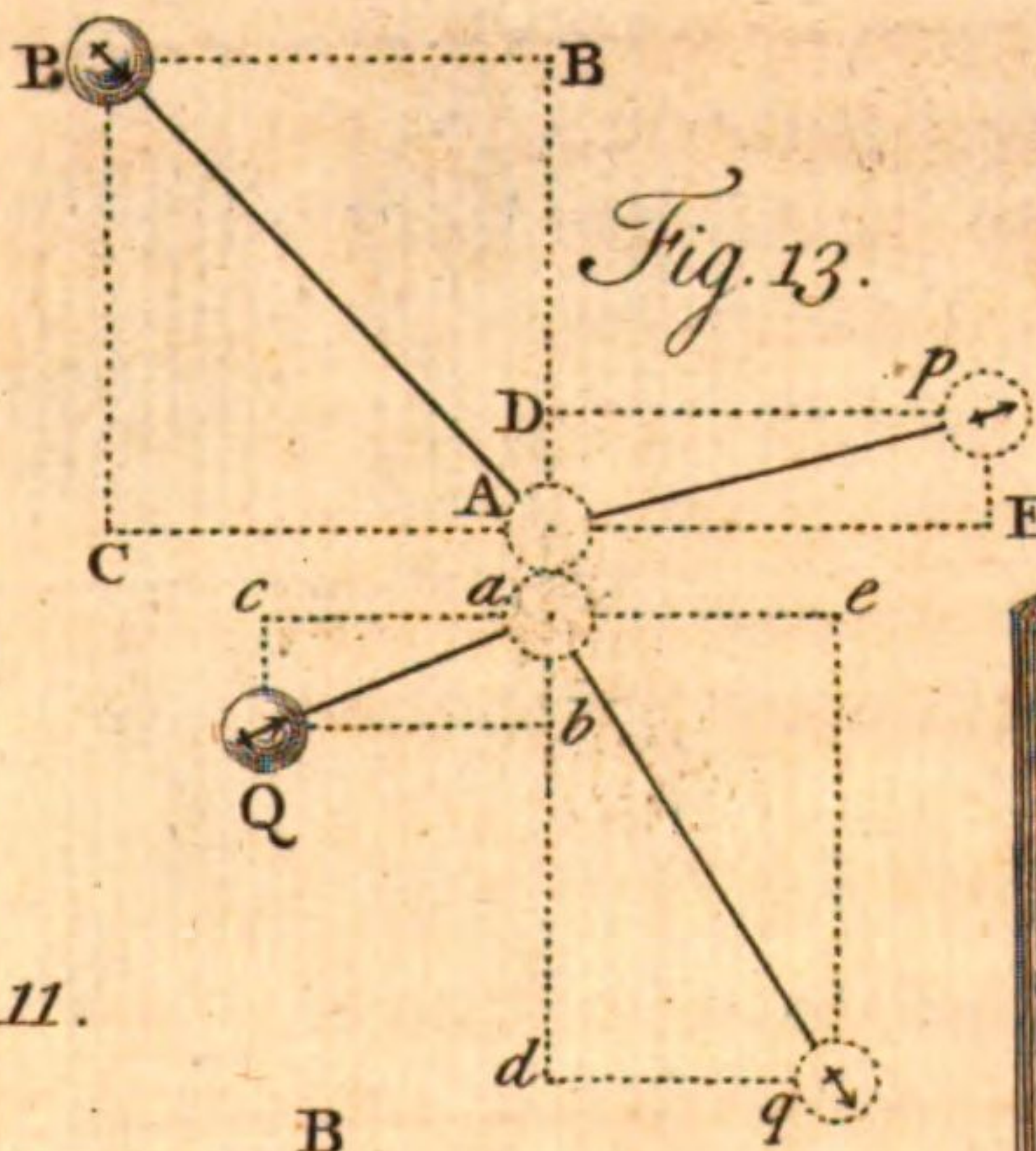
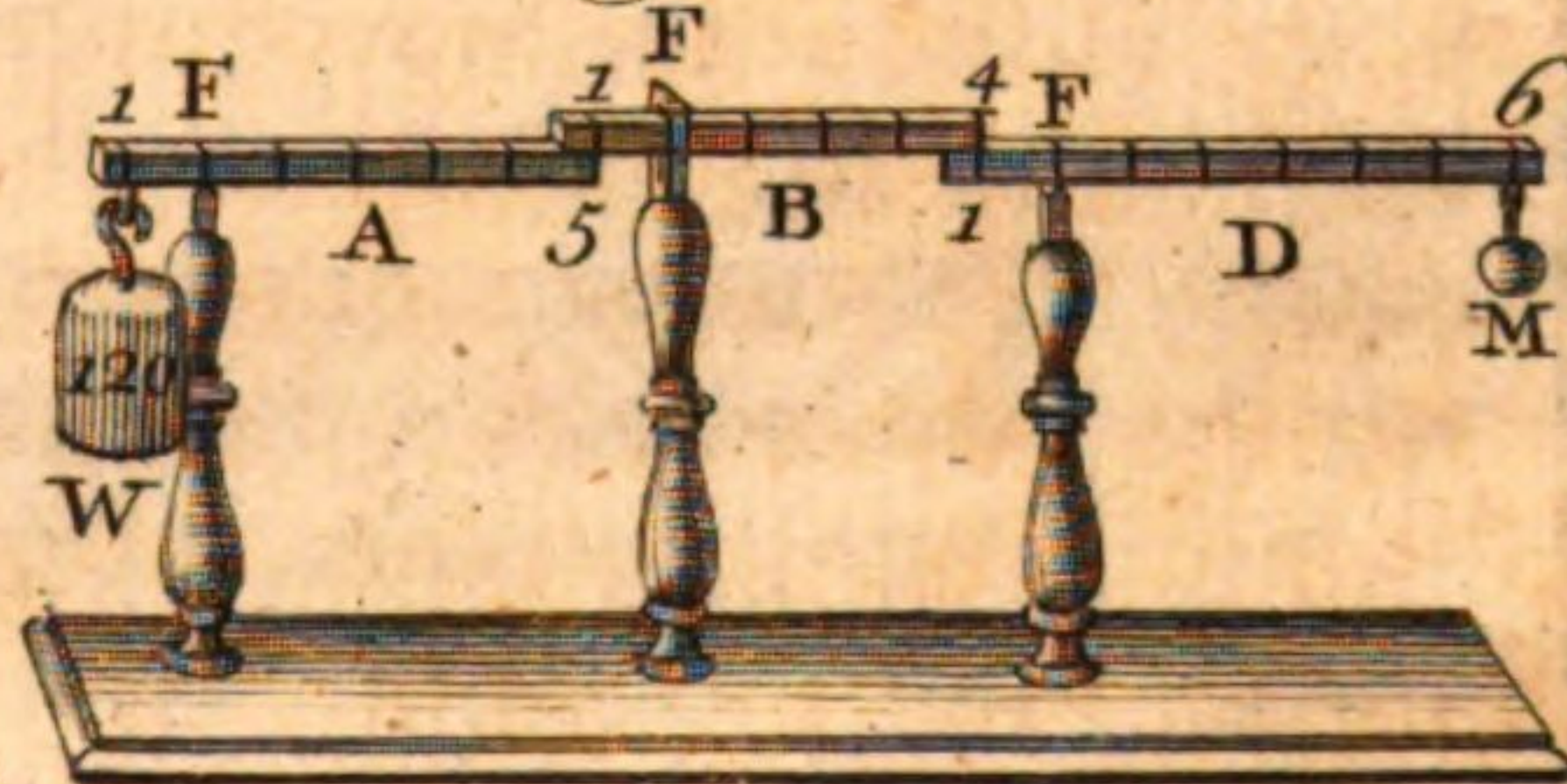


Fig. 9.

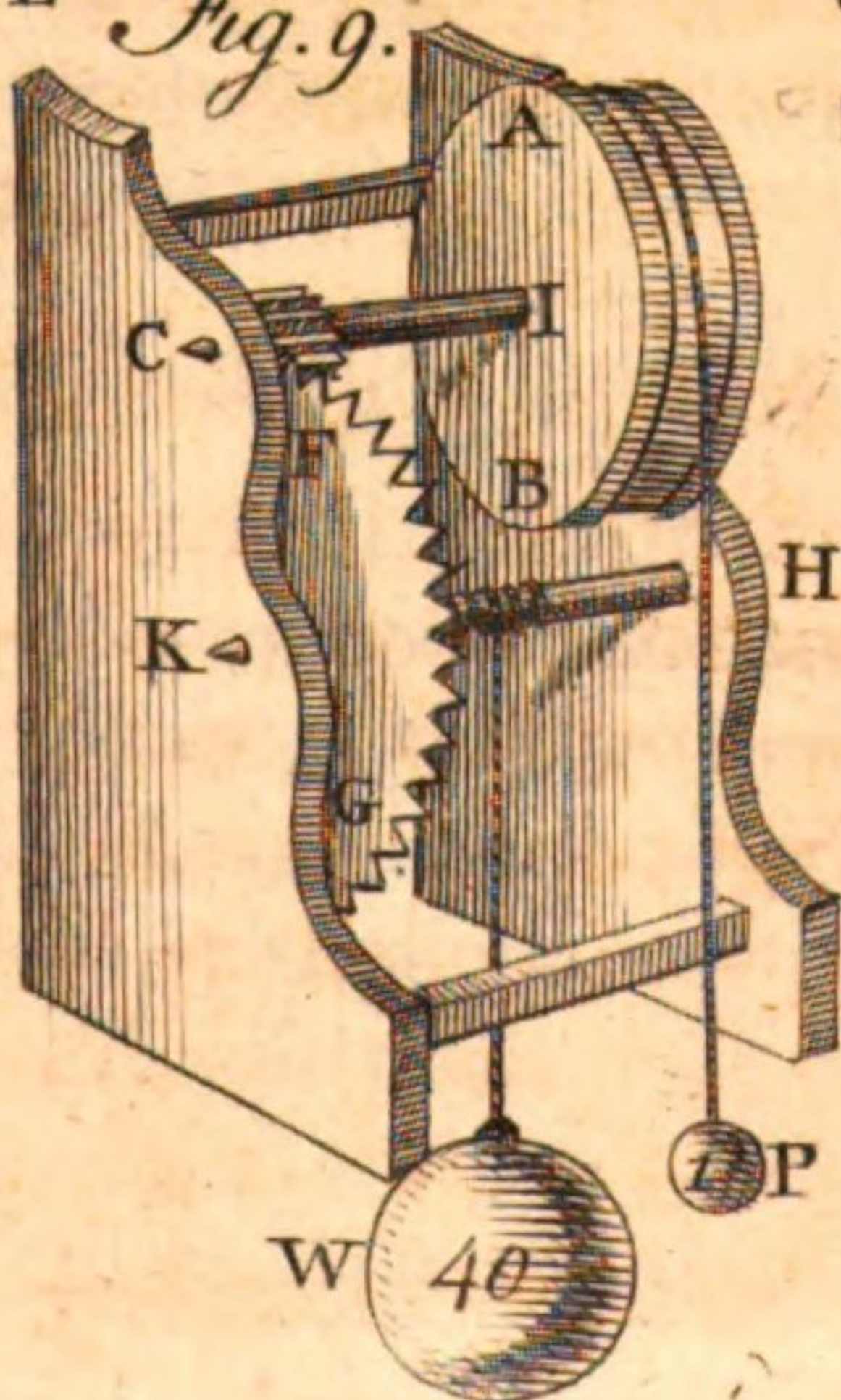


Fig. 10.

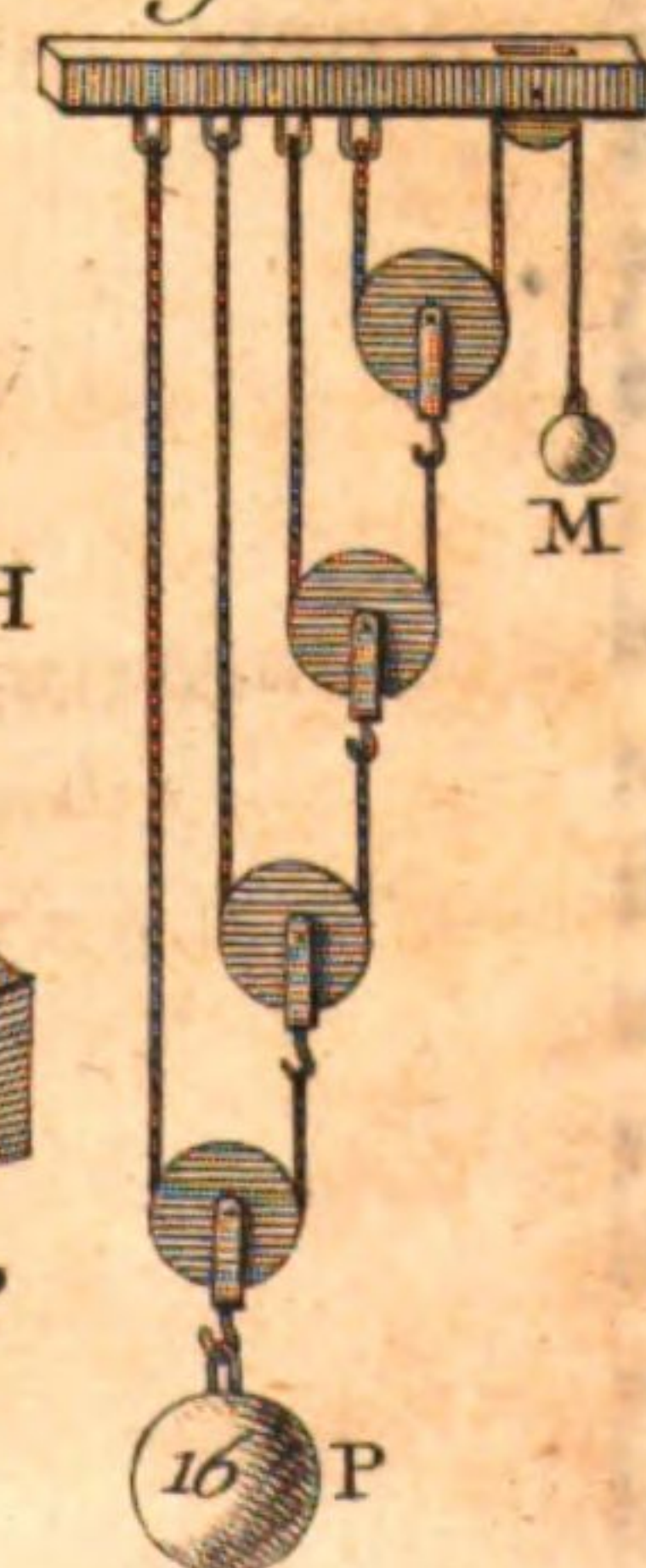
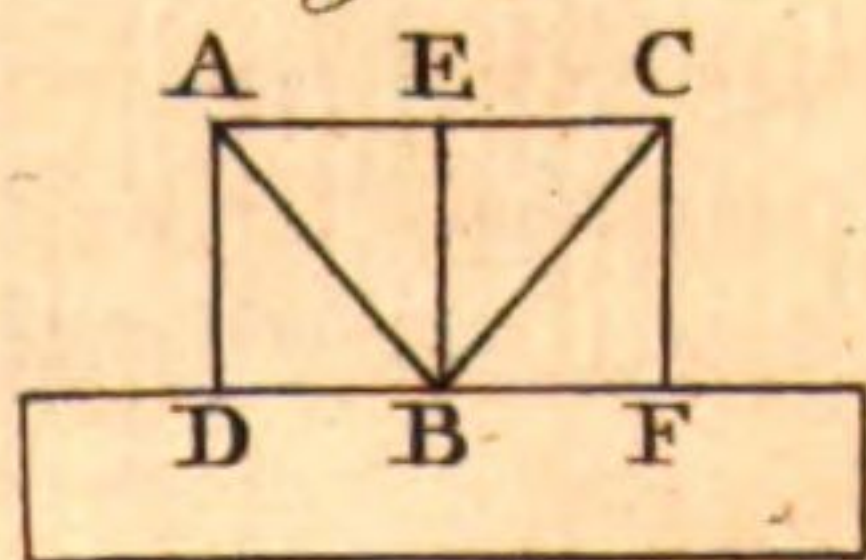


Fig. 14.



Descending weights are applied to two purposes; they are to repare the motion in the machine, and likewise to raise the other weights which were down as low as they could go. And these weights, when they are raised, are to descend again and in their turn to answer the same purposes. For this end some one or more of the mechanical powers is made use of, by the assistance of which the descending weights may be made to move always with a greater velocity than the ascending ones: and then, if the quantities of matter in the weights be equal, their moments will be as their velocities, and consequently the descending weights may by this means have a moment given them so much greater than the ascending ones have, as to be sufficient to answer the two purposes already mentioned.

It would be endless to examine all the inventions of this sort. The principle, upon which they depend, is applied in the neatest manner in the wheel described by Desaguliers: and by examining the principle in this one instance, we shall see the fallacy of it, and how little hopes of success there is in any other application of it. The wheel *Plat. IV. fig. 1.* has two rims or circumferences *DeAB* and *rbif*. The space between the two rims is divided into cells by the spokes *qA, ft, pD, be, &c.* which are bent in such a manner as to cause weights placed loose in the cells to descend on the side *As t B* from *A* to *B*, which is the lowest point, on the outer rim of the wheel, and to ascend again from *B* on the other side of the wheel *BD* in the line *B n m p r*, which comes nearer the center *C* and touches the inner rim. Thus the weights on the descending side of the wheel being farther from the center of motion than those on the ascending side, it is imagined that the moment of the former will always be greater than the moment of the latter. The weight *A* in particular descends to *B* in the arc *As t B* which is a quarter of a circle; but it rises from *B* to *D* in the curve *B n m p r*, which is less than a quarter of a circle. And, since the velocity is as the space described in a given time, the velocity, and consequently the moment of *A*, when it descends, is greater than its moment, when it ascends. This will be the case of every weight in its turn: therefore the descending weights will always have a greater moment than the ascending one.

The fallacy is this. The velocity with which the weight *A* descends to *B* is estimated by the line *As t B*, whereas the velocity of all bodies in motion is to be estimated in the line of their direction, by the second law of motion. For it is only in the line of direction that any force acts so as to communicate motion to a body: and it is in the same line only that the body in motion acts, when it communicates motion and is itself considered as a moving force. From hence it follows, that the velocity

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with which a weight descends, if the weight is considered as a moving force, must be estimated in the line of its direction, that is, in a line drawn from its center of gravity to the center of the earth. Or, the velocity of a weight is proportional to the approach of it in any given time to the earths center. In like manner the velocity, with which the weight ascends, is to be estimated in the line of its direction, and in any given time it is as the distance to which it is raised from the earths center in that time. From hence it appears, that, though the weight A descends to B in the curve $AstB$, yet the velocity, with which it descends, being as its approach to the earths center, is as $AG = CB$. And though the same weight ascends in the lesser curve $Bnmpr$, yet the velocity, with which it ascends, being as the distance to which it is raised from the earths center in that time, is as $Er = CB = AG$. Therefore the descending and ascending velocities are equal, and consequently the moment of the descending weights will be no greater than the moment of the ascending ones.

I do not know whether this will be made more intelligible by distinguishing between the weights velocity whilst it descends, and the velocity with which it descends. This may appear to the reader too nice a distinction, but I hope he will understand my meaning, if he observes, that whilst A descends, it does not move directly downwards in the line AG, but moves partly downwards and partly in a horizontal direction, and by both these motions together is carried to B. Since therefore, whilst it descends, it is likewise carried horizontally, I think it is evident that it does not descend with its whole velocity. Therefore its velocity whilst it descends, is different from the velocity with which it descends, as different as the whole is from its part. Or otherwise; whilst A descends to B in the line $AstB$, we may consider its motion as resolved into the two sides of a parallelogram AG and $AC = GB$. Where the velocity with which it descends is AG, and the other part of its velocity AC or GB carries it horizontally. In like manner we may distinguish between the velocity which the weight has whilst it ascends, and the velocity with which it ascends. For, if, as before, the weight ascends in the curve $Bnmpr$, we may consider its motion as resolved into the two sides of a parallelogram $BC = Er$ and BE; where the velocity with which it ascends is Er, and the other part of its velocity BE carries it horizontally. Now the descending velocity AG is equal to the ascending velocity Er: and therefore the moment with which the weight descends, will be equal to the moment with which it ascends. The horizontal velocity GB of the descending weight is greater indeed than the horizontal

zontal velocity BE of the ascending one. But, I suppose it is unnecessary to prove that this cannot possibly contribute any thing towards turning the wheel, or towards making the moment on the descending side greater than the moment on the ascending side. For, I imagine, the reader will easily see that a motion in the direction GBE or AC r will not contribute in the least towards making the point A descend, or towards turning the line ACD round upon the center of motion.

100. *The force of elasticity cannot produce a perpetual motion.*

Elastic springs, as has been observed already, by expanding themselves grow weak; and, when they are quite expanded, become entirely unable to produce any motion at all. But in proposition 83 it was shewn that if an elastic ball A strikes upon another B, which is greater than itself, the moment of B after the stroke will be greater than that of A was before it. In like manner, if B was to strike upon C another elastic body greater than itself, the motion might still be increased. And thus by placing more elastic bodies in the row, of which each should be greater than the last before it, the motion might be increased in what proportion we pleased. Now why might not motion produced in this manner be applied to repara what is lost in a machine by the friction of its parts, and to make the motion of the machine a perpetual one? I intended to supply the reader with an answer to this question when I desired him to remember in the instance of two bodies, in proposition 83, that though there is motion produced if we consider the greater body alone, yet if both bodies are taken together there is no motion produced in the same direction; but the moment in the same direction is exactly the same after the stroke, that it was before it. This would be the case were there ever so many bodies. But a machine can only be moved by moving forces in the same direction. For equal moving forces in contrary directions would destroy each other and produce no effect on the machine, and the effect produced by unequal ones in contrary directions will be as their difference. Since therefore the moving force, by which the motion of the machine should be repared, is the moment of these elastic bodies, and since their moment in the same direction is the same after the stroke that it was before; it follows, that no such moment is produced by the stroke as can keep up the motion of the machine and make it perpetual. For instance, suppose the machine consists of four wheels; call them c , d , e , and f . Let c be the first mover, and let f be the last. Now if c begins to move with 8 degrees of moment and by the friction 2 degrees are lost when the motion comes to the wheel f ;

then f can return only 6 degrees of motion on c the first mover. And thus the motion, as it decays in every round, will at last be entirely lost, unless some method could be found for repairing it. Suppose therefore that there were two elastic balls A and B , and that A is to B as 2 to 4: then, by proposition 83, if A strikes B with a moment 6, B will gain by the stroke a moment 8. Now if f the last wheel, which has only the moment 6, was to strike upon A , and A to strike upon B , and B to strike upon the first wheel c , then this first mover would have the same motion 8 communicated to it, with which it began to move. And as the two elastic bodies would always repara the 2 degrees of motion that are lost by friction in conveying the motion from c to f , why might not this contrivance be so applied as to make the machine a perpetual motion? I answer, because, though B strikes c with 8 degrees of motion, yet A will at the same time be reflected with 2 degrees against f , and striking f the last mover by reflection will stop the machine with a force 2, whilst B is moving it with a force 8. Therefore by the action of both bodies together, of B impelling c with the moment 8, and of A reflected back upon f with the moment 2, the machine will be moved only with the difference or $8 - 2 = 6$. So that if c began to move at first with the moment 8, and this moment by being carried round through d and e to f was reduced to 6, the two elastic balls A and B will not repara the moment lost; but the machine will have only 6 degrees of moment, notwithstanding the seeming increase of motion by A 's striking upon B .

CHAP. VIII.

Of the perpendicular ascent and descent of heavy bodies.

100. *Motion is said to be accelerated, if its velocity continually increases, and to be uniformly accelerated, if its velocity increases equally in equal times.*
101. *Motion is said to be retarded, if its velocity continually decreases; and to be uniformly retarded, if its velocity decreases equally in equal times.*
102. *When heavy bodies fall perpendicularly, near the earths surface, they are uniformly accelerated by the force of gravity.*

THOSE, who are learning philosophy, having been taught beforehand, that the force of gravity increases as a body approaches to the earths center, are apt to look upon this as the reason why a heavy
body

body is accelerated in falling, and imagine that it moves faster at the end of the fall than it did at the beginning of it, only because the force of gravity, which acts upon it, is constantly increasing. But this is so far from being the true reason; that, though a force, which is greater at the end of the fall than at the beginning, would accelerate a body, yet it would not accelerate it uniformly. Upon this account, I restrained the proposition to bodies falling near the earth's surface: for if they fall from inconsiderable heights, the force of gravity is the same through the whole line of descent; it is no greater when the bodies come to the ground than when they first begin to fall thither. The semidiameter of the earth is about 4000 miles long, and the greatest heights from whence we can let a body fall, bear a small proportion to this length. Suppose the body let fall from some eminence which is 100 yards from the earth's surface; then at first it is 4000 miles + 100 yards from the center, and when it comes to the ground it is 4000 miles from thence. But this difference is so inconsiderable in respect of the whole that we may look upon the force of gravity as the same from the beginning to the end of the fall: and the continued action of such a force as this will produce an uniform acceleration.

For if we conceive the whole time of descent to be divided into infinitely small instants, which are equal to each other; then the continued force of gravity by acting equally upon the body in each of those instants, will communicate equal velocities to it in each of them; that is, the force of gravity will increase its velocity equally in equal times or will accelerate it uniformly. To explain this a little farther, let us suppose that one single impulse of gravity by acting upon the body in the first instant of its descent communicated to it 1 degree of velocity, but that after this impulse the force of gravity ceased entirely and acted no more upon the body: then, according to the first law of motion, the body once moved downwards by this single impulse would continue to move uniformly in the same direction, or to fall without being accelerated or retarded. Thus without any second impulse of gravity the velocity of the body in the second instant of descent, would be the same that it was in the first. But the force of gravity does not cease acting after one impulse: it acts as strongly, therefore produces as great a velocity, in the second instant as it did in the first. Therefore the velocity produced by this second impulse being added to that which the body would have had without it, will make the velocity in the second instant double what it was in the first. In like manner, if at the end of the second instant, the force of gravity was to cease, then the body would continue

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to fall uniformly with this double velocity. But in the third instant another impulse equal to either of the former acts upon it, and produces a third degree of velocity, which added to what the body had before makes the velocity in this instant triple what it was in the first. An equal impulse in the fourth instant, will add an equal degree of velocity and make the whole velocity 4 times what it was in the first. And by the same means in the fifth instant, the velocity will be increased so as to be 5 times what the first impulse of gravity had produced. And thus, in every instant 1 degree of velocity being added to what the body had before, equal velocities are produced in equal times, or the body is uniformly accelerated.

103. *The velocity acquired by heavy bodies falling perpendicularly is as the time in which it is acquired.*

This is a necessary consequence of the foregoing proposition. For if equal velocities are produced in equal times, then in a double time a double velocity will be produced, and in a triple time a triple one: that is an uniformly accelerated velocity increases in the same proportion with the time, or is directly as the time, in which it is produced.

104. *When a body in a given time has fallen from a certain height, and has acquired any degree of velocity: this last acquired velocity, if it was to continue uniform, would carry the body in the same time through a space equal to twice the height from whence it fell to acquire it.*

For if we call the last acquired velocity V , then $\frac{1}{2}V$ will be the middle velocity between the first and the last. Now a body moving uniformly with half the velocity acquired by falling from a certain height, or with the middle velocity between the least and the greatest, in a time equal to what it has spent in falling, will move just as far as it has fallen. But since $\frac{1}{2}V$ would carry it through a space equal to the height from whence it has fallen, twice this velocity or V , that is the last acquired velocity, will carry it in the same time through twice the height from whence it fell to acquire it.

Or otherwise. Let AB Plat. IV. fig. 2. represent the whole time of descent, and let BC perpendicular to AB represent the velocity acquired in that time, and suppose the time AB divided into infinitely small instants, such as ei , im , mp , po , &c. then, to find what velocity a falling body will have at the instant ei , complete the triangle ABC and draw ef parallel to BC . The velocity BC is acquired in the whole time AB ,

AB, and the velocity that we are seeking is acquired in a part of that time Ae . But, by the foregoing proposition, the velocities acquired in different times are to each other as the times themselves, that is, as AB the whole time of descent is to Ae , so is BC the last acquired velocity, to the velocity acquired in the time Ae . And since the triangles ABC and Aef having one angle at A common and the angles at e and B right ones are equiangular, Eucl. b. I. pr. 32. corol. 2. the sides of these triangles are proportional, Eucl. b. VI. pr. 4. or as AB is to Ae so is BC to ef . Therefore ef is the velocity of the body at the instant ei , or the velocity acquired by falling for the time Ae . For the same reason the velocity at the instant im , which is acquired in the time Ai , is proportional to the line ik , that is drawn parallel to BC or to the base of the triangle ABC: mn likewise is the velocity in the instant mp acquired in the time Am . And universally, when the whole time of descent is represented by AB the height of a rightangled triangle and the velocity by BC the base of it; then a line drawn parallel to the base will be as the velocity acquired in such a part of the whole time as is represented by the part of AB, which lies between the point A and that line. Now though the body falls with an accelerated motion, so that the velocity is constantly increasing and is greater in any instant than it was in the preceding instant; yet we may suppose it to continue the same during the time of one single instant, especially if we take the instant infinitely short. Therefore let ef represent the velocity at the beginning of the instant ei , and because ei is infinitely short a line equal to ef will likewise represent the velocity at the end of it. And because the spaces described, where the velocity is uniform, are, by proposition 10, as the times and velocities conjointly, it follows that the space described in the time ei with the velocity ef will be as the area of a rectangle, the sides of which are ei and ef . For the same reason the space described in the instant im with the velocity ik will be as the rectangle mk . And the space described in the instant mp with the velocity mn will be as the rectangle pn . But if we suppose the whole time AB divided into instants in the same manner, then the sum of all the rectangles, to which the spaces described in each of these instants are proportional, will fill up the whole triangle ABC; except only the small triangular spaces about the line AC. But these small spaces may be neglected, since they arise from the construction of the figure not agreeing exactly with the suppositions that were made. For we supposed ei to be an infinitely short instant, and the triangular space fk arose from not making ei infinitely short, since if e and i had been close together, f and k would have

have been so too, and then there would have been no such small triangle at fk . And as we may account for all these other small triangular spaces in the same manner; the sum of the spaces described in the several instants of the whole time AB to acquire the velocity BC is proportional to the area of the triangle ABC . Therefore universally, the lines described by heavy bodies falling perpendicularly are to each other as the areas of rightangled triangles whose heights are as the times of descent and whose bases are as the last acquired velocities. Thus Plat. IV. fig. 3. the space or line described in the time AF to acquire the velocity FG , is to the line described in the time AB to acquire the velocity BC as the area AFG is to the area ABC . But to return to the proposition and to fig. 2. If a heavy body is falling for the time AB and acquires the velocity BC , then the space described is as the area of the triangle ABC . But if with the last acquired velocity BC , it was to move on uniformly for an equal time AB then, by proposition 10, the space described would be as the area of the rectangular parallelogram $ABCD$. But the parallelogram $ABCD$ is double the triangle ABC upon the same base BC and between the same parallels AD and BC . Eucl. b. I. pr. 41. Therefore a body by falling from a certain height in a given time acquires a degree of velocity, which, if it was to continue uniform, would carry the body in the same time through a space equal to twice the height from whence it fell to acquire this velocity.

Near the surface of the earth a heavy body descends through $15\frac{1}{12}$ French feet or $16\frac{1}{10}$ English feet in 1 second of time. For the sake of expressing this fractional quantity $16\frac{1}{10}$ feet by an integer, suppose it $16\frac{1}{2}$ feet, then the space through which a heavy body falls in one second of time will be equal to an English measure that is called a rod or pole. Therefore, by what has been proved, the velocity that is acquired by a heavy body in falling through 1 pole, or for 1 second of time, would carry it uniformly through 2 poles in an equal time.

The same thing may be expressed arithmetically. If a body is in motion with 6 degrees of velocity, and moves on uniformly for 7 instants, then by proposition 10, the space described will be as $6 \times 7 = 42$, that is, the space described will be as the velocity multiplied into the time, or as the sum of all the velocities, which the body had in each instant. But if the body from being at rest had been uniformly accelerated for 7 instants then its velocity in the several instants reckoning from the beginning of the fall would have been 0. 1. 2. 3. 4. 5. 6. and the space described in the 7 instants being as the sum of all these velocities would be as $0 + 1 + 2 + 3 + 4 + 5 + 6 = 21$. Whereas we have seen already that

that the last velocity, or 6, will in 7 instants carry the body through a space as 42, which is double 21, or double the height from whence the body fell to acquire this velocity. And universally, the space which a body describes with a velocity uniformly accelerated is as the sum of a number of terms in arithmetical progression, beginning from 0 and ending with the greatest velocity; and the space which it would describe in the same time with the last velocity continuing uniform is as the sum of an equal number of terms, each of which is equal to the last in that progression. And since this latter sum will always be double the former, the space described with an uniform velocity in a given time will always be double the height from whence the body fell to acquire that velocity.

105. *The spaces described in unequal times by a heavy body falling perpendicularly are, if we reckon from the beginning of the fall, as the squares of the times.*

Thus, if a body falls through 1 pole in 1 second of time, then the number of poles, that it will fall through in 2 seconds, is to the number it falls through in 1 as 2^2 to 1^2 or as 4 to 1. For the same reason in 3 seconds, it will fall through 3^2 or 9 poles, in 4 through 4^2 or 16, &c.

For let AB Plat. IV. fig. 3. represent the whole time of a bodys descent, and AF a part of that time; let BC represent the last acquired velocity, and FG the velocity acquired in the time AF. Then, by what was said in proving the foregoing proposition, the space described in the time AF is to the space described in the time AB as the area of the triangle AFG is to the area of the triangle ABC. But since these triangles are equiangular, the area AFG is to the area ABC as the square of AF to the square of AB, Eucl. b.VI. pr.19. that is, the spaces described being as AFG to ABC are as the square of the time AF to the square of the time AB.

The spaces described by bodies in motion are as the time multiplied into the velocity, or, calling the time T and the velocity V, as $T \times V = TV$. But in uniformly accelerated motions V is as T, by proposition 49. Therefore TV or the space described is as TT the square of the time.

I cannot leave this proposition without applying it to one case, where it is of singular use; and that is to determine how far an heavy body, which is as far from the earth as the moon is, would be made to fall towards the earth in 1 minute. A heavy body near the earths surface falls through 1 pole in 1 second of time. Therefore, by this proposition, in

1 minute, or 60 seconds, it will fall through the square of 60 or through 3600 poles. Now the distance of the body from the earth's center, when it is near the surface, is only 1 semidiameter of the earth; but if it was as far off as the moon is, then the distance from the center would be 60 semidiameters. But, by proposition 23, the force of gravity at different distances from the center is as the squares of those distances reciprocally. Therefore, if we call the force of gravity at the earth's surface 1, where the distance is 1 semidiameter, then at the distance of the moon or 60 semidiameters this force will be as much less as $60^2 = 3600$ is greater than $1^2 = 1$, that is the force of gravity at the earth's surface is to the same force at the distance of the moon as 1 to $\frac{1}{3600}$. And consequently the force of gravity, which at the earth's surface makes a heavy body fall through 3600 poles in a minute, will, at a distance equal to that of the moon from the earth, make a heavy body fall through $\frac{1}{3600}$ of this, or through 1 pole in a minute. So that a heavy body, which falls 1 pole in a second of time near the earth's surface, would take up a minute to fall 1 pole, if it was as far off as the moon.

106. *The spaces described by a heavy body falling perpendicularly are, if we reckon from the beginning of the fall, as the squares of the last acquired velocities.*

The spaces described to acquire the velocities GF and CB Plat. IV. fig. 3. are to each other as the triangle AGF to the triangle ACB. But these triangles being equiangular are to each other as the square of GF to the square of CB, Eucl. b. VI. pr. 19. that is, the spaces described are as the squares of the velocities which are acquired in describing them.

The spaces described by bodies in motion are as the time multiplied into the velocity; or, calling the time T, and the velocity V, as $T \times V = TV$. But in uniformly accelerated motions V is as T, therefore TV or the space described is as V V the square of the last acquired velocity.

If we can by any means find out what proportion the velocities, which two bodies have acquired in falling, bear to each other, then by this proposition we may determine the difference of the heights from whence they have fallen. For suppose their last acquired velocities to be in the proportion of 2 to 6, then the spaces described are as 2^2 to 6^2 or as 4 to 36.

107. *The time of descent from unequal heights, and also the acquired velocities, are as the square roots or in a subduplicate ratio of those heights.*

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This is a necessary consequence of the two last propositions. For since the spaces described are as the squares of the times; therefore the times of descent must be as the square roots of the spaces. Thus, calling the space described S , and the time of descent T , since TT is as S , T will be as $\sqrt{S}$ or as the square root of the space described.

And since the spaces described are as the squares of the last acquired velocities, therefore those velocities are as the square roots of the spaces. Thus, calling the velocity V , since VV is as S , V will be as $\sqrt{S}$.

Suppose a body to fall 1 pole in a second, I ask how long it will be in falling 25 poles. The answer, by this proposition, is 5 seconds. For the time of descent through 25 poles is to the time of descent through 1 as the square root of 25 to the square root of 1; or as 5 to 1.

If two bodies, which contain unequal quantities of matter, are let fall from heights, which are as those quantities inversely; then, since by this proposition, the velocities acquired will be as the square roots of those heights, it follows, from what we have here supposed, that the velocities will be as the square roots of the quantities of matter inversely. Let the body A weigh 9 ounces and the body B 25. Then, if A falls from a height of 25 feet, and B from a height of 9 feet, the spaces which they describe in falling, will be as their quantities of matter inverted. Now, by this proposition, the velocity which A acquires in falling through 25 feet is to the velocity which B acquires in falling through 9, as 5 to 3. But the square root of A 's quantity of matter, or 9 ounces, is to the square root of B 's, or 25 ounces, as 3 to 5. Therefore the last acquired velocities being to each other as 5 to 3 are as the square roots of the quantities of matter inverted.

108. *The spaces which a heavy body falls through in equal parts of time, when those parts are taken separately, are, if we reckon from the beginning of the fall, as the uneven numbers 1. 3. 5. 7. &c.*

For since, by proposition 105, a heavy body falls 4 poles in 2 seconds, therefore, as it falls 1 pole of the 4 in the first second, it must fall 3 in the next. Since it falls 9 poles in 3 seconds, 1 of them in the first, and 3 in the next, therefore it must fall the remaining 5 in the third. Since it falls 16 poles in 4 seconds, 1 of them in the first, 3 in the next, and 5 in the third, it must fall the remaining 7 in the fourth.

109. *The motion of a heavy body, when it is thrown perpendicularly upwards, is uniformly retarded.*

When a body is falling the impulses of gravity act in the same direction in which the body moves, and therefore by communicating

equal degrees of velocity in equal times, they produce an uniform acceleration. But when the body is thrown upwards these impulses act in a direction contrary to the bodys motion, and therefore by destroying equal degrees of velocity in equal times, they will retard it uniformly.

110. *Whatever is the height to which a body ascends when thrown perpendicularly upwards, it would have ascended to the same height in half the time, if the first velocity had continued uniform.*

For as the last velocity, when the body has been uniformly accelerated, would in a given time, if this velocity was afterwards to continue uniform, carry the body through twice the space, which was described to acquire it; so for the same reasons, if the first velocity of an ascending body was to continue uniform, it would carry the body, in a given time, to twice the height to which it will ascend, when that velocity is uniformly retarded. Therefore this first velocity, which would carry the body to double the height in the same time, must, by proposition 9, carry it to the same height in half the time.

111. *The heights, to which heavy bodies ascend, when they are thrown perpendicularly upwards, are to each other as the squares of the times spent in ascending.*

For, as in uniformly accelerated motions, so in uniformly retarded ones, the spaces described are to each other as the areas of rightangled triangles, whose heights are the times of ascent and whose bases are the greatest velocity. The only difference is, that in uniformly accelerated motions this greatest velocity is the last, or that, which is acquired in falling; and, in uniformly retarded ones, it is the first, or that, with which the body is thrown upwards.

Let the velocities, with which two bodies are thrown perpendicularly upwards, be to each other as GF to CB, Plat. IV. fig. 3, then, because these velocities are uniformly retarded, that is, because they decrease as the times of ascent increase, it follows that the time in which the velocity GF is lost will be to the time in which CB is lost as GF is to CB, or as AF to AB. And the space described by the body whose velocity is GF, will be to the space described by that whose velocity is CB as the area AGF to the area ACB or as the square of AF to the square of AB, Eucl. b. VI. pr. 19. that is, the spaces described will be as the squares of the times, in which they are described.

The spaces described by bodies in motion are as the time multiplied into the velocity, or calling the time T and the velocity V, the spaces de-

described are as $T \times V = TV$. But in uniformly retarded motions V or the velocity lost in ascending is always as T or as the time of ascent, therefore TV , the space described is as TT or as the square of the time.

112. *When heavy bodies are thrown perpendicularly upwards the heights, to which they ascend, will be as the squares of their first velocities.*

After what has been said already the reader cannot want a distinct proof of this proposition, he need only be reminded, that when the velocities are as GF to CB , Plat. IV. fig. 3. the heights are as the areas AGF , ACB ; that is, as the square of GF to the square of CB . Or, that since the spaces described are as TV , and in uniformly retarded motions V is as T , therefore TV the spaces described are as VV the squares of the velocities lost in ascending.

From this proposition it follows, that whatever is the height to which any force will throw a bullet; a double force communicating twice the velocity to the same bullet would throw it to 4 times that height, and a triple force would throw it to 9 times that height. Because in all cases the height increases as the square of the velocity with which the bullet is thrown.

113. *The times of ascent, as also the first velocities, are as the square roots of the heights to which the bodies rise.*

This is a necessary consequence of the two last propositions; and needs no other sort of proof than what the reader will remember was made use of in proving proposition 107.

114. *A heavy body, when thrown perpendicularly upwards, ascends to such a height that in falling back again it will acquire a velocity equal to that, with which it was thrown.*

For the first velocity is 'as the square root of the line of ascent, by the last proposition; and the velocity, which the body acquires in falling back again, is as the square root of the line of descent, by proposition 107. But since the body falls in the same line through which it had risen, the line of descent is equal to the line of ascent. Therefore their square roots are equal; and consequently the velocities, which are as those roots, that is, the velocity acquired in falling, and the velocity, with which the body was thrown at first, must be equal too.

The same impulses of gravity, which retard the motion of the body in the line of ascent, accelerate it in the same manner in the line
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of descent. Therefore they will communicate in one case as much velocity as they destroy in the other, or the same velocity, that was lost in ascending, will be acquired again in descending.

115. *When a heavy body has fallen from any height, if its direction could be inverted without changing the velocity that it has acquired; this velocity would make it rise to the same height from whence it fell. And the time of ascent would be equal to the time of descent.*

For the line of descent is as the square of the last acquired velocity, by proposition 106, and when this acquired velocity, by inverting the bodies direction, becomes the first velocity of ascent, then the line of ascent will be as the square of the same velocity, by proposition 113. And since the same square is equal to itself the line of ascent will be equal to the line of descent; or the body will rise to the same height from whence it fell.

The same impulses of gravity, that accelerated the body in its descent, will retard it, when it is turned back again and made to rise with the last acquired velocity. Therefore the velocity, which was communicated by these impulses whilst the body descended, will not be lost in ascending but by an equal number of contrary impulses, that is, not till an equal time has passed, and the body has risen to the same height from whence it fell.

The time of ascent is, by proposition 113, as the square root of the line of ascent, and the time of descent as the square root of the line of descent; and since by the former part of this proposition the body rises to the same height from whence it fell, that is, since the line of ascent is equal to the line of descent; therefore the square roots of those lines will be equal or the time of ascent will be equal to the time of descent.

From this proposition it follows, that, because the spaces through which a body descends in equal parts of time, when taken separately, are as the uneven numbers 1, 3, 5, 7, &c. by proposition 108; the spaces through which a heavy body thrown perpendicularly upwards ascends in equal parts of time, when taken separately, are, if we reckon from the instant of projection, as the uneven numbers taken backwards. For the velocity in the line of ascent, and consequently the space run through in a given time, is every where the same as in the line of descent. So that if a body was 5 seconds in falling, the spaces run through in each second taken separately, would be as 1, 3, 5, 7, 9. And if it was thrown upwards with such a force as would make it 5 seconds in rising to its
greatest

greatest height, the spaces run through in each second would be as 9, 7, 5, 3, 1.

116. *When the times are known, which heavy bodies have spent in falling, the heights from whence they fell may be found.*

For if they fall 1 pole in 1 second of time, then, by proposition 105, they fall 4 poles in 2 seconds, 9 poles in 3 seconds. And universally the number of poles, through which they have fallen, is always equal to the square of the number of seconds, which they have spent in falling.

The reader is desired to observe, that in this proposition and the following one, I suppose the bodies to meet with no resistance. So that neither of them can be applyed with exactness, to measure either the heights from whence bodies fall, or the heights to which they rise in such a resisting fluid as the air is.

117. *Supposing a body, that is thrown perpendicularly upwards, to move in a medium void of resistance, the height, to which it ascends may be determined,*

The time spent from its first setting out to the moment it returns to the place it was thrown from, is the time of ascent and descent together. But as the time of ascent is equal to the time of descent; half of this is equal to the time of descent alone. And, when this is known, the height from whence the body has fallen, which is likewise the height that it ascended to, may be determined by the last proposition. If the time of ascent and descent together is 6 seconds: then the time of descent alone or the half of this is 3 seconds. Therefore, by the last proposition, the height from whence it has fallen, or the height to which it had ascended, is 9 poles.

118. *The moment of a given body, which is uniformly accelerated, is not as the square of the last acquired velocity.*

The foreign philosophers differ from Sir Isaac Newton in the manner of estimating the moments of bodies. His rule has been explained already, when I shewed that the moment is as the quantity of matter, when the velocity is given; as the velocity, when the quantity of matter is given; and universally in a ratio compounded of the quantity of matter and velocity. They on the other hand maintain, that the moment is indeed as the quantity of matter, when the velocity is given, but as the square of the velocity, when the quantity of matter is given, and therefore uni-

universally in a ratio compounded of the quantity of matter and square of the velocity. One of the arguments, which they make use of to prove this theory of the moment will be best understood now we are speaking of uniformly accelerated motions.

They say that if you put a body in motion and give it any degree of velocity by the action of some external force such as the pressure of your hand against it; then in order to double this degree of velocity you must double the force of your hand. For if your hand was to keep pressing the body only with the same force that it did at first, they would both keep moving on together in the same direction and with the same velocity, but by doubling the force you will double the body's velocity. In like manner to triple the velocity of the body, your hand must press it with a triple force; to give a fourth degree of velocity, you must press it with four times the force that you made use of to give it one degree; and universally the force with which you press the body must always be as the velocity, which you communicate to it. In the triangle ABC Plat. IV. fig. 2. let ef , drawn parallel to the base BC, represent any degree of velocity communicated to a body by the pressure of your hand against it: then to increase this velocity and make it ik , the force of your hand must be increased in the proportion of ik to ef : to give it the velocity mn , the pressure must be as mn , to give it the velocity pq , the force acting upon it must be as pq ; and to give it the last velocity BC the moving force of your hand must be as BC. Therefore if in the time AB the body, by the pressure of your hand against it, is uniformly accelerated and acquires at last the velocity BC, the sum of the moving forces, which have acted upon it in that time, will be as the sum of innumerable lines drawn from A to B in the triangle ABC parallel to the base of it, in the same manner that ef , ik , mn , &c. are drawn. But all these lines together will fill up the area of the triangle ABC. Therefore the sum of all the moving forces are as the area of the triangle, whose height is the time of acceleration, and whose base is the last acquired velocity.

Suppose therefore that one body in the time AF, Plat. IV. fig. 2. acquires the velocity GF, and that another body in the time AB acquires the velocity BC, then, by what has been said, the sum of the moving forces that have acted upon the former of them, is to the sum of the moving forces that have acted upon the latter; as the area AGF to the area ACB; that is, since the triangles are equiangular, as the square of GF to the square of CB. Eucl. book VI. prop. 19. Therefore the moving force or rather the sum of the moving forces, by which a body is ac-

accelerated, is as the square of the last acquired velocity. And, since the quantity of motion or moment is, by the second law of motion, as the moving force, the moment must likewise be as the square of the last acquired velocity.

What has been already said concerning the acceleration of a body by the force of gravity might serve to confute this whole argument. For when the first impulse of gravity has communicated one degree of velocity to it, a second impulse, which is only equal to the former, will double the velocity, and a third equal impulse will triple it. So that in the case of this acceleration we find that to double the velocity, which a body has, a double force need not be made use of, nor a triple one to triple it: neither is the sum of the accelerating forces as the squares of the last velocity, but as the velocity itself.

But the defenders of the opinion, against which I am disputing, have an answer ready for this objection. They distinguish between an internal force and an external one. Gravity, they say, is an internal force; it acts upon bodies in motion, just as it does upon bodies at rest: and therefore as one impulse would produce one degree of velocity; so the second impulse, though only equal to the former, will have the same effect upon the body in motion as the first had upon it before it began to move, and will produce a second degree of velocity; and a third equal impulse will for the same reason produce a third degree. But an external force does not act upon a body in motion and at rest in the same manner. A force which presses on the outside of a body may produce one degree of velocity, but if it continues to press with the same force, then instead of doubling or tripling the bodys velocity, it will move on close to the body without altering its motion at all. To double the velocity you must press with twice the force, and to triple it the force itself must be tripled.

This mistake in the case of an external force, such as our hand is, has arisen from not distinguishing between the force with which our hand moves, and the force with which it strikes or impells the body. For though the sum of the moving forces in our hand is as the square of the velocity last acquired by the body in motion, yet the moment of the body will not be in this proportion, unless the sum of the forces, with which the hand has impelled it, is likewise as the square of this velocity. But it is easy to shew, that the sum of these impulses, or forces acting upon the body and communicating motion to it, are in a simple ratio of the velocity, which it acquires. For suppose with a force or velocity of your hand as 1, you communicate 1 degree of velocity to the
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body impelled by it : then to double this velocity in the body you double the force, or, which is the same thing, you double the velocity of your hand. But though your hand has the moving force 2, yet the moving force with which it acts upon the body will not be 2. Because they both move in the same direction the body with the velocity 1 your hand with the velocity 2 ; and consequently the quantity of the stroke, or moving force which acts upon the body, will, by proposition 94, be as the difference of these velocities, or as $2 - 1 = 1$. In like manner, when you would triple the bodys velocity you triple the velocity or moving force of your hand ; but as the body is already moving in the same direction with 2 degrees of velocity, the quantity of the stroke, the impulse, or moving force, will be as the difference of the velocities in the same direction or as $3 - 2 = 1$. If when the body is moving with 3 degrees of velocity you want to communicate a fourth, you must move your hand against it with 4 degrees of velocity. But here again though your hand has the velocity and consequently the moving force 4, yet it will act upon the body with only the force 1. Because the body and your hand are moving in the same direction one with the velocity 3 the other with the velocity 4 ; therefore, by proposition 94, the quantity of the stroke of your hand against the body is as the difference between 4 and 3, or as $4 - 3 = 1$. Thus each impulse or moving force of your hand against the body is only 1, whatever may be the moving force in your hand itself ; and as each of these impulses communicates 1 degree of velocity ; the number of impulses, that is, the sum of the moving forces which have acted upon the body, will be proportional to the number of degrees of velocity communicated. And consequently the moment or motion produced by these forces will be as the sum of them or as the velocity of the body.

CHAP. IX.

Of the descent of heavy bodies upon inclined planes.

119. *A power will support a weight upon an inclined plane, if the power is to the weight, as the height of the plane is to the length of it.*

IF a power at R, Plat. IV. fig. 4. that acts upon a weight B in the direction BR parallel to the inclined plane AC, supports the weight and keeps it at rest ; then the power R is to the weight B as AD the perpendicular height of the plane to AC the length of it.

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The weight upon the inclined plane is acted upon by three forces. The first is its own gravity in the line of direction BE. The second is the force of the inclined plane, which, by throwing it out of the line of direction or preventing it from falling in the line BE, makes it fall in the line BC. This force acts in the direction BH perpendicular to the plane, as all bodies, by the second law of motion, act in lines perpendicular to their surfaces. The third force is the power R, which acts in the direction BR. But, by proposition 21, if a body is acted upon by three forces at the same time and rests between them, then these forces are to each other as the three sides of a triangle respectively parallel to the directions in which they act. In the triangle GBF the side BF is parallel to BH or to the direction in which the plane acts, the side BG is parallel to BR or to the direction in which the power acts, and the side GF is parallel to BE or to the line of direction in which the gravity of the body acts. Therefore since these three forces are to each other respectively as the three sides of this triangle, the power is to the weight as GB to GF. We must therefore prove that GB is to GF as AD the height of the plane to AC the length of it. Now BE is perpendicular to DC, because DC is a horizontal plane and BE is the line of perpendicular descent. AD is likewise perpendicular to DC, because AD is the perpendicular height of the plane. From whence it follows that AD and BE are parallel. Eucl. b. XI. pr. 6. And as GF, by the construction, is parallel to BE, it is likewise parallel to AD. Eucl. b. I. pr. 30. Therefore CA, crossing GF and AD, will make the external angle CGF equal to CAD the internal opposite one on the same side. Eucl. b. I. pr. 29. Again ADC is a right angle because AD, by the construction, is perpendicular to DC and GBF is a right one, because HF, by the construction, is perpendicular to AC. Therefore in the triangle GBF there are two angles respectively equal to two angles in the triangle ADC, from whence it follows, that the third angle in each will be equal, Eucl. b. I. pr. 32. corol. 2. and the sides about the equal angles will be proportional, Eucl. b. VI. pr. 4. GB to GF, or the power to the weight, as AD to AC, or as the height of the plane to the length of it.

We may prove the same proposition in an easier manner. For when the weight is drawn quite up to the top of the plane from C to A, the power in its line of direction must have described a space equal to CA; but the weight in the mean time, though it has passed over CA, has described in its line of direction no more than a space which is equal to AD, or to the distance through which it has risen from the earths center. But the velocity of the power and weight are to each other as the spaces which

they have described in the same time in their respective lines of direction. Therefore the velocity of the power is to the velocity of the weight as CA the length of the plane to DA the height of it. And consequently, if the power is to the weight as DA to CA, or as the height of the plane to the length of it, then they will be to each other as their velocities inversely, and by proposition 14, their moments will be equal, or the power will be sufficient to support the weight.

120. *The accelerating force of a bodys gravity, when it descends upon an inclined plane, is to the accelerating force of its gravity, when it falls perpendicularly, or to the whole force of its gravity, as the height of the plane is to the length of it.*

The force wherewith a body endeavours to descend, or the accelerating force of its gravity upon an inclined plane is equal to the power that is able to support it. But, by the last proposition, this power is to the whole weight of the body, as the height of the plane is to the length of it. Therefore the force, with which it endeavours to descend on an inclined plane, is to the whole weight in the same proportion.

121. *Heavy bodies descend upon inclined planes with an uniformly accelerated motion.*

The whole accelerating force of a bodys gravity, or the force wherewith it endeavours to descend perpendicularly, is always the same; and the accelerating force of gravity upon every part of the plane, whilst the body is descending from A to C, has the same proportion to the whole force. For, by the last proposition, the gravity upon an inclined plane is to the whole gravity as the height of the plane is to the length of it. Since therefore the whole gravity is given and the gravity upon the plane has always the same proportion to it, it follows that the gravity upon the plane must be given too. But the constant action of an uniform force, such as the gravity of a body upon an inclined plane has been proved to be, will produce equal degrees of velocity in equal times, that is, will cause an uniformly accelerated motion.

From hence it follows that all the propositions, which have been already proved in relation to the uniformly accelerated motion of a body falling perpendicularly, are true likewise of a body falling down an inclined plane. Therefore, without repeating either the propositions or the demonstrations, I shall refer the reader to the foregoing chapter. But must desire him to remember that the accelerating force upon an inclined

clined plane is less than the accelerating force of perpendicular descent, in the proportion of the planes height to its length.

122. *The velocity, which a heavy body acquires in any given time upon an inclined plane, is to the velocity, which it would acquire in the same time by perpendicular descent, as the height of the plane is to its length.*

In an uniformly accelerated motion the velocities produced in equal times are as the causes, which produce them, that is, as the accelerating forces. But, by proposition 120, the accelerating force upon an inclined plane is to the accelerating force in perpendicular descent as the height of the plane to the length of it. Therefore the velocities produced in equal times are in the same proportion.

123. *The space, which a heavy body describes in a given time upon an inclined plane, is to the space, which it would describe in the same time by falling perpendicularly, as the height of the plane is to the length of it.*

The spaces described in a given time are as the velocities, by proposition 10. And the velocities produced in a given time in two bodies, one of which falls upon an inclined plane and the other perpendicularly, are, by the foregoing proposition, as the height of the plane to the length of it. This therefore will likewise be the proportion of the spaces, which they describe in the same time.

Let AB, Plat. IV. fig. 5. be an inclined plane and suppose a heavy body was to fall perpendicularly from A to C, or through a space equal to the height of the plane. How far would another heavy body fall in the mean time upon the plane from A towards B? The spaces described in the same time will be to each other as the height of the plane to the length of it. Therefore the space sought is to AC, or to the space described by the body which falls perpendicularly, as the same line AC is to AB, or as the height of the plane is to the length of it. Now the angle ACB is always a right one; because CB is an horizontal plane, and AC is the perpendicular height of the inclined one. And, if from the angle C in the rightangled triangle ACB you draw the line CD perpendicular to AB, then ADC is similar to ACB; Eucl. b. VI. pr. 8. and AD is to AC as AC to AB. Eucl. b. VI. pr. 4. Therefore AD is the space sought. So that, if the two bodies were let fall from A at the same instant, when one had fallen perpendicularly to C, the other would have fallen upon the plane no farther than D.

124. *An heavy body takes up the same time in falling either perpendicularly through the diameter of a circle, or upon an inclined plane through any chord of it.*

A heavy body will fall through AD, Plat. IV. fig. 6. or AE, or any other chord of the circle ADBH in the same time, that it would fall perpendicularly through the diameter AB.

For if from B, the lowest point in the diameter, a line be drawn to D, the lowest point of the chord; then the angle, which this line makes with the chord, being in a semicircle will always be a right one. Eucl. b. III. pr. 31. Upon which account from the answer to a question, that we were led to by the last proposition, it follows, that, considering the chord AD as part of an inclined plane AC, the chord will always be to the diameter as AB, which is the height of the plane, to AC, which is the length of it. Therefore, by the last proposition, AD the chord and AB the diameter will be described in the same time.

In like manner, if the chord AE is continued to G, it might be proved, that a body will fall in the same time either through the chord AE, or through the diameter AB. And if we draw a chord HB parallel to AD; then, because they are parallel, they are equally inclined: therefore the accelerating force of gravity, when a body is to descend upon HB, will be the same as when it is to descend upon AD. And, because they are drawn from opposite ends of the diameter and are parallel, therefore they are at equal distances from the center, and HB is equal to AD. Eucl. b. III. pr. 14. But since the two inclined planes AD and HB are equal to each other and the accelerating force of gravity is the same upon both, it follows, that a heavy body will take up an equal time to describe either of them, and that the time of oblique descent through HB as well as through AD is equal to the time of perpendicular descent through the diameter. The same may be proved of the chord FB, which is parallel to AE; and of any chord, that can be drawn in the circle ADBH.

125. *The time in which a body describes an inclined plane is to the time in which it would fall perpendicularly from the same height, as the length of the plane is to the height of it.*

The time that a body takes up to run from A to B, Plat. IV. fig. 5. is just as much longer than the time it would take up to fall perpendicularly from A to C as AB is longer than AC.

To prove this we must remember, what has already been taken notice of, that AD is to AC as AC is to AB, or that AD, AC, and AB are in continued geometrical progression. From whence it follows, Eucl. b. V.

b. V. def. 10. that the first of these quantities AD is to the third AB as the square of the first, or AD^2 , is to the square of the second, or to AD^2 . The reader may illustrate this conclusion by trying it in numbers. 3. 6. 12. are in continued geometrical progression for 3 is to 6 as 6 is to 12. And 3 the first number is to 12 the third, as 9 the square of the first is to 36 the square of the second. For as three is contained 4 times in 12 so is 9 in 36.

This being premised, let us first enquire what proportion the time of descent through a part of the plane AD bears to the time of descent through the whole plane AB. Now, by proposition 121, the motion of a body descending upon an inclined plane is uniformly accelerated; and, by proposition 105, the spaces described by uniformly accelerated motions are as the squares of the times, in which they are described. Therefore as AD is to AB so is the square of the time in which AD is described to the square of the time in which AB is described. So that if we can find any two quantities, whose squares are to each other as AD to AB, these two quantities are as the times of descent through AD and AB. But by what was premised AD^2 is to AC^2 as AD to AB. Therefore the time of descent through AD is to the time of descent through AB as AD to AC. And by what has been shewn already AD is to AC as AC to AB. From whence it follows that the time of descent through AD is to the time of descent through AB as AC the height of the plane is to AB the length of it. And, since, by proposition 123, a heavy body will take up the same time to descend either through AD upon the inclined plane or through AC perpendicularly, we may conclude that the time of perpendicular descent through AC is to the time of oblique descent along the inclined plane AB, as AC the height of it is to AB the length of it.

126. *The velocity, which a body acquires in falling from a given height is the same; whether it has fallen perpendicularly or along a plane inclined in any manner whatsoever.*

The velocity acquired by a heavy body is the same when it has fallen from the point A to the line CB, Plat. IV. fig. 5. whether it has fallen perpendicularly through AC or along the inclined plane AB. The velocity produced in a given time is as the accelerating force that produces it, by propositions 12. 31, and the velocity produced by a given accelerating force is as the time of producing it, by proposition 103. Therefore universally the velocity is as the accelerating force and time conjointly; and consequently if the accelerating force is inversely as the time then the velocity

velocity produced will be a given quantity. But the accelerating force upon an inclined plane is to the accelerating force in perpendicular descent, by proposition 120, as the height of the plane to the length of it, and the times of descent, by the last proposition, are to each other as the length to the height. Therefore the accelerating forces are inversely as the times, and the velocities produced will be equal.

This may be otherwise expressed. The force of gravity, which acts upon a body descending perpendicularly from A to C is greater than the force which acts upon it when it descends from A to B. But then as much as this latter force is less than the former, just so much longer it acts, and therefore in the end both will produce an equal effect, that is, an equal velocity in the body.

From what has been said it follows that a body by falling from the line AK, Plat. IV. fig. 7. into the line CB, that is, by falling from a given perpendicular height, will acquire the same velocity, whether it falls through the plane AD, or AB, or KB, since in falling through any one of them it acquires the same velocity as in falling through AC, which is the perpendicular height of them all.

127. Heavy bodies by falling from a given perpendicular height acquire the same velocity, whether they fall through one plane, or through more, which are contiguous to each other.

The perpendicular distance of the line HE, Plat IV. fig. 8. from the line DF is HD: and a body will acquire the same velocity in falling through this distance either along the inclined plane ED or along the three contiguous ones AB, BC, CD. For first the body acquires the same velocity in falling to B either through GB or AB, by the last proposition. Therefore secondly, the velocity which it has, when it comes to C, will be the same whether it fell to C through the two planes AB and BC or through the one continued plane GBC. Because by the first step the velocity at B will be the same in both cases, and the line BC, which it descends through afterwards is the same in both. But thirdly, the body acquires the same velocity by falling to C either through EC or GC, by the last proposition, that is either through EC or through AB and BC by the second step of this demonstration. Therefore fourthly, the velocity which it has when it comes to D will be the same whether it fell to D through the one continued plane ECD, or through the two contiguous ones GC and CD, or through the three contiguous ones AB, BC, and CD. Because by the third step the velocity at C is the same in all the three cases, and the line CD, through which it descends afterwards, is the same in all of them.

128. *If*

128. *If two heavy bodies descend upon two or more planes, which are similarly inclined and proportional; the time taken up to run down those planes will be as the square roots of their lengths.*

Let one heavy body descend upon the two inclined planes AB, BC; and the other upon DE, EF. By the planes being similarly inclined is meant that AB and DE, as likewise BC and EF make equal angles ABT and DES with the horizontal planes BT and ES or that AB is just as much inclined as DE is, and likewise BC just as much as EF. From hence it follows, that ABG and DEH are equi-angular triangles. For, since AB and DE make equal angles ABT and DES with the horizon, they will likewise make equal angles at A and D with AG and DH, two lines that are parallel to the horizontal planes BT and ES, Euc. b. I. prop. 29. because the angle at A is the complement of ABT and the angle at D is the complement of DES to two right ones. And, since BC and EF make equal angles BCR and EFP with the horizon, they will likewise, if they are continued to G and H, make equal angles at G and H with AG and DH two lines that are parallel to the horizontal planes CR and FP, Euc. b. I. prop. 29. because the angle at G is the alternate of that at C, and the angle at H is the alternate of that at F. But, since there are two angles in the triangle AGB respectively equal to two angles in the triangle DHE, and the three angles of every triangle are only equal to two right ones, it follows Euc. b. I. prop. 32. coroll. 2. that the third angle at B in one of them is equal to the third angle at E in the other.

By the planes being proportional is meant, that AB bears the same proportion to DE that BC bears to EF. The proposition affirms, that when two planes AB and BC are similarly inclined and proportional to two other planes DE and EF, then the time, which a heavy body takes up to descend along ABC is to the time, which any other heavy body takes up to descend along DEF as the square root of ABC is to the square root of DEF.

The reader will remember that when I use the word root in the following demonstration, I always mean square root. From the planes being proportional, and from the similarity of the triangles ABG and DEH, which is a consequence of the planes being similarly inclined we have the eight following proportions. First, AB is to DE as BC is to EF; because the planes are proportional. Secondly, AB is to DE as GB to HE; because the triangles, of which these are the sides, are equi-angular. Euc. b. VI. prop. 4. Thirdly, Because AB is to DE as BC to EF; therefore AB is likewise to DE as AB added to BC is to DE added to EF,

P

or

or as ABC is to DEF . Euc. b. V. prop. 12. Fourthly, Because AB is to DE as GB to HE , and as BC to EF ; therefore AB is likewise to DE as GB added to BC is to HE added to EF , that is as GC to HF . Euc. b.V. prop. 12. The other proportions are deduced from these. For fifthly, since by proportion the first, AB is to DE as BC is to EF ; therefore the root of AB is to the root of DE as the root of BC to the root of EF . Sixthly for the same reason, by proportion the second, the root of AB is to the root of DE as the root of GB to the root of HE . Seventhly, By proportion third, the root of AB is to the root of DE as the root of ABC to the root of DEF . And eighthly, by proportion the fourth, the root of AB is to the root of DE as the root of GC is to the root of HF .

Now as AB and DE are two planes equally inclined, the accelerating force of gravity will, by proposition 120, be the same upon both of them; and two heavy bodies will descend upon them, as if they were different parts of the same plane. Therefore by propositions 121. 107. The time of descent along AB is to the time of descent along DE as the root of AB to the root of DE . For the same reason, the time of descent of one heavy body along GC is to the time of descent of any other heavy body along HF , as the root of GC to the root of HF , that is, by proportion eighth, as the root of AB to the root of DE . And again for the same reason, the time of descent along GB is to the time of descent along HE as the root of GB to the root of HE , that is, by proportion sixth, as the root of AB to the root of DE . But, since the time of descent along the whole plane GC is to the time of descent along the whole plane HF as the root of AB to the root of DE ; and since the time of descent along the part GB is likewise to the time of descent along the part HE as the root of AB to the root of DE ; it follows that the time of descent along the remainder BC is to the time of descent along the remainder EF , as the root of AB to the root of DE . Euc. b.V. prop. 19. This is the case when one of the bodies has descended to B along GB , and the other to E along HE . But, by the last proposition, the velocity acquired at B and E and likewise the descent afterwards along BC and EF will be the same, whether the bodies descended first down GB and HE or down AB and DE . From hence it follows, since the time of descent down AB is to the time of descent down DE as the root of AB to the root of DE , and since afterwards the time of descent down BC is likewise to the time of descent down EF as the root of AB to the root of DE , therefore the time of descent down AB added to BC or ABC is to the time of descent down DE added to EF or DEF as the root of AB to the root of DE , Euc. b.V. prop. 12. But by proportion the seventh

as

as the root of AB to the root of DE so the root of ABC to the root of DEF. Therefore the time of descent along ABC is to the time of descent along DEF, as the root of ABC to the root of DEF.

129. *The time of descent of a heavy body along similar curves, which are in similar position, is in a subduplicate proportion, or as the square roots, of their lengths.*

Let AB and DE, Plat. IV. fig. 10. be two similar curves, suppose them for instance to be two similar arcs of two circles; and let them be in similar positions or equally inclined to the horizon. Now every curve may be considered as consisting of innumerable infinitely short right lines. Therefore AB and DE may be considered as consisting of innumerable infinitely short inclined planes. But because AB and DE are in similar position, all the planes in the whole curve AB and all those in the whole curve DE are similarly inclined. And because AB and DE are similar arcs therefore they consist of an equal number of planes, each of which planes are proportional to the respective arcs AB and DE, or to their respective wholes of which they are similar parts, and consequently all these parts are proportional amongst themselves. Since therefore the times of descent of heavy bodies along similar and proportional inclined planes are as the square roots of their length, by proposition 128, it follows, that the time of descent along BA is to the time of descent along ED, as the square root of BA to the square root of ED.

130. *When a heavy body has descended along one or more inclined planes; if its direction could be changed, without changing the velocity that it has acquired; this velocity would make it rise to the same height from whence it fell; and the time of ascent upon the same planes would be equal to the time of descent.*

When a body has fallen down the planes ABCD Plat. IV. fig. 8. if it could be turned back without destroying any part of the velocity, which it acquired in descending; this velocity, with which the body would begin to rise from D, would be sufficient to carry it up the planes DCBA to the same height from whence it fell. The force of gravity to accelerate a body in its descent and to retard it in its ascent is always the same, when the body descends and ascends upon the same plane. For the whole force of gravity in perpendicular descent and ascent is always the same: and the force of gravity upon a given inclined plane always bears a given proportion to the whole force, because, by proposition 120, it is always as the height of the plane is to its length. From hence

it follows that the same impulses of gravity, which accelerate the body in its descent along ABCD, will retard it again in its ascent along DCBA, when it is turned back again, and begins to rise with the last acquired velocity. Therefore this proposition may be proved in the same manner as proposition 112, for the velocity which was communicated by the impulses of gravity whilst the body descended, will not be lost in ascending but by an equal number of contrary impulses, that is, not till it has risen to the same height from whence it fell, and has ascended for the same time that it spent in descending.

Let AB Plat. IV. fig. 11. be a plate of brass, exactly polished and bent into the arc of a circle. This may be considered as innumerable inclined planes, that are contiguous to each other. And, by this proposition, a brass ball by rolling down them will have acquired such a velocity when it comes to B, as is sufficient to carry it up again to the same height from whence it fell. Therefore if just such another plate BC was to receive the ball when it came to B, the acquired velocity would make it rise to C, supposing A and C to be at the same perpendicular distance from B. Now when the ball was risen to C and had lost all its velocity, there being nothing to stop it, its own weight would make it fall back again to B, and in descending it would, as before, acquire such a velocity as would carry it up to A the place from whence it fell at first. And thus the ball would continue alternately descending and ascending along the curve ABC. For in descending either from A or from C to B it always acquires velocity enough to carry it up to the same height again on the opposite side.

The reader must observe, that we here suppose that the ball is not resisted by the air, in which it moves, and that none of its motion is lost by friction against the planes ABC. For unless these two causes are supposed away, they will constantly retard the ball, and so stop that motion at last, which would otherwise have been a perpetual one.

131, *The squares of the times, in which equal spaces are run through, are inversely as the accelerating forces.*

If two bodies descend upon two planes differently inclined; or if two bodies, one at the poles where the force of gravity is greater, the other at the equator, where it is less, descend perpendicularly, or upon planes equally inclined; or in any other case, where the accelerating forces are different, by which the bodies are put in motion; then the squares of the times, in which equal spaces are described by them, will be inversely as the accelerating forces. Call the space described S, the accelerating force A, and

and the square of the time T^2 . Now, from proposition 105 and the manner of proving it, we see, that where the accelerating force is given the space described is as the square of the time in which it is described; that is, where A is given S is as T^2 . And, since the velocities produced in a given time are as the cause which produces them, or as the accelerating forces; it follows, by proposition 10, that where the square of the time, or, which amounts to the same, where the time itself is given the space described is as the accelerating force, that is, where T^2 is given S is as A . Therefore universally where neither A nor T^2 are given, S will be in a ratio compounded of both, as A directly and as T^2 directly, or as $T^2 \times A = T^2 A$. But if $T^2 A$ is as S , then $T^2 A$ divided by A will be as S divided by A . That is, $\frac{T^2 A}{A} = T^2$ is as $\frac{S}{A}$. But because by the terms of the proposition we are speaking of equal spaces, therefore S is a given quantity: and all fractions, whose numerators are given, are inversely as their denominators: therefore $\frac{S}{A}$ is inversely as A . But T^2 is as $\frac{S}{A}$, therefore, where S is given, T^2 is inversely as A . That is, where the spaces described are equal, the squares of the times, in which they are described, are inversely as the accelerating forces.

C H A P. X.

Concerning pendulums.

132. *A pendulum is a heavy body hanging to a small cord or wire, which is moveable upon a center.*

133. *The oscillations of a pendulum are produced by the force of gravity.*

FOR if the ball D Plat. IV. fig. 12. is drawn up to A and let fall from thence, its own weight will make it descend, and the string CD will prevent it from going to any greater distance from C the center of motion; therefore it will descend in the arc of a circle AED keeping always at the same distance from C . But in descending to D the body acquires velocity, and with the acquired velocity, by proposition 130, it will rise on the opposite side to B the same height from whence it fell. In falling back from B to D it will, as before, acquire a velocity sufficient to make it rise to A : and thus it will alternately by descending acquire velocity and by ascending lose it, that is, it will continue to oscillate or vibrate backwards and forwards, till the resistance of the air and the friction of the string upon the center of motion stop it.

Thus

Thus the vibration of a pendulum is performed in the same manner and by the same laws, as the motion of a heavy body descending along the innumerable inclined planes AB, Plat. IV. fig. 11. and ascending along the innumerable inclined planes BC. And the same causes which prevent one of these motions from being a perpetual one, prevent the other from being so.

A pendulum may indeed be made to vibrate in a vacuum, and then the resistance of the air will be removed. But still there will be the friction of the string upon the center of motion. Or suppose the string was to be tyed fast to the center, so that there would be no friction; yet still there would be something equivalent to it. For then the pendulum could not vibrate without bending and unbending the string. And it will be easy to see that this will destroy the motion of the pendulum, by imagining the string to be a wire so stiff that the weight of the ball could not unbend it. For in this case if the ball was at A it must continue there, since the stiffness of the wire would prevent it from descending. Now if the string is less stiff it will be a less hindrance to its motion, but still let it be ever so flexible, it will be some hindrance every time it is to be bent or unbent: and this hindrance will in time destroy the motion of the pendulum as effectually as a friction upon the center.

134. *If a pendulum was to vibrate in the chords of a circle, instead of vibrating in the arcs of it; the time of one oscillation would be equal to the time, that a heavy body would take up to fall through 8 times the length of the pendulum.*

If a pendulum was to descend in the right line or chord AD Plat. IV. fig. 12. instead of descending in the arc AD, and was to rise in the right line or chord DB instead of the arc DB, the time of descent and ascent together or of one oscillation from A to B would be equal to the time, which a heavy body would spend in falling perpendicularly through 8 times CD or 8 times the length of the pendulum. For, whilst the pendulum is descending along AD a chord of a circle, a heavy body would fall perpendicularly through the diameter of the same circle, by proposition 124. And, since the time of the pendulum's ascent along DB is equal to the time of its descent along AD, by proposition 121. 130. therefore the time of descent and ascent together, or the time of an oscillation, is double the time of descent alone, that is, the time of descent is to the time of an oscillation as 1 to 2. Now in the time of the pendulum's descent a heavy body would fall perpendicularly through 1 diameter,

ameter, and the spaces described by a body falling perpendicularly, are to each other as the squares of the times, by proposition 105. Therefore the space described by a body falling perpendicularly in the time of descent alone is to the space described by it in the time of descent and ascent together, or in the time of an oscillation, as the square of 1 to the square of 2, that is as 1 to 4. But the body would fall perpendicularly through 1 diameter in the time of descent alone; and consequently in the time of an oscillation it must fall through 4 diameters of the circle in which AD and DB are two chords. Now the length of the pendulum CD is only a semidiameter of that circle, and 4 diameters = 8 semidiameters. Therefore, whilst the pendulum is performing an oscillation upon the chords AD and DB, a heavy body would fall perpendicularly through eight times the length of the pendulum.

135. *The vibrations of the same pendulum in unequal arcs are performed nearly and as to sense in the same time, provided the arcs are very small.*

The same pendulum, whether it vibrates in the arc ADB or in the lesser arc EDF, will perform its vibrations nearly and as to sense in the same time, provided those arcs are very small. By the last proposition, the time of vibration along the two chords AD and DB would be equal to the time taken up by a heavy body to fall perpendicularly through 8 times the length of the pendulum. And for the same reason the time of vibration along the two chords ED and DF would be equal to the time taken up by a heavy body to fall perpendicularly through 8 times the length of the pendulum. Therefore since we suppose it to be the same pendulum, which vibrates either in the chords AD and DB or in the chords ED and DF, 8 times the length and consequently the time of perpendicular descent of a heavy body through 8 times its length is a given quantity. From whence it follows that the time of vibration in the chords AD and DB is equal to the time of vibration along the chords ED and DF. Now very small arcs and their chords, for instance the arcs and chords ED and DF, differ but little either in length or declivity. Therefore vibrations either in chords or in very small arcs will be performed nearly in the same manner. And as the times of vibration in the unequal chords AD and DB, ED and DF have been proved to be equal, it follows that the times of vibration in the unequal arcs AD and DB, ED and DF, will be nearly and as to sense equal.

This is the reason why clocks with long pendulums should go more exact for any long course of time than those with short ones. For
where

where a pendulum vibrates in an arc of a given length, this arc will be so much less in proportion to the whole circle, as the circle is larger of which that arc is a part. But since circles are as their semidiameters, the circle will be larger as the pendulum, which is the semidiameter of it, is longer. From whence it follows that where pendulums vibrate in arcs of a given length, this arc is so much the less in proportion, as the pendulum is longer. Or where two pendulums of different lengths swing through an equal space the arc in which the longer pendulum vibrates, though equal in length to that in which the shorter vibrates, may be looked upon as a smaller arc, since it is less in proportion to the whole circle of which it is a part. Now when a clock is clean and the parts of it move more freely, upon that account the force, with which the swing-wheel acts upon the pallets of the pendulum, will be so much the greater; and when it is foul and the parts are clogged this force will be so much the less. For this reason the pendulum of a clock, when it is clean, being thrown out more forcibly will swing farther or vibrate in a longer arc, than when the clock is foul. And unless the arcs, when they are the longest, are very small in proportion to the whole circle of which they are parts, a pendulum, will take up more time to vibrate in a long arc than in a short one. But the longer the time is which a pendulum takes up to vibrate so much the slower the clock will move. Therefore unless a pendulum vibrates in very small arcs, the clock, if it goes right, when it is clean, will go too fast when it is foul. But, where other circumstances are equal, long pendulums vibrate in smaller arcs than short ones do. Therefore a clock with a long pendulum will go more exactly alike, when it is clean and when it is foul, than a clock with a short pendulum.

136. *The vibrations of the same pendulum in unequal arcs of a cycloid are performed in the same time.*

If the circle BS, Plat IV. fig. 13. is rolled along the right line BH like a wheel upon a plane surface, till it has gone once round, or till the point *b* which touched that line at B, when the circle began to move, is brought round and touches it again at H, so that the generating circle as it is called, is got to HT. The curve BGH which this point *b* has described in one revolution of the generating circle is called a cycloid.

To make a pendulum vibrate in a cycloid the string of it AF, Plat V. fig. 1. must be pliable, and it must swing between two plates ACB, and AED, which are semicycloids: by which means the string constantly applying itself to the plate towards which the ball moves, and bending itself into the form of the plate, will make the ball P, that is fastened to it, vibrate in the curve BFD.

For

For the use of those, who are not much acquainted with the mathematics, it will be necessary to mention two properties of the cycloid. First. If a tangent CF is drawn to any point C in the cycloid, Plate IV. fig. 13. and a right line CE is drawn from the same point perpendicular to GA the axis of the cycloid, and from D , where the right line intersects the generating circle, a chord DG is drawn in the generating circle to G the vertex of the cycloid. Then CF the tangent is parallel to DG the chord. Secondly. If from any points H and K the lines HM and KR in a cycloid ACD Plat V. fig. 2. are drawn perpendicular to EC its axis, and the chords GC and LC are drawn in the generating circle as before. Then the arc HC in the cycloid is double the chord GC , and the arc KC is likewise double the chord LC .

If GC and LC were two inclined planes, in the same time that one heavy body fell down GC any other heavy body would fall down LC . For the time of descent down either of them, though they are of different lengths, must be the same; because as they are chords of the same circle the time of descent down either of them is equal to the time of perpendicular descent down EC the diameter of that circle, by proposition 124. But when two unequal spaces are described in the same time, the velocities of the moving bodies must be as those spaces, the body which falls down GC must move as much faster than the body which falls down LC as GC is longer than LC , by proposition 10. And, since the cause is proportional to the effect, the forces which produce those velocities, that is, the accelerating forces of gravity must be as the velocities themselves; or the accelerating force of gravity on the plane GC is to the accelerating force of gravity on the plane LC as GC to LC . And since twice GC is to twice LC as GC to LC , these accelerating forces are to each other as twice GC to twice LC , that is, by the second property of the cycloid, as HC to KC . Thus we have proved that the force of gravity, that acts upon a body to make it descend along GC is as much greater than the force of gravity, that acts upon a body to make it descend along LC , either as GC is greater than LC or as HC is greater than KC . But since, by the first property of the cycloid a tangent at H would be parallel to GC and a tangent at K would be parallel to LC , therefore the point H and the plane GC , because they are parallel to each other, are equally inclined, and so likewise are the point K and the plane LC . But since H and GC , as likewise K and LC are planes equally inclined, the accelerating force which acts upon a heavy body at H will be the same as that which acts upon a heavy body at G : and likewise the accelerating force upon a heavy body at K will be the same as upon a heavy body at L . Therefore the force of gravity,

that acts upon a heavy body at H is to the force of gravity that acts upon another heavy body at K as HC to KC. The same might be proved in every point of one bodys descent from H to C down the arc HC of the cycloid, and likewise in every point of the other bodys descent from K to C, down the arc KC, that is, it might be proved in the same manner that the accelerating force is always as the bodys distance from C. But the velocities produced are as the causes which produce them, that is as the accelerating forces. Therefore the velocity of the body which descends along HC is to the velocity of the body which descends along KC as HC is to KC. And since their velocities are as the spaces to be described in descending, that is, since as much longer as HC is than KC, so much faster the body moves which is to describe HC, than the body which is to describe KC; it follows that one of these two bodies will describe HC and the other KC in the same time.

In the same manner it might be shewn, that the time of ascent will be the same either through $CB = CH$ or through $CF = CK$. For the force of gravity, which retards the body in its ascent will always be as the distance from the point C. Therefore the time of a vibration, or the time of ascent and descent together, through HCB is equal to the time of a vibration through KCH.

137. *The times in which pendulums perform their vibrations, where the accelerating force of gravity is given, are as the square roots of their lengths.*

Suppose two pendulums AB and CD, Plat.V. fig. 3. to vibrate in similar arcs of circles EF and GH, that is, in arcs, which bear the same proportion to each other that the circles do, of which they are parts. Now the circumferences of all circles are to each other as their diameters or as their semidiameters, and similar arcs are in the same proportion; therefore the arc EF is to the arc GH as AB the semidiameter of one circle or length of one pendulum, to CD the semidiameter of the other circle or length of the other pendulum. But if EF and GH are similar arcs their halves or EB and GD must be similar too, and they are likewise in similar position. Therefore, by proposition 129, the time of descent along EB is to the time of descent along GD as the square root of EB to the square root of GD, that is, since EB is to GD as AB to CD, as the square root of AB to the square root of CD. But the time of descent is half the time of a vibration. Therefore since the whole times of vibration must be in the same proportion with their halves, the time of a vibration in the arc EF is to the time of a vibration in the arc GH, in a subduplicate ratio of the lengths

lengths of the pendulums, that is, as the square root of AB to the square root of CD.

But, if AB vibrates in a cycloid, then, by the last proposition, the time of vibration will not be altered, whether the arc of vibration is longer or shorter than EF. So likewise if CD vibrates in a cycloid the time of vibration is not altered, whether the arc of vibration is longer or shorter than GH. Therefore in a cycloid the times of vibration, and consequently the proportion which the times of vibration bear to each other, will be the same, though EF and GH by lengthening or by shortening either of them are so altered as not to be similar. From whence it follows, that when pendulums vibrate in cycloids, whether the arcs of their vibration are similar or not, the times of vibration will be as the square roots of their lengths. So that if the lengths of the pendulums are as 16 to 9, the times of vibration will be as the square roots of those lengths or as 4 to 3.

From hence we may find how long a branch is which hangs down from the roof of a church, and consequently, by measuring from the ball of the branch to the floor and adding this to the length of the branch, we may find how high the church is. For suppose the branch, when you swing it, to vibrate once in 3 seconds, then, since the time of vibration is as the square root of a pendulums length, it follows, that the lengths of pendulums are as the squares of their times of vibration, and that the length of the branch is to the length of a pendulum, which performs a vibration in 1 second, as the square of 3 to the square of 1, or, as 9 to 1. Now the length of a pendulum which vibrates seconds, being 39 inches and 2 tenths, this multiplied by 9 will give 352 inches and 8 tenths for the length of the branch. And so universally, if the length of a pendulum, which vibrates seconds, is multiplied by the square of the number of seconds contained in the time of one vibration of the branch, the product will be the length of the branch.

138. *The times, in which a pendulum of a given length performs its vibrations, are inversely as the square roots of the accelerating forces.*

If the pendulum vibrates in the cycloid ACD. Plat. V. fig. 2. the time of a vibration either in the arc HCB or KCF, or any other arc, will be inversely as the square root of the accelerating force of gravity, which produces the motion of the pendulum. The reader will remember, that hanging a different weight to the pendulum does not alter the accelerating force. For since, by proposition 26, bodies of different weights fall with the same velocity, the accelerating force, which is

proportional to the velocity produced in a given time, is no greater when the ball of a pendulum weighs 10 pounds than when it weighs only 1 pound. The moving force of gravity is indeed different, for this is as the quantity of motion produced in a given time, that is, since the velocity is given, as the quantity of matter : and consequently the moving force of gravity in one ball would be 10 times what it is in the other, though the accelerating force is the same. The accelerating force can then only be greater or less, when heavy bodies in the same circumstances are made by the force of gravity to descend with a greater or a less velocity. Thus, as will be proved in its proper place, a heavy body acquires a greater velocity in a given time, when it falls perpendicularly at either of the poles than when it falls perpendicularly at the equator. Therefore the accelerating force of gravity at the poles is greater than at the equator. And the times of vibration of the same pendulum at the poles and at the equator will be inversely as the square roots of those accelerating forces. The square of the time of descent along the given space HC, Plat.V. fig. 2. is inversely as the accelerating force, by proposition 131; and consequently the time of descent is inversely as the square root of the accelerating force. But the time of an oscillation in the arc HB is double the time of descent along HC. Therefore the time of an oscillation in the arc HB is likewise inversely as the square root of the accelerating force. And since the time of vibration of the same pendulum in the arc KF or in any other arc of a cycloid is equal to the time of vibration in the arc HB, it follows that the time of vibration of the same pendulum in any arc of a cycloid is inversely as the square root of the accelerating force.

139. *If the lengths of two pendulums are to each other as the accelerating forces, which act upon them; they will perform their vibrations in the same time.*

Call the length of one pendulum L and the length of the other l. Call the accelerating force, which acts upon the former A, and the accelerating force which acts upon the latter a. Then by the supposition L is to l as A is to a. Now, by proposition 137, 138, the times of vibration are as the square roots of the lengths directly, and as the square roots of the accelerating forces inversely. But L divided by A, is to l divided by a, or $\frac{L}{A}$ is to $\frac{l}{a}$, as L to l directly and as A to a inversely. Therefore the times of vibration are as the square root of $\frac{L}{A}$ to the square root of $\frac{l}{a}$. And, since

since L is to l , as A is to a , the first quantity, or L , divided by the third, or by A , is equal to the second, or l , divided by the fourth, or by a , that is, $\frac{L}{A}$ is equal to $\frac{l}{a}$: therefore the square roots of $\frac{L}{A}$ and $\frac{l}{a}$ are equal; and consequently the times of vibration are equal.

140. *The center of oscillation is that point of a pendulum on each side of which the quantities of motion are equal, or in which all the gravity of the pendulum might be collected without altering the time of its vibrations.*

It is evident that this center of oscillation must be different from the center of gravity. For if we imagine a plane perpendicular to the string CD , Plat. IV. fig. 12. to pass through the center of the ball D and to divide it into two equal parts: then, as the ball oscillates upon the center of motion C , the lower half of it being farther from C must in the time of one vibration describe a greater arc than the upper half. Therefore the velocity of the lower half, and consequently the quantity of motion, must be greater than that of the upper half. From whence it follows that the quantities of motion on each side the center of gravity being unequal, this cannot be the center of oscillation. But the center of oscillation must be farther from C than the center of gravity is; so that a plane passing through it may divide the ball D into two unequal parts in such a manner as to make the greater quantity of matter above it compensate for the greater velocity below it, that the quantity of motion on each side may be the same.

141. *The center of oscillation is likewise the center of percussion.*

Any weapon will strike the greatest blow, when the stroke is given with that part of it, where the whole motion acts: now this is the part on each side of which the quantities of motion are equal, or the common center of oscillation of the weapon and our arm. For if an obstacle stops this center, the whole motion is destroyed at once, and consequently the whole force of that motion is spent upon the obstacle. That end of a cane, which is farthest from our shoulder, will not strike the greatest blow, because, it doth not strike with the accumulated weight of the cane and our arm, since this weight all acts in their common center of gravity. Neither will this common center of gravity strike the greatest blow, because there are other parts which move with a greater velocity. But it will be the center of oscillation, which is a point between

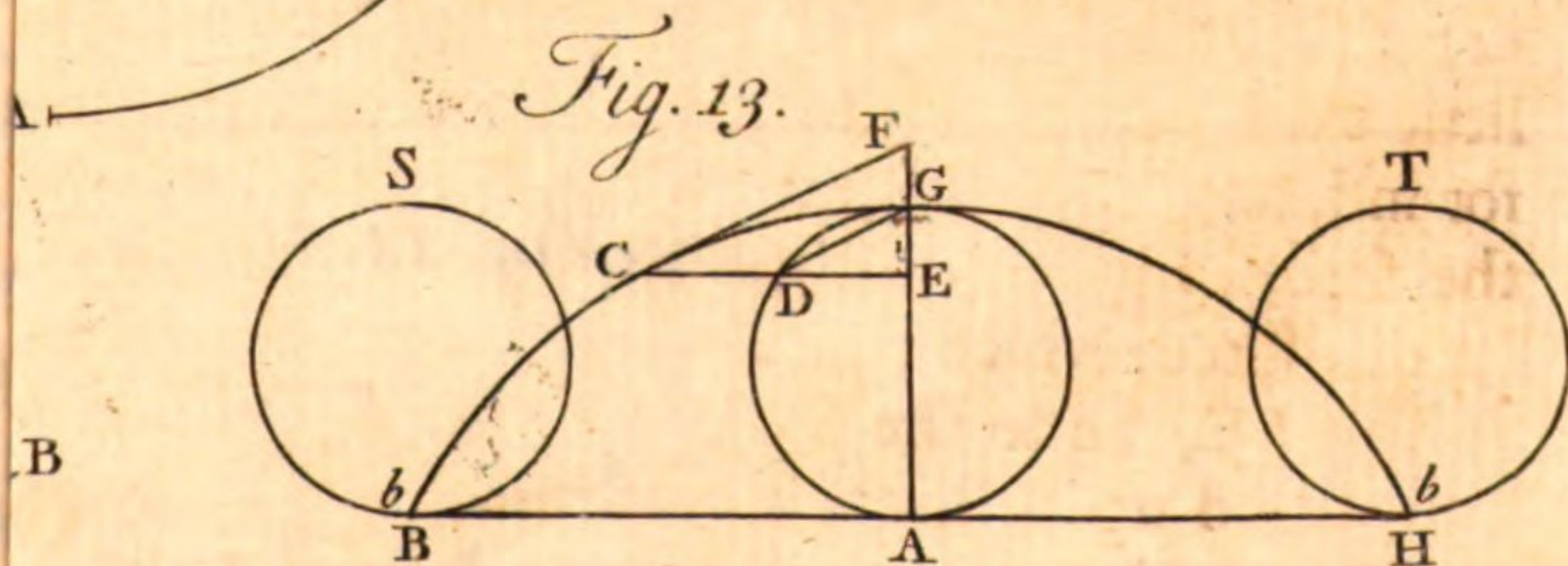
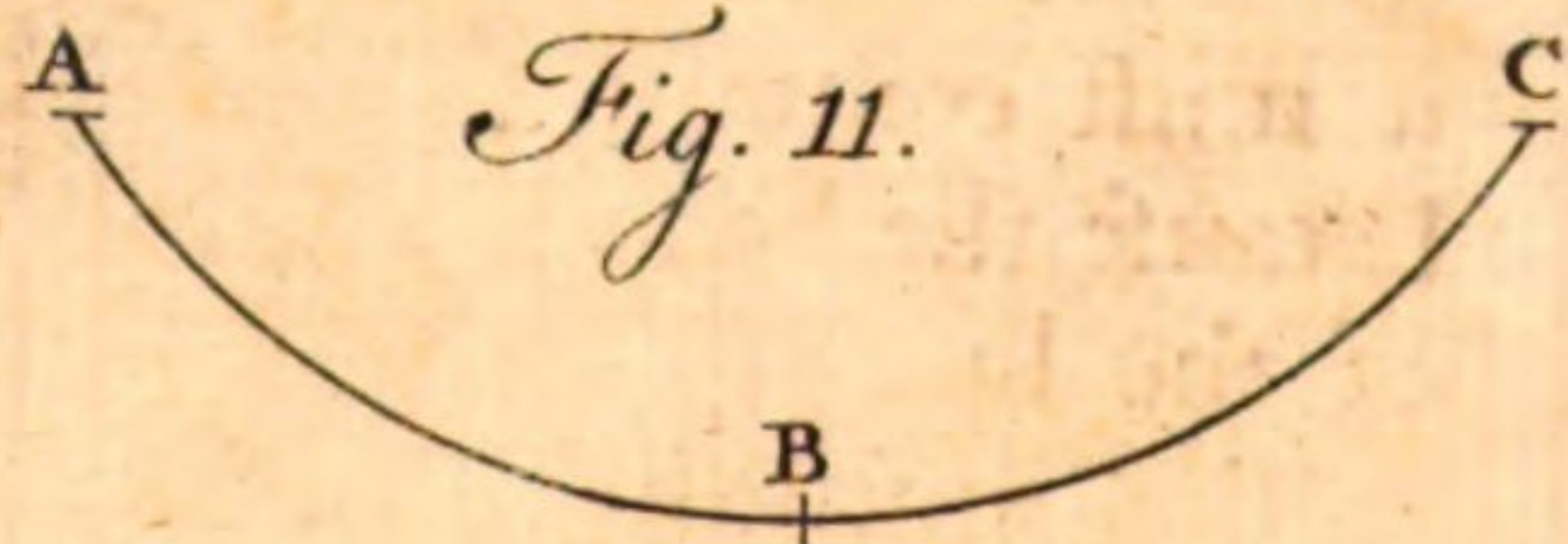
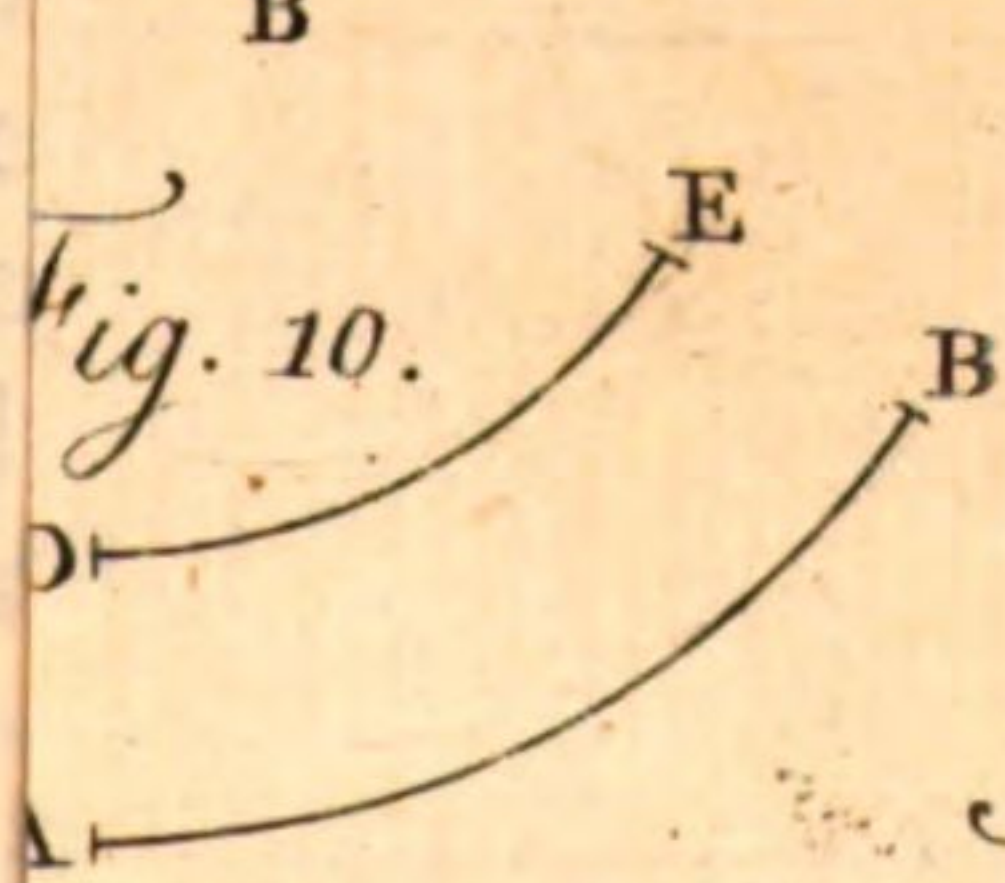
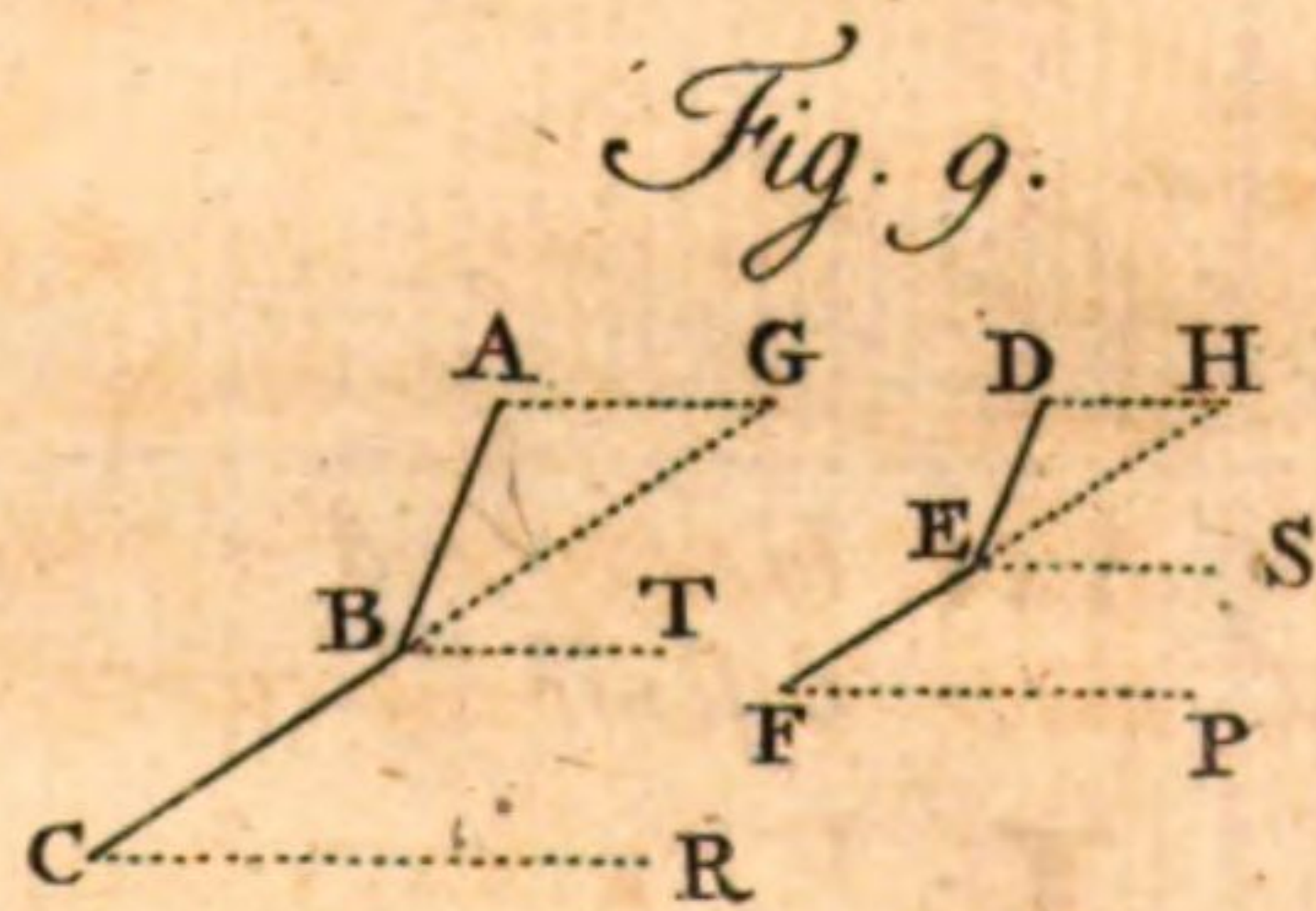
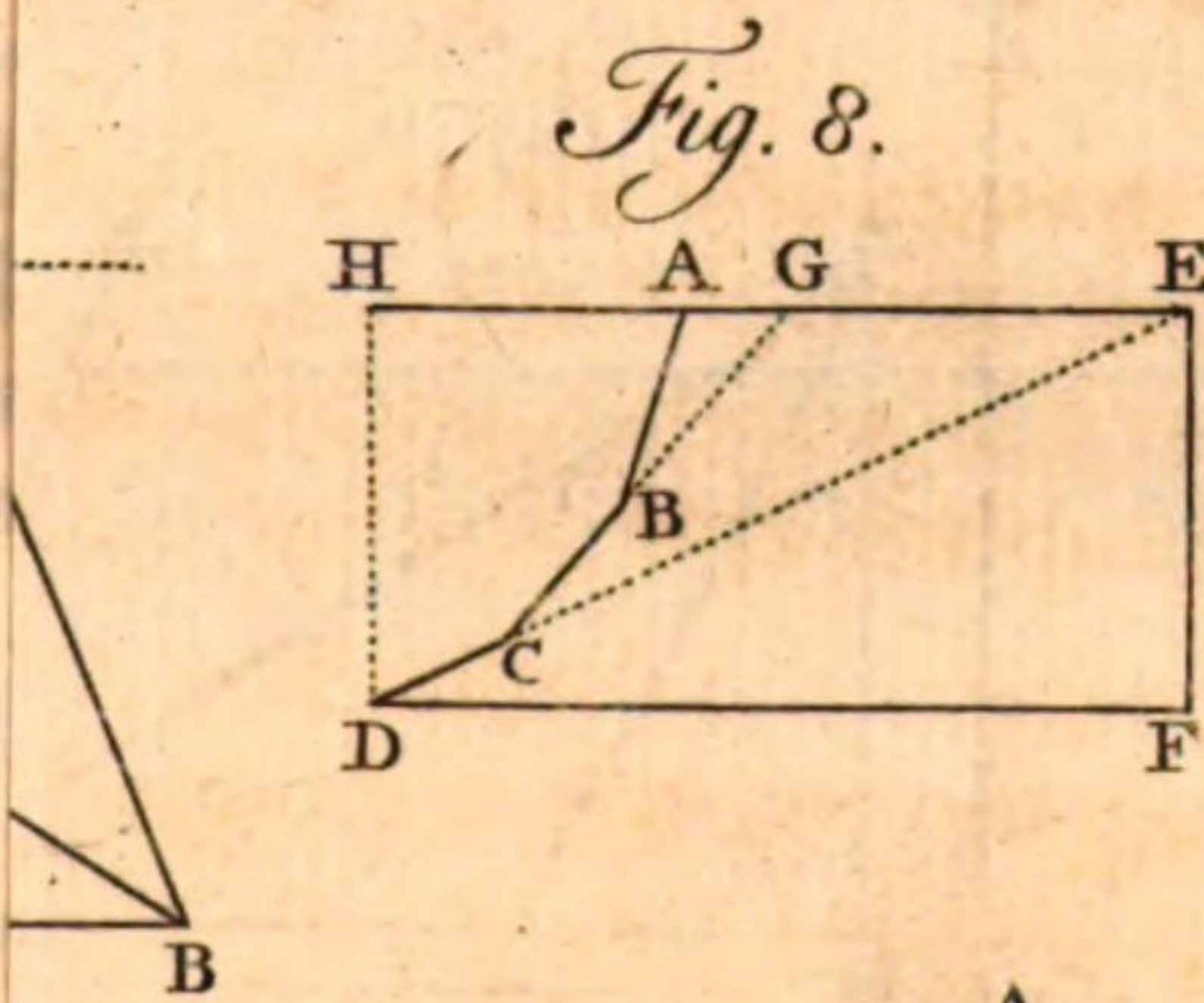
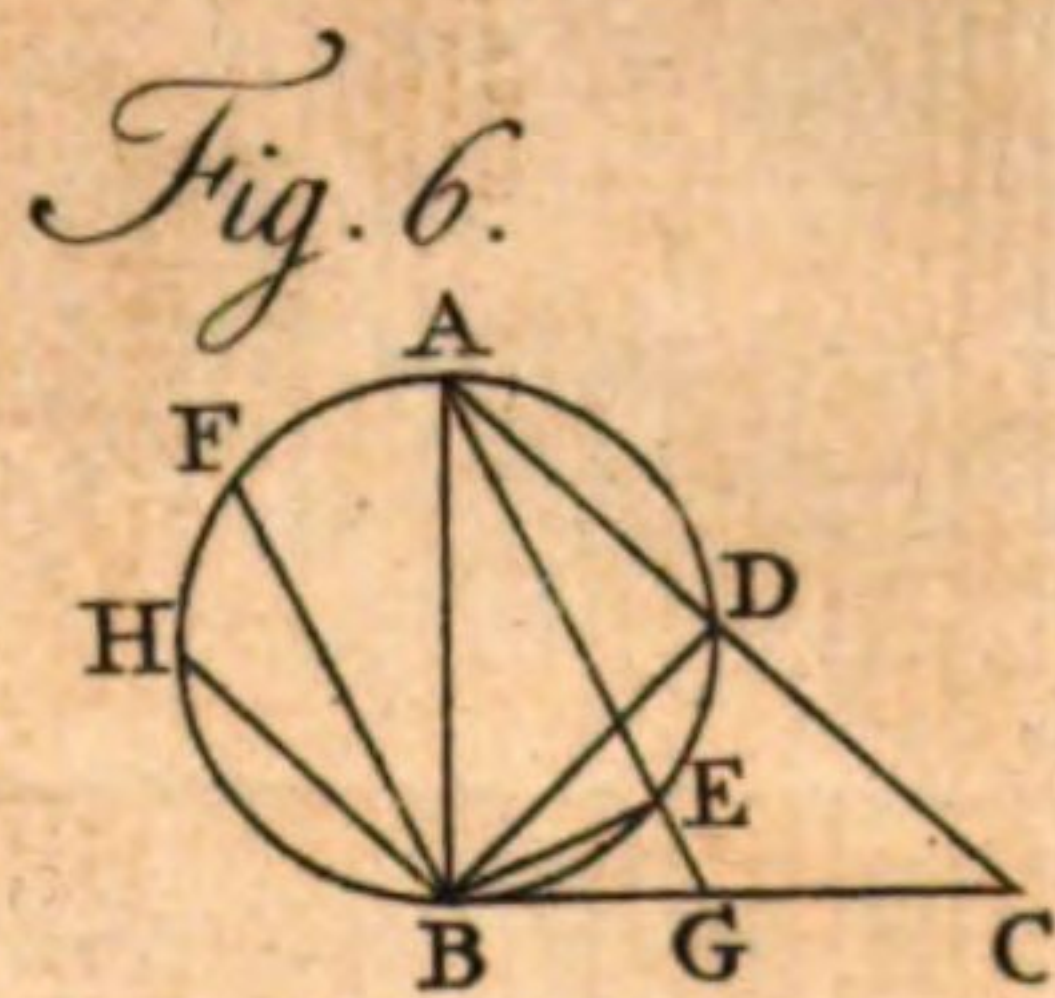
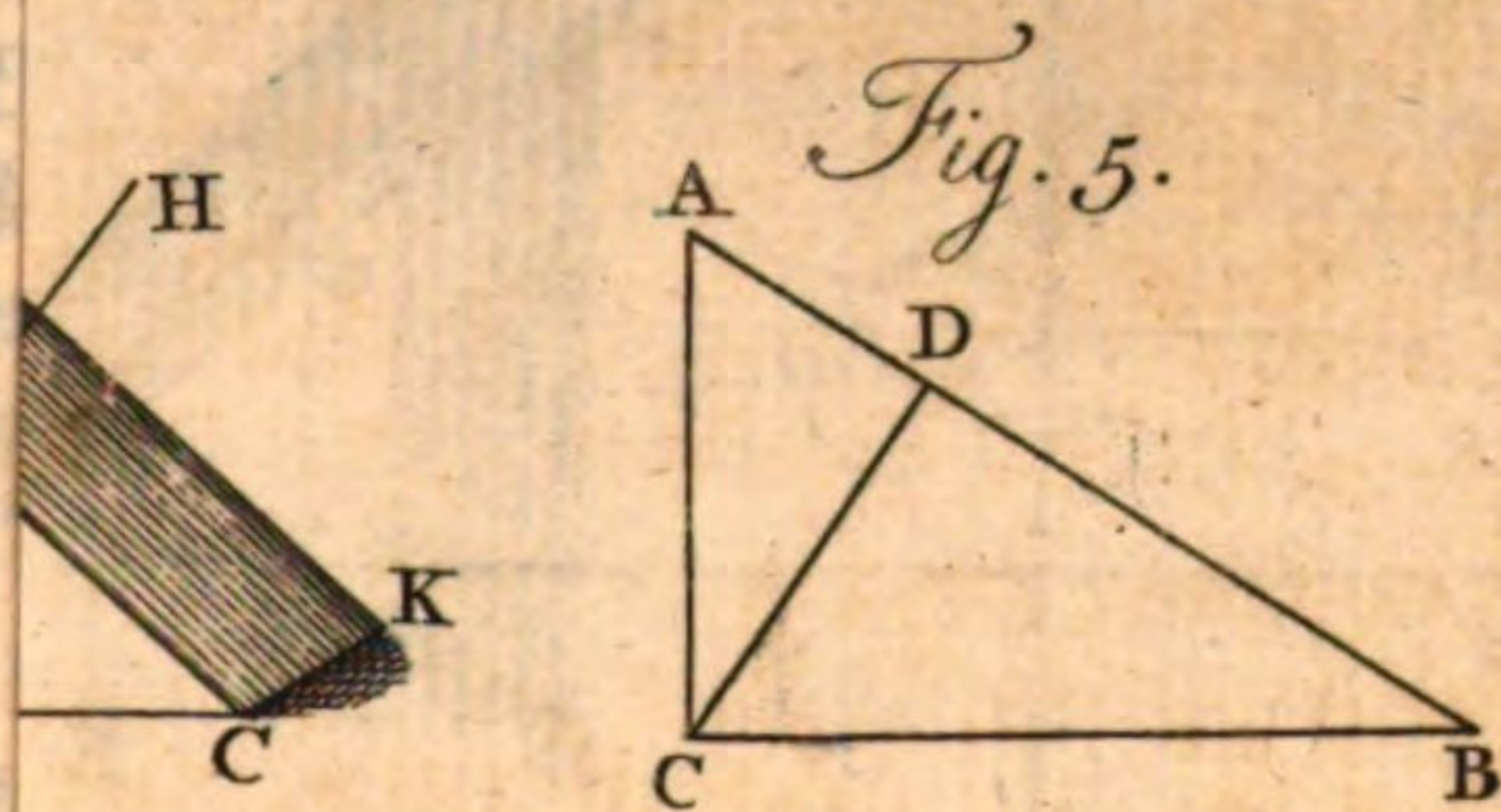
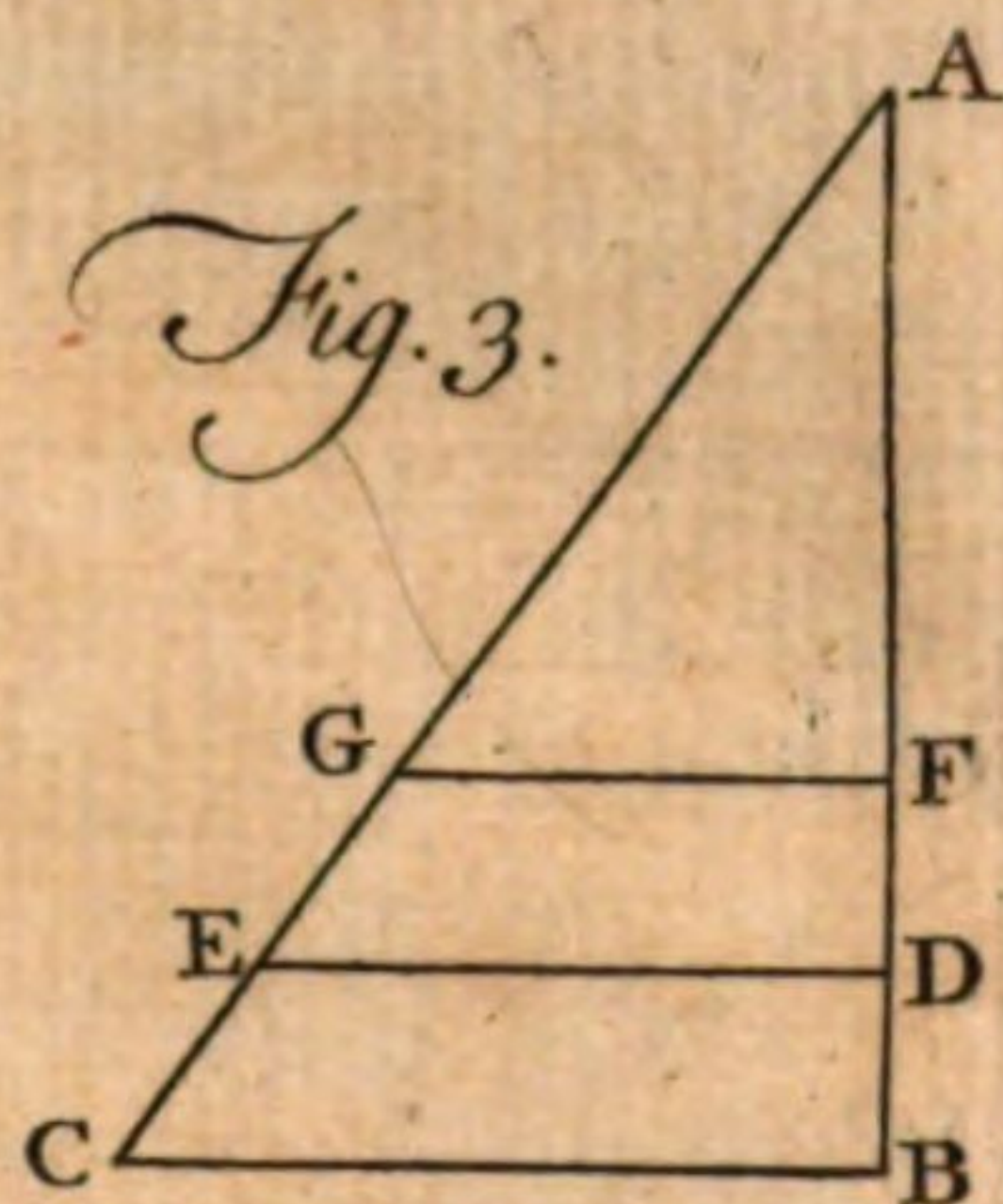
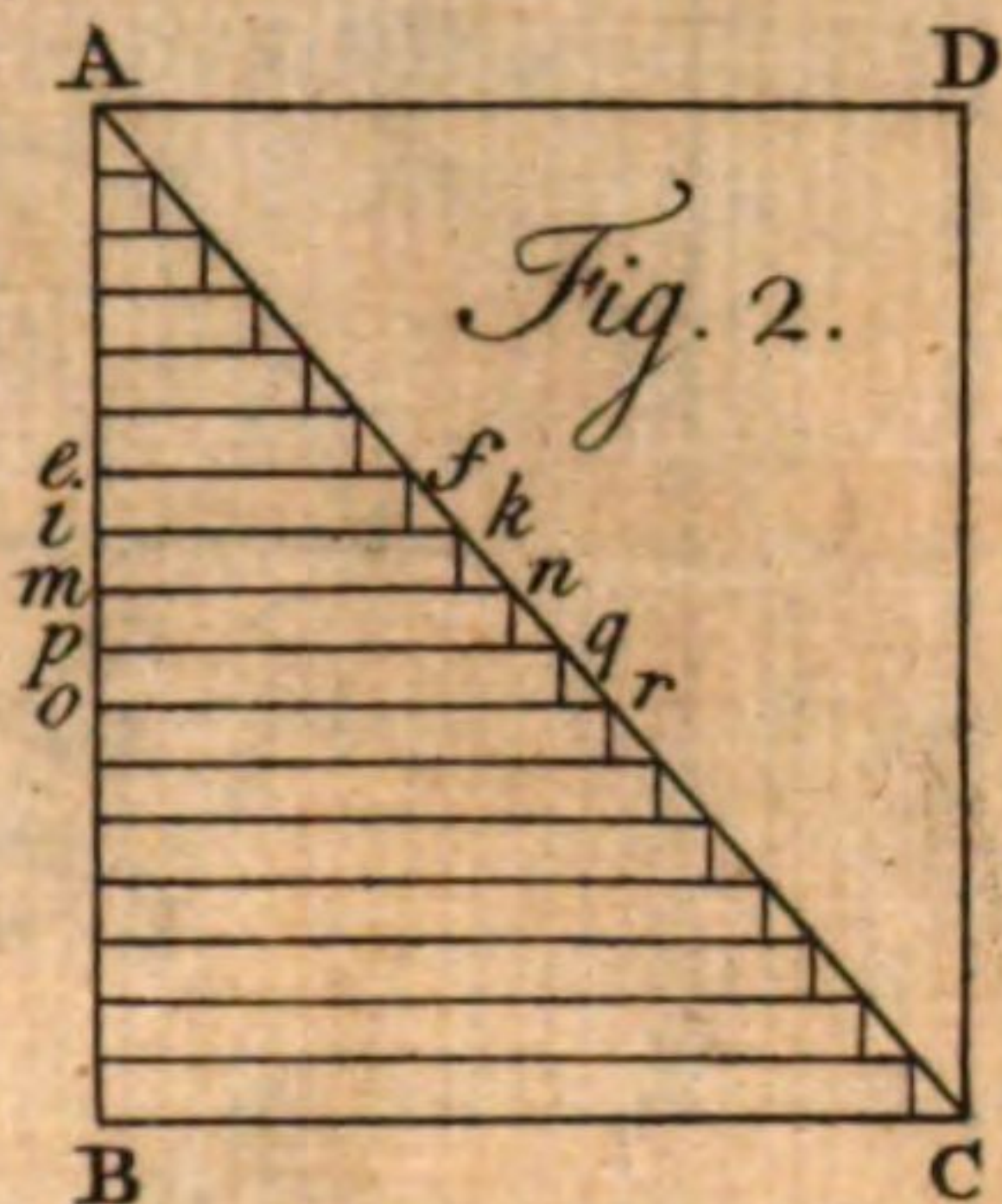
tween both so situated that the greater quantity of matter on one side of it makes amends for the greater velocity on the other side. If we strike an obstacle with any part of the cane, which is farther from our shoulder than the center of oscillation, as if we strike with the end of it; then the whole motion is not spent in the blow: but the part of the cane between us and the obstacle will continue to move after the end is stopped by the obstacle; therefore the cane will be bent and may be broken. In like manner, if we strike with any part of the cane, which is nearer to us than the center of oscillation, so that this center is beyond the obstacle, then the whole motion is not spent in the blow, but the part of the cane beyond the obstacle will continue to move, after the part we struck with is stopped by the obstacle, and in this case too the cane will be bent and will be in danger of breaking. But if we strike with the center of oscillation, there will be no danger of breaking the cane, since the whole motion of it is spent in the blow upon the obstacle, and is stopped by it at once.

CHAP. XI.

Of the motion of projectiles.

142. *The random of a projectile is the horizontal distance to which a heavy body is thrown.*
 143. *Heavy bodies, when they are thrown either obliquely or horizontally, describe a curve.*

LET a heavy body be thrown in the direction AV, Plat. V. fig. 4. then, by proposition 11, or the first law of motion, if no other force acted upon the body but the projectile force, or force with which it is thrown, it must continue to move on uniformly in the right line ABCV. But because the body is a heavy one, the force of gravity will draw it out of the line AV or will make it constantly fall from the line of its direction towards the earths center. Suppose that the force of gravity was not a continued one, but that it acted at certain intervals, so that some little time should pass between one of its impulses and the next. Thus for instance, suppose that, when the body was thrown in the line AV, the force of gravity did not act upon it, till it was come to B; and then let this force, which acts always towards the earths center, or in the direction BE, be to the projectile force in the line AV as BE to BC. The body acted upon by these two forces would, by proposition 16, describe the diagonal BD of a parallelogram whose two sides are BE and BC. Suppose



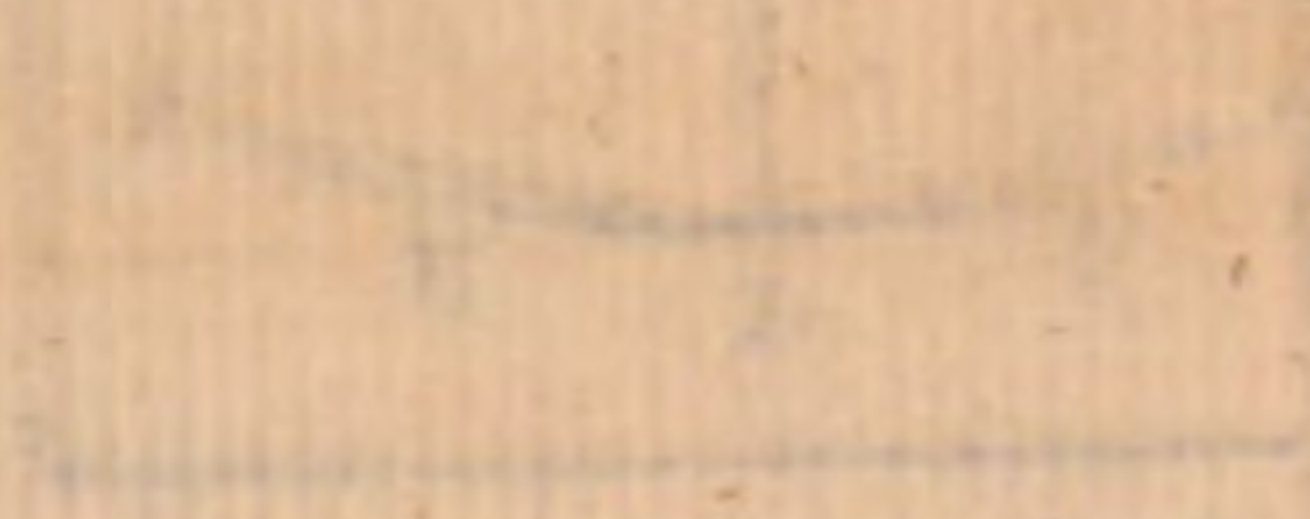
ever on one side of
 side. If we strike
 it from our shoul-
 der end of it, then
 the cane be-
 comes bent is stop-
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 But which is never
 is beyond the ob-
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to which a draw
 is made, or horizontally,

Plat. V. Fig. 4 then,
 no other force acted
 which it is drawn,
 it line ABC V. But
 it will draw it out
 in the line of an di-
 rect force of gravity was
 applied, so that force
 and the next. Then
 point at the line AV,
 point to B, and then
 point, or in the di-
 rect to BP to BC. The
 point is, or the
 point is, or the

Fig. 12



pose the interval of time between this first impulse of gravity and the next to be long enough for the body to pass from B to D. Now if no other impulse besides the first was to act upon it at all, then, by the first law of motion or proposition 11, it would continue to move uniformly in the line BDF. But when the body comes to D there is a second impulse of gravity supposed to act upon it in a direction parallel to FG, and let the force of this impulse be to the projectile force as FG to DF, then making these two lines the two sides of a triangle and drawing DG the third side, it is evident that this triangle is half a parallelogram of which DF the projectile force and FG equal to the force of gravity are two sides and DG is the diagonal. Therefore the body acted upon by two forces in the directions DF and FG the two sides of the triangle will describe the third side DG, by proposition 16. The body, when once moving in this direction, would, if left to itself, go on uniformly in the right line DGH. But a third impulse of gravity towards the earths center, or in a direction parallel to HI, being supposed to act upon it, when it comes to G, if this impulse is to the projectile force as HI to GH, then, as before, making GH and HI two sides of a triangle, the body acted upon by two forces as GH and HI will describe the third side GI. And if, after another interval of time sufficient for the body to move as far as I in the new-acquired direction GL, a fourth impulse of gravity was to act upon it, the projectile force IL and an impulse of gravity acting at I in a direction parallel to LK would make it describe IK. Thus by the projectile force, or force with which it is thrown, and the force of gravity together, supposing an interval of time to pass between the impulses of gravity, the body is made to describe the polygon ABDGIK. Now the length of the sides of this polygon as AB, BD, DG, GI, and IK, depends upon the interval of time between any one impulse of gravity and the next. For if the action of gravity is a continued one, that is, if the intervals between any one impulse and the next are infinitely short, the body in such an interval or instant of time could only move to an infinitely small distance. And since the length of these sides is the space described by the body between any two impulses, it follows that if these impulses were continued ones, or the intervals between them were infinitely short, these sides must be infinitely short too. Since likewise in the time, which the body takes up to move from A to K, there must be innumerable such instants, therefore when the sides of the polygon are infinitely diminished in length, they will be infinitely increased in number. But the impulses of gravity are continued, and do not act at any interval of time from each other: therefore the body will describe a polygon
of

of innumerable and infinitely short sides; and such a polygon is a curve; such an one as is represented Plat. V. fig. 7. Therefore a heavy body, which is thrown obliquely, is by the projectile force and force of gravity together made to describe a curve.

We need not be particular in proving this proposition when the body is thrown horizontally. For it is only supposing the body to be thrown originally in the direction DGH Plat. V. fig. 4. and then the proof will be the same as that already made use of in tracing out the line DGIK, which a heavy body describes in the latter part of its motion, when it is thrown obliquely.

144. *A heavy body, which is thrown either horizontally or obliquely, describes a parabola.*

Suppose the body to be thrown in the direction AF Plat. V. fig. 5. The projectile velocity is uniform, by the first law of motion, for all bodies acted upon by only one force move uniformly, and consequently this velocity would carry the body in the line AF through equal spaces in equal times: so that if it described AB in one second of time, it would describe BC in the next, CD in the third, DE in the fourth, and EF in the fifth. Upon supposition that all these lines are respectively equal to each other. Or if the body moved from A to B, or through a space, which we will call 1, in 1 second of time, and at the end of that second was at B, then in 2 seconds it would move twice as far from A or through the space $AC = 2$ and at the end of this next second would be at C. In the same manner in 3 seconds it would have moved from A through the space $AD = 3$. In 4 seconds through the space $AE = 4$. And in 5 seconds through the space $AF = 5$. Thus the spaces described by the force of projection in the times 1, 2, 3, 4, 5, are to each other as 1, 2, 3, 4, 5, or as the times, in which they are described. But the body is not carried on by the projectile force alone, but will, because it is heavy, fall towards the earth's center. And the motion produced by the force of gravity is uniformly accelerated, by proposition 102. So that if we compute the spaces through which the body falls from the line AF in each of the 5 seconds of time, these spaces are to each other as the uneven numbers 1, 3, 5, 7, 9, by proposition 108. Thus if the body falls from A to G or through 1 pole in the first second, it will fall through GH or 3 poles in the next, through HI or 5 in the third, through IK or 7 in the fourth, and through KL or 9 in the fifth, by proposition 108. Since therefore if we reckon from the beginning of the fall the body falls from A to G or 1 pole in 1 second, from A to H or 4 poles in 2 seconds, from A to

to

to 1 or 9 poles in 3 seconds, from A to K or 16 poles in 4 seconds, and from A to L or 25 poles in 5 seconds; it follows that in the unequal times 1, 2, 3, 4, 5, the spaces described are as the squares of the times, that is as 1, 4, 9, 16, 25. Let us next trace out the line, which the two forces of projection and gravity together will make the body describe. By the force of projection it is carried in the first second from A to B and by the force of gravity it falls through AG or 1 pole, and by both forces together it describes Ab the diagonal of a parallelogram, of which these two forces are the sides. It cannot in the next second be carried in the line of projection from B to C, because at the beginning of the second it is at b and not at B: but however the force of projection acts upon it, when it is at b , just as it would have done if it had been at B, and therefore will make it describe in that second bh parallel and equal to BC. But the force of gravity in this second will make it fall through 3 poles or through $bb = GH$: and consequently both forces together will make it describe bc . In the same manner the projectile force will carry it in the third second through $cd = CD$ and the force of gravity will make it fall through $ci = HI$ or 5 poles; and both forces together will make it describe cd . In the fourth second it will describe $de = DE$ by one force, and $dx = IK$ or 7 poles by the other, that is de by both together. And in the fifth second it will describe $ef = EF$ by one, and el by the other, that is ef by both together. Thus by the projectile force and force of gravity together the body will describe the curve $Abcdef$. Now the reader will remember that the unequal spaces AB, AC, AD, AE and AF, described by the projectile force are as the times in which the body describes them: and that the spaces Bb , Cc , Dd , Ee , and Ff , the spaces through which the body is made to fall by the force of gravity, are as the squares of the times in which the body describes them. But since the spaces described in the line AF are as the times, and the spaces through which the body falls from AF to the curve $Abcdef$ are as the squares of the times, it follows that these latter spaces are to each other as the squares of the former, or that Bb , Cc , Dd , Ee , and Ff , are to each other as AB^2 , AC^2 , AD^2 , AE^2 , and AF^2 , or taking AB, AC, AD, AE and AF, in the proportion of 1, 2, 3, 4, 5, Bb , Cc , Dd , Ee , and Ff will be in the proportion of 1, 4, 9, 16, 25. If the reader understands this he will not find any great difficulty in the following demonstrations.

Suppose a heavy body to be thrown horizontally, or in the direction AB, Plat. V. fig. 6. then, by what has been proved already, taking AC, AD, AE, AF, AB, in the proportion of 1, 2, 3, 4, 5, these will be the spaces described in the line AB by the projectile force alone, in the times 1, 2, 3, 4, 5. And taking CH, DI, EK, FL, BP, in the proportion of 1, 4, 9, 16, 25, these will be the spaces through which the body would

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fall

fall by the force of gravity alone in the times 1, 2, 3, 4, 5. But when both forces act together the body describes a curve, by proposition 143.

Let AB be a tangent to A the highest point or vertex of the curve: from A draw AN perpendicular to this tangent; and draw HG, IM, KN, parallel to the tangent. AN is called the axis of the curve, which the projectile describes. HG, IM, and KN are called ordinates to the axis. And AG, AM, AN, the parts intercepted between the vertex and each of these ordinates, are called abscissæ in the axis. Now, since AC is equal to GH, and MI is equal to AD, and NK is equal to AE; because they are opposite sides of a parallelogram, Euc. b. I. prop. 34. since likewise, for the same reason, AG is equal to CH, AM equal to DI, and AN equal to EK, it follows, that if HG, IM, and NK, &c. are taken in the proportion of 1, 2, 3, &c. then AG, AM, AN, &c. will be in the proportion of 1, 4, 9, &c. that is, the abscissæ in the axis AG, AM, AN, &c. are to each other as the squares of the ordinates GH, MI, NK, &c. But the mathematicians prove this to be a property of no curve but the parabola. Therefore the curve which a heavy body describes, when it is thrown horizontally, is a parabola.

If the body is thrown obliquely, as in the line AB Plat. V. fig. 7. then let AB be a tangent to the curve at the point A from whence the body is thrown, draw AN parallel to the axis and HG, IM, KN, parallel to the tangent AB. AN is called a diameter of the curve, HG, IM, and KN are ordinates to the diameter, and AG, AM, AN are abscissæ in the diameter. Now if AC, AD, AE, AF, and AB, are to each other as 1, 2, 3, 4, 5, or if the spaces described in unequal times by the projectile force are as the times, then from what has been said already it appears, that CH, DI, EK, FL, and BP, or the spaces through which the body falls by the force of gravity are as the squares of the times, or as 1, 4, 9, 16, 25. And since, as before, AC = GH, AD = MI, and AE = NK; since likewise AG = CH, AM = DI, and AN = EK, it follows, that if GH, MI, and NK are to each other in the proportion of 1, 2, 3, then AG, AM, and AN, will be as the squares of GH, MI, and NK, or as the squares of 1, 2, 3, that is as 1, 4, 9. Therefore in the curve, which a heavy body describes, when it is thrown obliquely, the abscissæ in the diameter are as the squares of the ordinates. But this property belongs only to the parabola. And consequently a parabola is the curve, which a body projected obliquely will describe.

145. *Though a heavy body, when it is thrown perpendicularly upwards, appears to ascend and descend in a right line; yet, supposing the earth to turn round its axis, the body does really describe a parabola.*

If a stone upon the earths surface would move in a certain time, as in 5 seconds, from f , Plat.V. fig. 5. to L , this motion, which the stone has in common with the earth, is not destroyed by throwing it in the direction fF perpendicular to the earths surface. But after the projection the stone will be moved by two forces, one in the direction fL , the other in the direction fF ; and by both of them together will begin to rise in the diagonal of a parallelogram whose sides are fF and fL . So that though the projectile force acts perpendicularly to the earths surface, yet this compounded with the force that carries the stone along with the earth produces a motion in it, as it rises, in the direction fe , which is oblique to the earths center. And thus the case of a stone, though thrown perpendicularly, is reduced to the case of an oblique projection as explained in the last proposition, and consequently the stone must describe a parabola; which it will do by rising in the half parabola fA and by falling again in the other half of the same parabola.

To be more particular; suppose that the earths surface and stone along with it, if the stone was not thrown upwards but rested on the surface, would move from f to P in one second, from P to O in another second, from O to N in a third, from N to M in a fourth, and from M to L in a fifth. But suppose, when the stone was at f , it was to be thrown upwards with such a force as would carry it in 5 seconds to its greatest height fF ; then the spaces through which it would ascend in each of the five seconds taken seperately, would be to each other as the five first uneven numbers taken backwards, or as 9, 7, 5, 3, 1. by proposition 115. That is, it would ascend through 9 poles in the first second, through 7 in the next, through 5 in the third, through 3 in the fourth, and through 1 in the fifth. Now to trace out the line which the stone will describe, we must compound these two motions together. For the motion, which the stone had in common with the earth, is not destroyed by throwing it perpendicularly upwards, but continues the same, whilst the stone is in the air, that it was, whilst it laid upon the earths surface. By one of these motions it is to be carried in a second from f to P and by the other it is to ascend through $f\phi$ supposing $f\phi$ equal to 9 poles, and therefore by both together it will describe fe the diagonal of a parallelogram, whose sides are fP and $f\phi$. In the next second the motion, which the stone has in common with the earth, will carry it through a space equal to PO , but the stone at the beginning of this second is not at P but at e , therefore it will be carried through $ex = PO$: and since it will in the mean time by the projectile force ascend through ee equal to 7 poles, by both forces together it will describe ed , by proposition 16. In the third second it will be carried in the same direction as the earth from d to i or through a space equal to ON , and will rise from d to δ or 5 poles; therefore by both

motions compounded together it will describe dc . In the fourth second, the motion common to the stone and earth will carry it through $cb = NM$, and the projectile force will make it ascend through ck or 3 poles, therefore by the compound motion it will describe cb . In the fifth second it moves as if it was still on the earths surface through $bG = ML$ and ascends 1 pole, by both which motions together it will be carryed in the line bA , and will then be arrived at its greatest height. After this it will descend again but will in the mean time still be carryed on by the motion common to itself and the earth: and therefore for the same reason that it ascended in the semi-parabola $fedcbA$ it will descend in the other half of the same parabola. But to any spectator on the earths surface, as particularly to the person who throws the stone, it will appear to ascend and descend in a right line. For the spectator is carryed along with the earth just as fast as the stone is and in the same direction: so that when the stone was at e the spectator would be at P , when the stone was at d he would be at O ; and in every point either of the stones ascent or its descent, it would be directly over his head; and it will fall at last on the point f from whence it was thrown, because this point has moved as fast as the stone has and in the same direction. The spectator in the mean time considering his own place as at rest, does not attend to the motion, which is common to himself and the stone, but seeing the stone always over his head and attending only to the ascent and descent of it, it appears to him to have risen and fallen in a right line.

If fF is the mast of a ship, and an arrow is shot directly up the side of it; then supposing the ship to move uniformly in a right line fL and to describe this line, whilst the arrow is ascending to the top of the mast; if we divide the line fL into 5 equal parts, when a fifth part of the whole time is over, the mast will be at PE , and the arrow, as appears from what has been said already, will, from its motion upwards and the motion which it has in common with the ship, be at e ; and by the same sort of reasoning as was used above, when the mast is come to OD the arrow will be at d , and so on till when the mast is at LA the arrow will be at A ; and thus in the course of its motion it will have described the line $fedcbA$. But it has all along ascended up the side of the mast: and, will for the same reason, as the ship goes on, descend again down the side of it, and will alight at the foot of it: and consequently to those, who are in the ship and do not attend to the motion of the mast, the arrow will appear to have described a line as strait as the side of the mast is.

146. *When a heavy body is thrown obliquely with a given velocity; if the space, through which it must have fallen perpendicularly to acquire that velocity, is made the diameter of a circle, the height to which the body will rise is equal to the versed sine of double the angle of elevation.*

Suppose a heavy body to be thrown, or a bullet to be shot from a gun, in the direction BE with a certain velocity; and suppose this velocity to be the same that this or any other heavy body would acquire by falling perpendicularly down AB, Plat.V. fig. 8. Now as the bullet moves in the direction BE it partly ascends and is partly carried forwards in an horizontal direction: and if AB is made the diameter of a circle the greatest height to which the bullet will ascend in this oblique projection will be BD.

This I shall prove immediately, after having shewn that BD is the versed sine of double the angle of elevation. IBL is in the plane of the horizon, and the angle EBI, which the projectile's direction makes with the horizon, is called the angle of elevation. But since the line IL is a tangent to the circle, and BE, which is drawn from the point of contact, cuts the circle, therefore the angle EAB in the alternate segment is equal to the angle EBI, which EB makes with the tangent, that is, EAB is equal to the angle of elevation, Euc. b. III. prop. 32. And ECB an angle at the center is double EAB an angle at the circumference, which stands on the same arc EB, Euc. b. III. prop. 20. Therefore ECB is double the angle of elevation EBI. But BD is the versed sine of ECB, that is, it is the versed sine of double the angle of elevation.

I am now to prove that if a body is thrown in the direction BE with such a velocity as might be acquired by falling down AB, the greatest height, to which it will ascend in this oblique projection, is equal to BD. Let the line BE represent, not only the direction of the body, but likewise the velocity with which it is thrown. Then, since this velocity is such as might be acquired by falling down AB, it follows by proposition 115, that if the body was thrown perpendicularly upwards with the velocity BE, so that the whole of the velocity might be employed in ascending, it would rise to the height BA. But the body is thrown obliquely; and therefore only part of the velocity will be employed in ascending, and the other part in carrying the body forwards horizontally. The proportion of these parts to each other and to the whole velocity may be thus found. Resolve the oblique velocity BE, by proposition 20, into two others, one in the direction BD perpendicular to the horizon, and the other in the direction DE parallel to it. Then the ascending velocity is to the horizontal velocity as BD to DE, and to the whole velocity as BD to BE. We know already how high the whole velocity would

would carry the body, it would carry it to the height BA. We only want to find, when the body is thrown obliquely, how high it will be carried by BD, or by only that part of the whole velocity which is then employed in its ascent. Now the heights to which different velocities will carry a body are to each other as the squares of those velocities, by proposition 112. Therefore as the square of BE to the square of BD, so would be the height of a body ascending with the whole velocity BE, to the height when it ascends with the part BD. But AEB is a right angle because it is in a semicircle, Euc. b. III. prop 31. and ED is perpendicular to AB, because it is parallel to the horizon. Therefore the triangle EDB is similar to the triangle AEB, Euc. b. VI. prop. 8. and BD is to EB as EB is to BA. Therefore since BD, BE, and BA are in continued geometrical progression, the first or BD is to the third or BA as the square of the first or BD^2 is to the square of the second or BE^2 . But the height to which the velocities BE and BD will make the body ascend are to each other as BE^2 to BD^2 , that is, as BA to BD. And it has been proved already that the first of these velocities or BE would make the body ascend to a height equal to BA, therefore the other or BD, which is that part of the whole velocity that the body ascends with when it is thrown in the direction BE, will make it ascend to a height equal to BD, which is the versed sine of double the angle of elevation.

In the same manner we might shew, if the direction had been BG, and the angle of elevation GBI, that GAB is equal to GBI, and GCB is double the angle of elevation, and that BH the versed sine of GCB, or of double the angle of elevation GBI, is the greatest perpendicular height to which a body can rise, when it is thrown in the direction BG with such a velocity as it might have acquired by falling down AB. Or if FBI, which is half a right angle, had been the angle of elevation, then BC, which is the versed sine of FCB, or of a whole right angle, or of double the angle of elevation, would have been the greatest height to which the same velocity of projection would make it ascend.

147. *When a heavy body is thrown obliquely with a given velocity, if the space, through which it must have fallen perpendicularly to acquire that velocity, is made the diameter of a circle, the random will be equal to four times the sine of double the angle of elevation.*

By what has been shewn in demonstrating the last proposition, if EBI is the angle of elevation, ECB is double that angle, and consequently DE, which is the sine of ECB, is the sine of double the angle of elevation. I affirm therefore, that if a body is thrown from the point B and in the direction BE, with such a velocity as might be acquired by falling
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ing perpendicularly down AB, then the random or horizontal distance from B, at which the body will fall, is equal to four times DE. For it has been observed already under proposition 146, that the whole velocity, if BE is put to represent it, may be resolved into two others, BD, which makes the body ascend, and DE, which carries it forwards in the direction DE or horizontally. Since therefore the ascending velocity is to the horizontal velocity as BD to DE, it follows by proposition 10, that, if these two velocities were to continue uniform, the spaces described in equal times would be as the velocities themselves, and consequently, whilst the body was rising to its greatest perpendicular height BD, it would be carried forwards through a space equal to DE. Now of these two velocities only the horizontal one or DE is uniform, or continues always the same; for the force of gravity can neither accelerate nor retard a motion in this direction, which neither carries the body farther from the earth's center nor brings it nearer to it. But the ascending velocity or BD does not continue the same that it was when the body first begins to rise, but is uniformly retarded, according to proposition 109; therefore, by proposition 110, it will be twice as long in ascending to its greatest height BD as it would have been, if the ascending velocity had been uniform or had continued the same that it was at first. But if the first ascending velocity had been uniform, then in the time of ascent through BD the body would have been carried forwards through once DE, as has been proved already, therefore in double the time, that is, in the time of ascent through BD with an uniformly retarded velocity, it will be carried forwards through twice DE. And since, by proposition 115, the time of descent is equal to the time of ascent, it will likewise in the time of descent be carried forwards through twice DE. And therefore in the times of ascent and descent together it will move forwards through four times DE. Therefore a body thrown in the direction BE with such a velocity as might be acquired by falling down AB the diameter of a circle, if it is thrown from the point B, will fall at the distance of four times the sine of double the angle of elevation.

In the same manner we might shew, if the angle of elevation had been GBI, equal to GAB, that the random would have been equal to four times HG, which is the sine of GCB or of double GAB, that is of double the angle of elevation. Or if FBI half a right angle had been the angle of elevation then the random would have been four times CF, which is the radius, whole sine, or sine of a right angle, that is of double the angle of elevation.

148. *When the velocity is given, the random of a projectile will be the greatest possible with that velocity, if the angle of elevation is half a right one or an angle of 45 degrees.*

For when the velocity is given, the height from whence a heavy body must have fallen to acquire that velocity, or the diameter of the circle AB is a given quantity. And in a given circle the longest line that can be drawn is the semidiameter, whole sine, radius, or sine of a right angle. Thus FC, Plat. V. fig. 8. is longer than GH or than ED, or than any other line which can be drawn in the circle whose diameter is AB. Therefore four times the radius is longer than four times any other line: and consequently the random will be the greatest, when it is equal to four times the radius or four times the sine of a right angle; and this will be the case, by proposition 147, when the double angle of elevation is a right one, or when the angle of elevation itself is half a right one.

149. *When the velocity is given, the random of a projectile will be the same at two different elevations, provided those elevations differ equally from half a right angle.*

Before we prove this it must be observed, that two angles, which differ equally from a right one, have the same or equal sines. Thus GCB and GCA, Plat. V. fig. 8. are two angles which differ equally from a right one. For GCB is greater than FCB a right one by the difference GCF: and GCA is less than FCA a right one by the same difference GCF. But GH is the sine both of the angle GCA and of the angle GCB. Therefore two angles, which differ equally from a right one, have the same or equal sines. Now, if the angles of elevation differ equally from half a right one, the double angles of elevation will differ equally from a whole right one. Thus, if EBI is an angle of 30 degrees, or 15 degrees less than 45, or than half a right one, then twice EBI is 60 degrees or 30 degrees less than 90, or than a whole right one. And if GBI is an angle of 60 degrees, or 15 degrees greater than 45 or than half a right one; then twice GBI is 120 degrees, or 30 degrees greater than 90, or than a whole right one. Therefore when the two angles of elevation differ each of them 15 degrees from half a right one, the double angles of elevation will each of them differ 30 degrees from a whole right one. And so it will be in all other cases. From hence it follows, that if two projectiles are thrown at different elevations, which differ equally from half a right angle, the double angles of elevation will have equal sines, because these angles differ equally from a whole right one: and consequently four times their sines, that is, by proposition 147, the randoms of the two projectiles, will be equal. Thus when the elevation is EBI the random is four times DE, and when the elevation is GBI the random is four times

times HG. But supposing EBI and GBI to differ equally from half a right angle, then double these angles or GCB and ECB will differ equally from a whole right angle; therefore the sines of these double angles, or GH and ED, will be equal, and consequently four times GH and four times ED, or the randoms of the projectiles thrown at these two different elevations, will be the same.

150. *When the velocity of a projectile is given, its greatest random is double the height, to which it would rise, if it was to be thrown perpendicularly upwards with the same velocity.*

The greatest random of a projectile is, when it is thrown at the elevation of 45 degrees, or when its direction makes half a right angle with the horizon. This random, when the velocity is given, is, by proposition 148, equal to four semidiameters of a circle whose diameter is the height from whence a heavy body must have fallen to acquire the velocity of projection. But if the body was to be thrown perpendicularly upwards with this velocity, the height, to which it would ascend being, by proposition 115, the same that it must have fallen from to acquire this velocity, would only be equal to the diameter of such a circle. Therefore the greatest random, or four semidiameters, is double this perpendicular height.

If a bullet was to be shot in the direction BF, Plat. VI. fig. 8. making half a right angle with the horizontal plane IBL, or from a gun elevated to an angle of 45 degrees; and, if the bullets velocity was equal to what a heavy body would acquire in falling down AB, then, by proposition 148, the random will be the greatest, and will be equal to four times CF, or to twice BA. But if the bullet had been shot perpendicularly upwards with the same velocity, then the height to which it would have risen would, by proposition 115, be equal to BA. Therefore, when the velocity is given, the greatest random is double the height to which the same velocity would carry a body, if it was to be thrown perpendicularly. The reader will not want to be put in mind, that when, by proposition 117, we have determined to what height a gun will carry, it will be easy to determine the guns greatest random from this proposition, when the charge of powder and the weight of ball is the same.

151. *The randoms of projectiles, whose elevation is given, are as the squares of their velocities.*

By proposition 147, if a body is thrown in any given direction, as suppose in the direction BE, its random will be equal to four times DE, or to

four times the sine of double the angle of elevation. But then we must remember that DE is always to be the sine of double the angle of elevation in a circle whose diameter AB is the height from whence a heavy body must fall to acquire the velocity of projection. And since, where we are comparing the randoms of projectiles which are thrown with different velocities, we consider the velocity as a variable quantity, greater in some bodies than in others; consequently the height AB from whence a heavy body must have fallen to acquire the velocity of projection must likewise be considered as a variable quantity. For a body must fall from a greater height to acquire a great velocity, than it need fall from to acquire a small one. But if AB was variable, if it was either longer than in the figure, or shorter, then DE would vary in the same proportion, or DE is directly as AB, longer as AB is longer, and shorter as AB is shorter. For in the triangle ECD the angle at D is invariable, whether AB increases or decreases: because, as the angle of elevation or EBI is supposed to be given, ECD which is double the angle of elevation must be given too. And DE by the construction is perpendicular to AB, because it is the sine of ECD, for which reason the angle EDC in the same triangle is likewise given, for it is always a right one. And consequently, since the three angles of every triangle are only equal to two right ones, the third angle or CED is given. Euc. b. I. prop. 32. corol. 2. From hence it follows, that in all states of AB, whether it is longer or shorter, the triangle ECD is always equiangular and similar to itself: therefore its sides are always in the same proportion to each other, or ED varies in the same proportion with EC, that is, ED is directly as EC. Euc. b. VI. prop. 4. But EC is a semidiameter of the circle, whose diameter AB is the height from whence a heavy body must fall to acquire the velocity of projection; and the semidiameter varies in the same proportion as the diameter, or EC is directly as AB. Now since ED is directly as EC, and EC is directly as AB, ED or the sine of a given angle must be directly as AB the diameter. But four times the sine is directly as the sine itself; that is, the random at the given elevation being equal to four times ED is greater as ED is greater or less as ED is less. And ED is directly as AB, therefore the random at the given elevation will be directly as AB, or directly as the height from whence a heavy body must fall to acquire the velocity of projection. But, by proposition 106, the height from whence a heavy body must fall to acquire the velocity of projection, whatever it is, will be as the square of that velocity. Therefore the random of a given elevation will likewise be as the square of that velocity. Thus if two balls were shot at the same elevation but with different velocities, and these velocities were to each other as 2 to 3, the

the randoms would be in the proportion of the squares of 2 and 3, or of 4 and 9: so that if one of the balls was to alight at the distance of 400 feet the other would fly 500 feet farther; that is, it would not alight till it came to the distance of 900 feet from the place where it was discharged.

CHAP. XII.

Of central forces.

§ 52. *A body, which is constantly drawn or impelled towards any point, may be made to describe round that point as a center a curve returning into itself; if the body is thrown in a line that does not pass through the center.*

IF T, Plat. V. fig. 9. is the earths center and ADEI its surface; the force of gravity will constantly draw or impell any heavy body towards the point T. And if such a body is thrown in any direction as GH, which does not pass through the point T, it may describe round T as a center the curve GKMF which returns into itself; that is, after its first projection from G in the line GH it may continue to revolve round T in the orbit GKMF. If a ball was let fall from G it would drop downright to A: but if it was shot from a cannon in the direction GH, the projectile force and the force of gravity together would make it describe a curve GB, by prop. 143. 144. If the projectile force was to be encreased it would make the ball describe the curve GC, and if it was still farther encreased the ball would describe the curve GD. For the greater the force is with which the ball is shot, the greater must necessarily be the distance to which it will fly, before it falls to the earths surface. It is therefore possible to imagine a heavy body to be thrown with a force great enough to carry it half round the earth, as far as M, without suffering it to fall to the surface. But if the body has not fallen to the surface in going half round, it can never fall thither at all. For suppose it to have been constantly approaching the earth, so as to be nearer to the center, when it is at M, than when it was at G: then the body in its descent, for so its motion from G to M may be called, must have acquired velocity, and consequently will be moving faster when it is at M, than it moved at first, when it was thrown from G. But the velocity which it had at G was so great as not to suffer it to fall to the surface in going half round the earth. Therefore certainly the velocity, which it has at M, will be sufficient to carry it the other half round, and will

make it complete its orbit. As it fell or approached the center T in the curve GKM , so it will rise or depart from the center in the similar curve MFG . And since the velocity, which the body must lose in rising from M to G , must necessarily be the same that it acquired in falling from G to M ; its velocity upon the whole, when it returns to G , will be the same that it had, when it first set out from thence. And consequently this same velocity will produce the same effect as before, that is, it will make the body describe the orbit $GKMF$ a second time. And as the velocity, when it returns to G , will always be the same, it will continue to revolve always in the same orbit $GKMF$ round the center T .

To make the case we have been describing a little more intelligible, and to shew the reason why the body, when it has kept approaching to the center, whilst it moved from G to M , should afterwards depart from it and rise from M to G ; it will be necessary to consider another case of a body projected or thrown in like manner from G , but with a greater velocity than was supposed in the former case.

When the body is thrown in the line GH , the projectile force, which acts in that direction, carries the body farther from the center T ; and the force of gravity impells it towards the center. Now if the force, with which the body is carried from the center in the line GH , is exactly equal to that, with which it is impelled towards it; the body must always keep at the same distance from the center, and the curve that it describes must be a circle. And since, when the body has performed half a revolution and is arrived at N in the circle GN , it is neither nearer to the center T nor farther from it than it was, when it first set out from G , it cannot in its motion from G to N have either acquired any velocity or lost any; and consequently it will go on from N in the same manner as at first from G , and will describe the other semicircle, still keeping as before at the same distance from the center T . But if, when the body arrived at N , the velocity of it had been greater than when it first set out, that is, if the projectile force in the line NR after half a revolution was greater than in the line GH at first; then it is evident that as the body moved from N it would have been constantly carried off farther from the center, so that the distance NT would have kept encreasing as the body moved on.

Now this is the case, when the body performs the first half of its revolution in the curve GKM . For the velocity at M , after half a revolution, is greater than it was at first, when the body began to move from G . Therefore as the body moves on from M , the distance TM will keep encreasing. And since the body in its ascent from the center loses velocity

city in the same manner and by the same degrees that it acquired velocity in its descent; the curve MFG, in which it rises, will be similar and equal to the curve GKM, in which it fell. This may perhaps be more evident, if we consider how the body would move from M, supposing that when it comes thither, any force should, without altering its velocity, change the direction of its motion, and make it move back again. Upon this supposition there can be no doubt, but it would return to the point G in the same curve MKG, in which it had before descended to M. Now in order to turn the body back again, no alteration is made in the force which impells it to the center T: for this force acts in the same manner and in a line drawn from the body to T which way soever the body is moving. The only alteration, that is made, is in the direction of the projectile force. Thus, when the body is at M, it is impelled towards the center T, and is carried on at the same time by the projectile force in the direction ML. Just as, when it was G it was impelled towards the same center, and was carried on by the projectile force in the line GH. But if, when the body is at M, the projectile force acts in the direction ML, and to turn it back again nothing need be altered but the direction of the projectile force; then it is evident that, when the body is arrived at M, and is moving by the projectile force in the line ML, if by any means it is made to move in the contrary direction MP with the same velocity, this alteration together with the constant action of the force, that impells it towards T, will turn it back again in the curve MKG which it had described before. Now if a projectile force in the direction MP can make the body describe the curve MKG, then an equal force in the direction ML will make it describe a similar and equal curve MFG. But when the body, after descending in the curve GKM, arrives at the point M; its projectile velocity in the direction ML is the same as the velocity, with which we supposed it to be thrown back in the direction MP. Therefore when the body has descended in the curve GKM, it will go on to move in the curve MFG, and will return to the point from whence it set out.

We may now consider the third case of a heavy body thrown from G and impelled constantly towards the center T. We have seen already that a certain projectile velocity would make the body revolve in the curve GKMF; in which case it constantly approaches the center, till it arrives at M, and constantly departs from it, after it leaves M, till it arrives at G. Therefore in the orbit GKMF, the point M is nearest of all to the center T, and the point G, from whence we supposed the body to set out, is farthest of all from that center. We have seen likewise that

a greater projectile velocity would make the body revolve in the circle GN; all the parts of which are at the same distance from the center T. And from hence it is obvious, in the third place, that a projectile velocity still greater would make the body begin to depart from the center T at its first setting out; and might make it describe the curve GO. For as the body departs from the center its velocity will constantly decrease; because the force, which impells it to the center, acts in a direction contrary to the bodys motion and therefore must retard it. When the body has ascended in the curve GO; and the force, which impells it towards T, acts again at right angles to the projectile force in the line OQ, as it did in the line GH, when the body was first thrown from G, then the body will begin to descend towards the center T in a curve OG similar and equal to GO the curve in which it ascended from thence; so that this third case is like the first, with this difference only, that in the first case the point G, from whence the body sets out, is farthest of all from the center T, whereas in this third case the point G is nearest of all to that center.

The curve GKMF, as also the other curve GO, is an ellipsis. Such a curve is drawn upon two centers as a circle is upon one. The manner of drawing it is this. Plat.V. fig. 10. A string Fdf , which is longer than Ff , being fastened to two pins at F and f , is kept stretched by a pen d : and as this pen is moved round the two points Ff , in such a manner as to return to the place, from whence it sets out, it will describe the ellipsis $mbnb$. The two points F and f are called the focuses. A line mn drawn through the focuses F and f is called the longer axis of the ellipsis. The point c which divides the longer axis into two equal parts, is called the center of the ellipsis. And a line bcb , drawn through this center perpendicular to the principal or longer axis, is called the shorter axis.

Plat.VI. fig. 1. Let S be the earth, and let any heavy body, the gravity of which impells it constantly towards the earth, be thrown at A in the direction $A\iota$. Then if the force, with which the body is thrown, is to the force, which impells it towards the earth, as $A\iota$ is to AM; that is, if in the same time that one force would carry the body along $A\iota$, the other would make it descend through AM; the body by both forces together will describe the diagonal AB of a parallelogram whose sides are $A\iota$ and AM, by proposition 16. Now this diagonal is a part of the ellipsis ADPG, one of whose focuses is S, where the earth is supposed to be, and the other is s . Therefore if the moon gravitates towards the earth and was thrown in the direction $A\iota$, or was moving, according to

to the first law of motion, in the right line Al , it would by this force and the force of gravity together be made to set out in an ellipsis with the earth in one of its focuses: and at the point A from whence it is thrown it would be at its greatest distance from the earth. This is the first of the three cases, before explained.

If the heavy body was to be thrown with a greater force, such an one as should be to the force, that impells it to the center S , as Ar to AM ; then Ar and AM will be the sides of the parallelogram, and Am will be the diagonal in which the body will begin to move. Now Am is a part of a circle, which has S for its center. Therefore, if a certain degree of velocity would make the heavy body set out from A to move in such an ellipsis that A should be the greatest distance from the point S towards which it is impelled; a greater degree of velocity might make it set out to move in a circle having S for its center, and to revolve round S keeping always at the same distance from it. This is the second of the three cases before explained.

A still greater force, one for instance, that is to the force which impells the body to the point S , as At to AM , would make it describe the diagonal An , and this An is part of such an ellipsis as has S for one of its focuses, and AS for the least distance of the revolving body from the point S , towards which it is impelled. This is the last of the three cases before explained.

The first of these cases may easily be reduced to the last by only supposing the body to be thrown originally from P instead of A , in the direction Pp , and with the same velocity that it will acquire in moving from A to P , that is, with a velocity too great to suffer it to keep always at the same distance from S , or too great to make it describe a circle round S having PS for its semidiameter.

We may now answer some questions, which are usually asked in relation to a body that is made to describe an ellipsis by the action of two forces, one of which is a projectile force, that acts in a tangent to the orbit, as in the line At , when the body is at A , in the line Dd , when it is at D , in the line Pp when at P , and in the line Ff , when at F ; and the other force impells the revolving body, not towards the center of the ellipsis, but towards S one of its focuses. When the body is at A or at the greatest distance from S the two forces act at right angles to each other, one in the line At , the other in the line AS . They act likewise at right angles again, when the body is at P , or at the least distance from S . Now, if a body revolves in a circle, the projectile force, which acts in the tangent, is always in every part of the circle at right angles to the other force, which impells the revolving body towards the center of
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the circle, as is obvious from a bare inspection of Plat. VI. fig. 2. where EG the tangent or direction of the projectile force when the body is at B is perpendicular to BA, or to the direction in which it is impelled towards the center A. So likewise CF is perpendicular to CA, DH to DA, and LK to LA: and universally the tangent is always perpendicular to a line drawn from the point of contact to the center. Euc. b. III. prop. 18. Therefore, since this is the direction of the two forces in respect of each other, when the revolving body describes a circle; it may be asked, why the body setting out from A, Plat. VI. fig. 1. in the line At perpendicular to AS does not move in a circle having S in its center, and AS for its semidiameter, rather than in an ellipsis having S in its focus. Or why, when it is come to P, where the two forces are perpendicular again, does not it go on to move in a circle having PS for its semidiameter. The reader is, I imagine, by this time able to tell any one who should make such an enquiry, that the revolving bodys velocity at A is too small to make it describe a circle having AS for its semidiameter, and at P is too great to make it describe a circle having PS for its semidiameter. The force, with which the body, when it is at A, moves in the tangent At, is so small, that the other force, which impells it towards S, will make it approach nearer to S than it is at A, or the distance of it from S will keep decreasing as the body moves on. Till at last it arrives at P where the two forces are perpendicular again. But then, as in case the third, the velocity is so great in the tangent Pp, that the force, which impells the body towards S, will not be sufficient to prevent its running out to a greater distance from S than PS, so that its distance from S, will, as it goes on from P, keep constantly encreasing in the same manner that it decreased before, and the body will return from P to A in a curve PGA equal and similar to the curve ADP, in which it descended from A to P.

But since the velocity of the revolving body is too small, when at A, and too great, when at P, to make it describe a circle round S; it may be farther asked, whether, in some intermediate part of the orbit between A and P, the velocity is not such as would make it describe a circle; and if there is such a velocity, why does not the body, when it has that velocity go on to describe a circle for the future? There may, and indeed will be, such a velocity of the revolving body, in some point of its orbit, whilst it is moving from A to P. And suppose the body to be at the point D, when it has this velocity. Then the question is, why it does not go on from the point D to describe a circle round S at the distance DS?

Now it must be observed, in answer to this, that when the revolving body is at D, the projectile force acts in the line Dd, which is a tangent
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to the orbit at the point D : for if the force, which impells the body towards the center, was to cease acting, when it is at that point, it would then be acted upon by the projectile force only, and would, by the first law of motion, run out in a right line Dd . But since that force does not cease to act, the body, when it is at the point D , will be acted upon by both the forces, by one of them in the direction Dd , and by the other in the direction DS . These two lines Dd and DS make an acute angle with each other, and consequently the direction of the two forces is not such as it ought to be in order to make the body describe a circle: for when a body revolves in a circle, the projectile force and the force, which impells it towards the center, act at right angles to each other; as was just now observed.

When the body by the projectile force is moving in the line Dd , and the force, which impells it to the center, acts at an acute angle in the line DS , this latter force does in some sort conspire with the bodys motion, and consequently will accelerate it. For this reason, though the velocity of the body, when at D , was such as would make it describe a circle round S at the distance DS , if the two forces had acted at right angles to each other; yet as they act obliquely to each other, this velocity will immediately be changed, and the body will go on to describe an ellipsis. In like manner, when the body is ascending from P towards A , suppose the velocity of it at G should be such as would make it describe a circle round S at the distance GS , if the two forces acted at right angles to each other; yet as the projectile force acts in the line Gg , and the force, which impells it towards the center, acts in the line GS at an obtuse angle to Gg , this latter force does in some measure oppose the bodys motion and consequently retards it: for which reason the velocity, that would have made it move in a circle, if the two forces had acted at right angles, is by their acting at an obtuse angle immediately altered, and the body goes on to describe an ellipsis.

A stone whirled round in a sling describes a circle; and so do most bodies that are made by artificial motions to describe curves returning into themselves. And as we are most used to this sort of orbit, and scarce ever saw a body describe an ellipsis; we are apt to think a motion in an ellipsis less natural than the other, and to make the enquiries that have just now been explained. For, if it was not on account of this prejudice, we might as well ask why a body, when it is moving in a circle, does not change the form of its orbit, and move in an ellipsis, as why a body, when it is moving in an ellipsis, does not change the form of its orbit, and move in a circle.

We may explaine the motion of a body in an elliptical orbit, when it is impelled towards one of the focuses, in another manner, so as to give the reader a farther notion of the reason why such a body, after it has approached the focus towards which it is impelled, whilst it is moving from A, Plat. VI. fig. 1. to P in the line ADP, should again depart from that focus, notwithstanding the same force continues to impell it thither, and should go round from P to A in the line PGA.

If the body, instead of being thrown from A at first in the direction Al , had received, when it was at A, no impulse at all from any projectile force, but had only been impelled towards the center S, or which amounts to the same thing had been let fall from A, it would have descended directly to S, in the line AS. But if it is thrown in the line $Alrt$ which makes a right angle with AS, or is impelled in that direction by any projectile force, at the same time that the other force impells it towards S, these two forces by their joynt action will make the body describe a curve. One impulse in the line $Alrt$ will be sufficient: for, by the first law of motion, a body once put in motion will always continue to move on uniformly in the same direction, if no other cause prevents it. Now the force, which constantly impells the body towards S, is such a cause as must change its direction, and may prevent it from moving uniformly. That this force will necessarily change the bodys direction, is evident after what has been shewn in proposition 16. For when the body is thus acted upon by two forces, instead of moving in the direction of either of them, it will describe the diagonal of a parallelogram, whose two sides will lie in the direction of these two forces. And consequently the diagonal, which the body describes cannot lie in the same direction in which the projectile force acted at first. Thus if AB is the diagonal, and Al and AM the two sides, the body acted upon by two forces in the directions Al and AM, will move in neither of these two directions, but in the direction of the diagonal AB. If the force which urges the body towards S was to cease acting, when the body arrives at B, there is no occasion for any new impulse of the projectile force to carry it on strait forwards from B in the line Bb , for a body once in motion will, by the first law of motion, continue to move on in the same right line, unless some external cause prevents it. But, instead of running out in the tangent Bb , it is made by this projectile force and the other force together to describe the diagonal of a parallelogram, one of whose sides is in the direction Bb in which the projectile force acts when the body is at B, and the other is in the direction BS, in which the body is at the same time impelled towards S. By the
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joynt action of these two forces the body is made to revolve round S in the curve ADPG.

Here we are to observe that the angle $\angle AS$ which is contained between the line Al in which the projectile force acts, and the line AS , in which it is urged towards S, is a right angle, when the body is at A. And since Al is a tangent, and since, if the body moves in a circle, and is impelled towards the center of its circular orbit, AS would be a semidiameter of that circle, it follows that these two forces will always act at right angles to each other in every part of such an orbit. Because a tangent to any point in a circle is always perpendicular to a semidiameter drawn from the center to that point. Euc. b. III. prop. 18. We have already seen that if the body, when it is at A, was to be thrown with a force which is to the centripetal force as Al to AM , the diagonal of the parallelogram, which the joynt action of these two forces would make the body describe, is AB , or a part of the ellipsis ADPG. But if the projectile force is to the centripetal one as Ar to AM , then the diagonal will be Am , which is part of a circle having AS for its semidiameter. From hence then we may conclude that when the body sets out from A the projectile force, which could make it describe a circle, must be greater than that which makes it describe the ellipsis ADPG. For the former is to the latter as Ar to Al . Now if the body had not been thrown at all from A, but, instead of being put in motion in the direction Al , had been suffered to fall towards S, by the action of the centripetal force only it would, as we have observed before, descend in the line AS . And if this line AS would be the direction of the bodys motion, supposing it not to have been acted upon by any projectile force; it follows that the less the projectile force is, the nearer will the direction, in which the body moves, approach to AS . But such a projectile force as would be sufficient to make the body describe a circle, would keep the angle at A, or the angle made by the projectile and centripetal forces, in this or in any other part of the orbit, always a right one. Therefore, since with a less projectile force, or with such an one as would make the body begin to revolve in the ellipsis ADPG, the direction of the bodys motion will approach nearer to AS , it follows that as the body goes on from A the projectile and centripetal forces will approach nearer to each other, that is, the angle made by these two forces will become less than a right one, as it is at B, where Bb is the direction of the body as it moves by the projectile force only, and BS is the direction of the centripetal force, and the angle bBS is an acute one.

As long as the projectile velocity, or the velocity with which the body would run off in a tangent to its orbit, is less than the velocity that is requisite to make the body revolve in a circle round S ; so long the direction of its motion, for the reasons just now explained, will keep approaching to the line AS , if the body was at A , or to BS , if it was at B , or to CS , if it was at C . That is, as long as the projectile force is too small to make the body move in a circle, the direction of the projectile force will keep approaching to the direction of the centripetal force, or the angle made by these two forces will keep diminishing. Thus the angle bBS is less than aAS . And cCS is less than bBS . And dDS is less than cCS .

But suppose the body, by descending from A to D , to acquire such a velocity as would be sufficient to make it revolve round S in a circle whose semidiameter should be DS . Then, after the body is passed the point D , the angle contained between the projectile and centripetal forces will begin to encrease. So that, when the body is arrived at E , the angle eES will be greater than dDS . This is evident from what has been said before. For if when the velocity is less than what is required to make the body revolve in a circle, the direction of the projectile force will approach to that of the centripetal force, or the angle made by these two forces will constantly diminish, it follows, that when the velocity is greater than what is required to make it revolve in a circle, the direction of the bodys motion will constantly depart from the line DS , and the angle made by the projectile and centripetal forces will constantly encrease, as the body moves on from the point D .

When the body arrives at P , and this angle has encreased to SPp and so is become a right one; the body is then at its least distance from S . And as it was easy to understand, how the body, when thrown from A in the direction $A/$ at right angles to AS , might have a less projectile force in the line $A/$ than is requisite to keep it always at the same distance AS from the point S , and might upon this account constantly approach to the point S in the curve ADP . So it seems to be as easy to understand, that, when the body is at P , and the projectile force, which acts in the line Pp , is again at right angles to the centripetal force, which acts in the line PS , the projectile force in the line Pp , or the velocity, with which the body is then moving in that direction, may be greater than it ought to be for the body to continue to revolve round S in a circle having PS for its semidiameter. But if the velocity of the body in the line Pp is so great that the centripetal force, which impells it towards S , is not sufficient to keep the body always at the same distance from it; because the body by its projectile force is more carryed off from the cen-

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ter in the line Pp than it is impelled towards it in the line PS ; then one difficulty about a bodys moving in an ellipsis, whilst it is impelled towards one of the focuses, is quite removed. For though the body has kept approaching towards S , whilst it descended from A to P , yet it cannot possibly approach towards it any longer: since that force, which is not sufficient to keep the body always at the same distance PS from the point S and to make it describe a circle round that point, must certainly be insufficient to bring the body nearer to S , or to make it approach towards S , after it is passed P .

This may be a little farther explained, if we go on with considering the angle made by the projectile and centripetal forces. This angle is the least of all when the body is at D , and keeps encreasing constantly, till it becomes a right one, when the body arrives at P . The reason of its encreasing after the body has passed by D is, because the projectile velocity, or velocity with which the body endeavours to move in the tangent, is greater than what would be requisite to make the body revolve round S in a circle. But the same reason holds when the body is at P ; its velocity is still too great to suffer it to move in a circle having PS for its radius or semidiameter. Therefore the angle SPp , as the body moves on, will keep still encreasing, and consequently must become greater than a right one. From hence it is evident that after the body is arrived at P , that is, after the angle made by the projectile and centripetal forces is become a right one, it can approach towards S no longer. For if the body was to approach nearer to S , the angle made by the lines Pp and PS must decrease: just as the angle made by the lines Al and AS decreased in the approach of the body towards S , when it set out from A . But the angle at P will encrease, as has been proved, whilst the body moves forward from P . Therefore it will not approach nearer to S than when it is at P : but the point P , where this angle is a right one, will be its least distance from S .

When the body goes on from P , the velocity being too great for it either to approach nearer to S or to keep at the same distance from S , it will depart from S , and the angle made by the projectile and centripetal forces will encrease; so that when the body arrives at F it will be SFf greater than a right one. And because, as the body rises or departs from S , the centripetal force, though it is not sufficient to draw it back again, will retard its motion, the projectile velocity will keep decreasing. Till at last, in the return of the body towards A , this velocity will be such an one as would just be sufficient to make the body revolve in a circle round the point S . Now, whilst the bodys velocity was greater than this, the

the direction, in which it endeavours in every point of its orbit to fly off in a tangent to that point, that is, the direction of the projectile force departed farther from the direction of that force by which it is impelled towards the center, or the angle made by these two forces encreased. Thus when the body was at P, the angle was SPp ; when it was at F, the angle was at SFf ; and when it arrives at G, the angle becomes SGg . But since, as the body goes on from G, it will be still retarded, the tangent, or direction of the projectile force, will approach towards the line, in which the centripetal force acts, that is, this angle will constantly decrease; at H it will be SHh , less than SGg ; at I, it will be SIi less than SHh ; and at last at A, it will be so far diminished as to be a right one again.

Three questions may here be asked in relation to the bodys motion, when it is at or about the point D. The first is this. Since the velocity at D is, by the supposition, just such an one as is requisite to make it describe a circle round S with the radius or semidiameter DS, or to keep it always at the same distance DS from the point S; whence does it happen that in going on from D it does not keep at the same distance from S, but continues to approach to S? This question has been answered already; but we may now put that answer into another form. The body, when it has passed by the point D, continues to approach to S, not because its projectile velocity is then too small to keep it always at the same distance from S, but because this velocity lies in such a direction as would make it approach to S even though no force impelled it thither. For if the force, which impells the body thither when it is at D, was to cease acting, and the body was to run out from D in a tangent, this tangent would be Dd , and the body, when it arrives at d , would be nearer to S than when it set out from D. Therefore certainly the force, which impells the body towards S, and the projectile force together will make it approach S, after it has passed by D: and consequently the body cannot move in a circle, notwithstanding, when it is at D, it has such a velocity in the tangent Dd , as would be proper for that purpose provided this tangent was perpendicular to SD.

The second question which may be asked in relation to the body, when it passes by the point D is, why the velocity, which it then has, should not continue, till the body arrives at P, and the two forces act at right angles to each other? The most obvious answer to this question is, that the body, after it has passed by the point D, continues still to approach or fall towards S, and consequently in falling will acquire velocity. Or otherwise; when the body is at D, the force, which impells it towards S, and the projectile force act at an acute angle, the former in the line DS and

and the latter in the line Dd . Suppose therefore the former force to be to the latter as $D\delta$ to Dd . Make these the two sides of a parallelogram, and the diagonal DE will be the line, that the body will describe by the two forces together in the same time that it would have described Dd by the projectile force alone, agreeably to what has been proved in proposition 16, and the velocity produced by both forces together will be to the projectile velocity as DE to Dd . Now the diagonal DE , being drawn from an acute angle, is longer than either side of the parallelogram, and consequently longer than Dd . Therefore the body's velocity produced by both forces, or the velocity with which it moves in its orbit, will be greater than its projectile velocity. That is the body will be accelerated, as long as the two forces continue to act at an acute angle. For thus again at E , where the angle eEe is an acute one, if the force which impells the body towards S , is to the projectile force as Ee to Ee ; the body by the joint action of these two forces will describe EP , and the velocity with which it moves in the diagonal EP , will be to the projectile velocity as EP to Ee . And EP the diagonal drawn from an acute angle is longer than the side De . Therefore the body is still accelerated.

The third question that may here be asked is this. Since, if a body revolves in a circle the angle made by the two forces is a right one, and since as soon as the body has passed by the point D , its projectile velocity becomes too great for it to move in a circle having S for its center, why does not the angle made by these forces immediately become greater than a right one, and the body immediately begin to depart from S ? For as this is the reason given why the body, when it is at P , approaches to S no longer, but begins to depart from S , because the velocity is too great for it to continue to move in a circle; ought not the same reason to take place as soon as ever the body has passed by D ; and ought not the body then to begin to depart from S , since the velocity is then too great for it to continue to move in a circle? To this it may be answered; that if the angle made by the projectile and centripetal forces was a right one, as soon as the body is passed by D , then upon the velocity becoming too great for it to move in a circle, this angle would immediately become greater than a right one, and the body would immediately begin to depart from S . But the angle both at D and after the body is passed by D is an acute one, and the body by the projectile force alone will continue to approach S , instead of departing from it. Therefore, since it would not depart from S even by the projectile force alone, it is certain that the action of the other force which impells it towards S , cannot make it depart from thence. But both these forces
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acting together will cause the body still to approach to *S*. The immediate effect produced by the velocitys being too great for the body to move in a circle round *S*. is not to make the body depart farther from *S*, but to make the angle contained between the tangent and a line drawn to *S*, or the angle contained between the two forces, begin to encrease. And as the body moves on from *D* to *P* this angle keeps encreasing, till at *P* it becomes a right one, and then, and not till then, the body will begin to depart from *S*.

It is easy to apply these questions and the answer to them to the motion of the body as it ascends from *P* towards *A*. For first, though the body at *G* be supposed to have such a velocity as in proper circumstances would make it revolve round *S* in a circle at the distance *GS*; yet, when the body passes on from *G*, it ought not to keep at the same distance *GS* from the point *S*, because the tangent *Gg* or direction, in which the body would move by the projectile velocity only, is not perpendicular to *SG* the line in which the body is impelled towards *S*, but makes a greater angle with it than a right one: consequently the projectile force in this line *Gg* makes the body depart farther from *S* in any given time than such a projectile force would do as acted in a line perpendicular to *GS*. Secondly, since the body after it leaves the point *G* keeps departing from *S*, the velocity, which it had at *G*, will not continue till it arrives at *A*, for the body in its ascent will be retarded. Or otherwise. When the angle made by the two forces is so obtuse as it is at *G*, the diagonal drawn from the obtuse angle *SGg* will be shorter than either side of a parallelogram having one side in the direction *Gg* and the other side in the direction *GS*. For which reason the velocity in the diagonal, or the velocity produced by both the forces, will be less than the velocity in the side *Gg*, or than the velocity arising from the projectile force alone. Thirdly. Though the velocity of the body as soon as it is passed *G* becomes too little to make it move in a circle; yet it will not immediately begin to approach to *S*, as it does after it has arrived at *A*. But as the body goes on from *G* towards *A* the angle made by the projectile and centripetal forces will constantly diminish, till the body comes to *A*, at which time this angle will be a right one; and after that, as the angle diminishes still farther, the body having passed *A* begins to approach towards *S* as before.

If the revolving body describes the ellipsis *mbnb*, Plat.V. fig. 10. and *F* is the point, towards which it is constantly impelled; then *m* is the greatest distance of the revolving body, *n* is its least distance, and *b* is its middle distance from the point *F*.

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That m is the greatest distance, and that n is the least distance is apparent; and if the reader thinks it worth his while to attend to what follows, it will be no difficult matter to prove that when the body is at b it is at the middle distance from F or at a distance which differs equally from the greatest and the least. That is, bF is just as much less than mF as it is greater than nF .

When the pen d in drawing the ellipsis is at m ; that is, when d and m coincide, df is equal to mf and dF is equal to mF . But df added to mF is always equal to the whole length of the string. Therefore mf added to mF is equal to the length of the string. In the same manner it may be proved that nF added to nf is equal to the length of the string. Again: when the pen is at m the string is stretched out that way, and when the pen is at n the same string is stretched out the other way. Now we may take it for granted in this case, that the same string stretched out upon the same two points F and f will reach to an equal distance from those points either way; and consequently mf and nF are equal to each other. Therefore mf added to nf , that is the longer axis, is equal to mf added to mF . But mf added to mF has just been proved equal to the length of the string. Therefore mf added to nf , that is the longer axis mn , is equal to the length of the string. From hence it follows, since c divides the longer axis into two equal parts, that mc or nc are equal to half the length of the string.

Now as mc and nc are equal, and likewise mf and nF equal, $mc - mf$ or fc is equal to $nc - nF$ or Fc : that is, the center c divides the distance between the focuses F and f into two equal parts. When the pen in describing the ellipsis is at b , then bf and bF is the length of the string, and bf is equal to bF . For in the two triangles bcf and bcF the angles at c are both of them right ones and consequently equal to each other; the side bc is common; and the side fc in one is equal to the side Fc in the other. From whence it follows, Euc. b. I. prop. 4. that bf the third side in one is equal to bF the third side in the other; and since bf and bF together make up the whole string, bF must be half the string. But mc or nc are equal to half the string. Therefore bF is equal to either mc or nc . That is, bF , which I am to prove to be the middle distance, is equal to half the longer axis.

mF is the greatest distance, which is greater than mc , or than half the longer axis, by the quantity cF ; and nF is the least distance, which is likewise less than nc , or than half the longer axis, by the same quantity cF . Therefore half the longer axis, or bF which is equal to half the longer axis, differs equally from the greatest and least distances; and consequently is the middle distance.

The reader is desired to observe farther, that, if F and f the two focuses of the ellipsis, or the two pins, to which the ends of the string are fastened, were to be brought close together at c , then the pen, which describes the ellipsis, would in going round be always at the same distance from c , and this distance would be equal to half the length of the string, to bf or bF . Consequently the curve described in this case, would be a circle, whose semidiameter would be equal to bF , or, which amounts to the same to mc , or to nc . From hence it appears, that a circle may be considered as an ellipsis, the focuses of which coincide, or an ellipsis may be considered as a circle drawn upon two centers.

153. *A body that revolves in an orbit, endeavours in every point of the orbit to fly off in a right line, which is a tangent at that point, and to depart from the center, towards which it is impelled.*

Whilst a stone is whirled round in a sling, it describes a circle; because it is constantly impelled towards the hand, which is the center, and is kept from flying off by the force of the string. But if the string breaks, or if one end of it is slipped, then the stone, being no longer impelled towards the hand or kept from flying off, will continue its circular motion no longer, but setting out from the place where it was, when the string broke, and when the force, which before impelled it to the center, ceased to act upon it any longer, it will, by the first law of motion proposition 11, go on to describe a right line. If $BCDL$ Plat. VI. fig. 2. is the circle, which the stone describes, and one end of the sling was to be slipped, or the string was to break, when the stone is at B ; the stone, by the first law of motion, would run off from the point B in the right line BG , which is a tangent to the circle at B . In the same manner, if the string was to break, when the stone is at C , it would fly off in the line CF ; if at D , in the line DH ; if at L , in the line LK ; or if at any other point of the circle, it would fly off in a tangent at that point.

This is equally true of bodies that revolve in any other orbit, as of those that revolve in a circle. For if the force, which impells a body to the point S , Plat. VI. fig. 1. and makes it describe the ellipsis $ADPG$, was to cease acting, when the body is at P , or at H , or at C , or at any other place in its orbit, it would, by proposition 11, run out in a right line which is a tangent to the ellipsis at that point.

Now the same force with which the body would fly off, if it was not impelled towards the center, constantly acts upon it, whilst it is describing the curve and is impelled thither. Therefore since this force
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would make the body fly off in a tangent, if no other cause prevented it, the same force constantly acting, whilst the body revolves in an orbit, makes it endeavour to fly off in the same manner, and is prevented from producing its effect by the other force, which constantly impells the body towards the center.

Now as the body moves on in the tangent, it will be every where farther from the center, towards which it is impelled, than if it was to move in the curve. Thus, Plat. VI. fig. 3. if the body describes the tangent Ba instead of the arc Bn it departs from the center C , or is farther from that center, when it is at m , or at any other point in the tangent, than it would have been if it had described Bn . In like manner, Plat. VI. fig. 1. if a body is constantly impelled towards S and revolves in the curve $ADPG$ the force, with which it endeavours to fly off in a tangent, makes it likewise endeavour to depart from the point S . If the body is at D , then in describing the tangent Dd , though it approaches the center, and is nearer to it when it comes to d , than when it was at D ; yet by describing the tangent it does not approach S so much as it would do by describing the curve, or in the tangent it is every where farther from S than it would have been, if it had gone on in the arc DE . For though the body by describing the tangent Dd is nearer to the center S at d than at D , yet it is farther off from the center, when it is at d , than it would have been, if it had described the arc DE , and so had been at E . Therefore since the body will be every where farther from the center by describing the tangent than it will by describing the curve, it follows that the same force, which makes it endeavour to fly off in a tangent, makes it likewise endeavour to depart from the center, towards which it is impelled.

From hence we may see the reason, why we may turn a pail full of water over our heads, and none of the water be spilled, when the pail has its bottom upwards. For the pail and the water in it being turned round in a circle, the water will every where endeavour to depart from the center of that circle. Therefore when the pail in this circular motion is over our heads and its bottom is upwards, though the weight of the water would make it fall out of the pail, yet as it cannot fall without approaching to the center of the circle which it describes, the force with which it avoids this center will, if the pail is whirled round fast enough, be sufficient to keep it in the pail.

154. *The force, which impells a body towards the center, when it revolves in an orbit, is called the centripetal force.*

155. *The force, by which a body revolving in an orbit endeavours to depart from the center, is called the centrifugal force.*

156. *When a body describes a curve returning into itself, the centripetal force is equal to the centrifugal force; and they are called by one common name the central forces.*

If a body revolves in the circle BD, Plat.VI. fig. 3. then in the time that it describes the arc Bn, it will have been impelled towards the center through the space an; for if it had been left to itself and had not been impelled thither at all, the projectile force alone would have carried it in that time along the tangent from B to a. Therefore since the line an is the space described by means of the centripetal force, this force is proportional to an. But if, when the body was at B no centripetal force had acted upon it, then instead of describing the arc Bn it would in the same time have moved along the tangent Ba, and the line na would have been the space through which it would have departed from the center. Therefore the centrifugal force is proportional to na. And consequently, as both these forces are represented by one and the same line, they must be equal to each other.

Though three forces are here spoken of, the centripetal, the centrifugal, and the projectile force; yet two are all that are required to make a body revolve in an orbit. For the centrifugal and projectile forces differ from each other only as a part differs from the whole. The projectile force is that, with which the body would run out in a tangent to its orbit, if there was no centripetal force to prevent it; and the centrifugal force is that part of the projectile force, which carries the body off from the center, whilst it is describing the tangent. Thus if the body revolved in the orbit BD, Plat.VI. fig. 3. the projectile force is what would make it describe the tangent Ba, if the centripetal force was to cease acting. But in the mean time the whole force Ba does not carry it off from the center C; when it is arrived at a, it is farther from the center than when it was at B only by the quantity an; and as much of the projectile force as acts in this line an is what carries the body off from the center, and is called the centrifugal force.

157. *The periodical time of a body revolving in an orbit is the time that it takes up to perform one complete revolution.*

158. *The plane of an orbit, in which a body revolves, passes through the line of projection, and through the center to which the centripetal force is directed.*

Suppose ADPG, Plat.VI. fig. 1. to be the orbit, which the body describes, and S to be the point or center, to which the centripetal force is directed; for though S is not the center of the ellipsis but one of its foci,

cuses, yet as it is the point towards which the body is impelled, or the point in which the centripetal force acts, we therefore call it the center. What is affirmed in the proposition amounts to this. The plane of the orbit passes through Az the line of projection and S the center in which the centripetal force acts. Or, since the projectile force continues to act upon the body in every point of its whole revolution, and in a line which is a tangent to that point, by proposition 153, the orbit lies in the same plane with the line Cc in which the projectile force acts, when the body is at C ; or in the same plane with Pp , or with Gg , or Ii , lines in which the projectile force acts in different parts of the orbit; and likewise in the same plane with S the center towards which the centripetal force tends.

If S is the earth and $ADPG$ the moons orbit; upon supposition that the moon is impelled towards the earth constantly by the force of gravity, and that this is the centripetal force, which retains it in its orbit, and prevents it from flying off in a tangent; the plane of the moons orbit must pass through any line as At , Dd , Ff , or Hh , in which the projectile force acts, and through S the earth.

So likewise if S is the sun, and $ADPG$ is the orbit of any planet, as of Jupiter for instance; upon supposition that the planet gravitates towards the sun, and that gravity is the centripetal force, which retains it in its orbit; the plane of the orbit described by it passes through any line as At , Dd , Pp , or Gg in which the projectile force acts, and through S the sun, or point towards which the centripetal force is directed.

Since the whole motion of the body arises from two forces, of which one acts in the line Al , and the other in the line AS ; it seems to follow that the first of these forces or the projectile force can never make the body leave the plane, in which the line Al is, and that the other can never make it leave the plane, in which AS or S is. Therefore it must always be found in the same plane with Al the projectile force and with S the point towards which it is impelled by the centripetal force.

This may be proved more exactly in the following manner. Let $ABCD$, Plat. VI. fig. 4. represent a part of the orbit, which is described by a body that is constantly impelled towards S . BC , CD , or DE , or any other part of this orbit lies in the same plane with AZ the line of projection and with S the point, in which the centripetal force acts and towards which the revolving body is impelled. If the body begins to move by the projectile force from A in the direction ABZ ; then, according to the first law of motion proposition 11, if no other force was to act upon it, and to alter the course of its motion, it would continue
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to move on uniformly in a right line in the same direction AZ . But the centripetal force bends the course of its motion towards S , and makes it constantly change the direction of its motion so as to describe an orbit round S , of which $ABCDE$ is a part. Let us suppose that the action of this centripetal force is not continued, but that it exerts itself by separate strokes after certain intervals of time. Thus for instance; let us suppose, that the body, after it has set out from A , may move on uniformly by the projectile force for as long a time as is requisite to describe the line AB ; but when it is arrived at B , and would, if left to itself, move on in the same uniform manner, and in an equal time describe Bc , which is equal to AB ; then suppose one stroke of the centripetal force to act upon it, and to bend its course or alter its direction from Bc to BC , so that both forces together shall make it describe BC in the same time that the projectile force alone would have made it describe Bc . Again, when the body is arrived at C , suppose another stroke of the centripetal force to act upon it, and to bend its course towards S , so as to make it describe CD , instead of moving strait forwards from C in the direction Cd , as it would do according to the first law of motion, if none but the projectile force was to act upon it. And suppose in like manner, after an equal interval of time was passed and the body was arrived at D , that another stroke of the centripetal force was to act upon it, and to bend its course from the direction CD into the direction DE . In this case either BC , or CD , or DE , are in the same plane with AZ and with the point S . When the body is at B , there are two forces act upon it; the projectile force in the direction BZ and the centripetal force in the direction BS . Now as BZ and BS represent the directions of the two forces, so let Bc taken in the direction of the projectile force and BG taken in the direction of the centripetal force represent the proportion of these two forces to each other. That is; let the projectile force be to the centripetal force as Bc to BG . From what was proved in proposition 16, it follows that the body thus acted upon by two forces Bc and BG , will describe BC the diagonal of a parallelogram, of which Bc and BG are the sides. But BC the diagonal of a parallelogram is in the same plane with its two sides Bc and BG . Euc. b. XI. prop. 2. And Bc which is part of the right line AZ is in the same plane with the whole line, Euc. b. XI. prop. 1. as also BG , which is part of the right line BS , is in the same plane with the whole line. Therefore BC is in the same plane with AZ the line of projection and with BS , and consequently with S the point, to which the centripetal force is directed. In like manner it may be proved

ved that the next part of the orbit CD is in the same plane with Cd and CH the two sides of a parallelogram, of which CD is the diagonal. For here, as before, the projectile force alone would make the body move strait forwards from C in the direction Cd . But at C the centripetal force impell sit towards S in the direction CS . And, supposing these two forces to be to each other as Cd to CH , their joynt action will make the body move in the diagonal CD , according to proposition 16. Therefore CD is in the same plane with Cd and with CH , that is, with CS , and consequently with S . But Cd , being the right line BC continued, is in the same plane with BC . Euc. b. II. prop. 1. and BC has been proved to be in the same plane with AZ . Therefore CD is likewise in the same plane with AZ the line of projection, and with S the point to which the centripetal force is directed. The same thing may be proved in the same manner of DE and of every other part of the orbit. And consequently the whole orbit is in the same plane with the line AZ and with the point S .

$ABCDE$ is not a curve, but a polygon, which is owing to the supposition we set out with, that the centripetal force does not act continually, but by separate strokes repeated at certain intervals of time. For upon this supposition, the length of AB , BC , CD , or DE , will always be equal to the space, which the body can describe in the interval of time between any one stroke of the centripetal force and the next that follows it. Because between any one of these strokes and the next, the body is acted upon only by the projectile force, and therefore will, by the first law of motion, move on in a right line during that interval. But, if we correct this supposition and make the case of the revolving body what in fact it is; that is, if we consider the centripetal force as a continued action; then the interval between any two strokes of it will be infinitely short; and consequently the space described by the revolving body during such an interval will be infinitely short too. That is, the sides AB , BC , CD , DE , &c. of the polygon will be infinitely short. And, as in the time of a whole revolution there must be innumerable such strokes of a continued centripetal force, these sides will be innumerable, as well as infinitely short. But a polygon consisting of innumerable infinitely short sides is a curve. And if the proposition is true in one sort of polygon, it will necessarily be true in every other sort. Therefore, when a body revolves in a curve, its orbit is always in the same plane with the line of projection and with the point to which the centripetal force tends.

159. *If a body revolves in a curve returning into itself, it describes, by a radius drawn to the point where the centripetal force acts, equal areas in equal times, and in unequal times areas proportional to the times.*

If a body revolves in the curve *mbnb*, Plat.V. fig. 10. and *F* is the point where the centripetal force acts, or the center towards which the body is constantly impelled; and if a radius or line as *Fm* is drawn from the center *F* to the body at *m*, and this radius or line is supposed to turn round upon the point *F* and to attend the body in its motion, so that when the body is at *l*, the radius is *Fl*, when the body is at *r*, the radius is *Fr*, when at *s*, *Fs*, and when at *n*, *Fn*. Then it is obvious, that whilst the body moves from *m* to *l* the radius *Fm* by moving at the same time from *Fm* to *Fl* will describe or sweep the area of the mixt triangle *mFl*: in like manner, whilst the body moves from *l* to *r*, a radius drawn from the body to *F* describes or sweeps the area or space *lFr*: whilst the body moves from *r* to *s*, the radius describes or passes over the area *rFs*: and whilst the body moves from *s* to *n*, the radius describes the area *sFn*. The proposition affirms, that the areas, which the radius thus describes in equal times, are equal to each other. For instance, if the body takes up just as much time to move from *m* to *l* as it does to move from *r* to *s*, then *ml* and *rs* are the arcs described in equal times by the body, and *mFl* and *rFs* are the areas described in equal times by a radius drawn to the center; now we are not to prove that the arcs *ml* and *rs* are equal, though they are described in equal times, for these arcs are lines and not areas; and in the figure before us they are plainly unequal. But we are to prove that the area or space contained within the triangle *mFl*, which is swept by the radius in a certain time, is equal to the area or space contained within the triangle *rFs*, which is swept by the radius in an equal time. The proposition farther affirms that the areas described in unequal times by the radius round the point *F* are proportional to the times, in which they are described. By this we mean, that if a radius or line drawn from the revolving body to the point *F*, where the centripetal force acts, describes the area *mFl* in a certain time; then in twice that time, it will describe, an area double to *mFl*; in thrice that time it will describe a triple area; and in all cases the area described by such a radius will be greater or less in proportion as the time is greater or less, which is taken up to describe it.

Having thus explained the meaning of the proposition, we may apply it to an instance or two, that the reader may understand the use of it, before we go on to prove it. If *mbnb* is the moons orbit, and *F* the place of the earth round which the moon revolves. Then suppose the moon

to be heavy as other bodies are and to gravitate towards the earth; and suppose likewise that its weight or gravity, is the centripetal force, by which it is constantly impelled towards the earth, and made to describe an orbit instead of moving on in a right line, as it would do, if no centripetal force acted upon it. A right line or radius drawn from the moon to the earth must, upon these suppositions, describe round the earth equal areas in equal times. In like manner, if F is the sun and $mbnb$ is the orbit of any planet revolving round the sun; upon supposition, that the planet is continually impelled towards the sun by its own weight or by the force of gravity, it will follow, if this proposition is true, that such a planet by a radius drawn to the sun describes equal spaces round it in equal times. A farther use of this proposition will appear in proposition 161.

We have seen already that as the revolving body approaches the point F it moves faster than it did, when it was at m . Therefore if two arcs as ml and rs are described in equal times, it is evident that ml the base of the triangle mFl , where the body moves slowly, must be shorter than rs the base of the triangle rFs where the body moves faster. For this reason though the triangle mFl is higher than the triangle or area rFs , yet the former will be narrower, or will have a shorter base in proportion; and consequently one of these areas or triangular spaces will be equal to the other.

To prove this more exactly let us consider $ABCDE$, Plat. VI. fig. 4. as part of an orbit described by a body, that revolves round the point S by means of a centripetal force, which impells the body towards that point. And let us suppose here, as we did in the last proposition, that the action of this centripetal force is not continued, but that its impulses are repeated after certain intervals of time. So that the body may move from A to B in a strait line, and not be impelled towards S by the centripetal force that acts there, till it is arrived at B . The body left to itself would move on from B , by the first law of motion, in the strait line Bc . But if it is there impelled towards S by the centripetal force, it will describe neither Bc which is in the direction of the projectile force, nor BG , which is in the direction of the centripetal force; but BC which is the diagonal of a parallelogram, whose two sides are Bc and BG the two forces that joyntly act upon the body. When the body has been just as long moving in the direction BC as it was before in the direction AB , that is, after an equal interval of time, suppose another impulse of the centripetal force to be exerted upon it and to draw it, when it is arrived at C , out of the direction Cd into the direction CD . And when, after an equal interval of time is passed, the body arrives at D ,

let it be impelled towards S by another stroke of the centripetal force, and be made to describe DE . Since $AB, BC, CD, DE, \&c.$ are the lines described in equal times by the revolving body, the areas swept round S in equal times by a radius drawn from the body to S are $ASB, BSC, CSD, DSE, \&c.$ These areas we must prove to be equal to each other upon supposition, that S is the point towards which the body is impelled. We will therefore shew that upon this supposition, the triangle ASB is equal to the triangle BSC . Now AB is the space, which the body describes in a certain time by the projectile force only: and if only this force acted upon the body, it would go strait forwards in the same right line and would in an equal time describe Bc , which is equal to AB . For, by the first law of motion, a body acted upon by only one force continues to move on uniformly in the same right line. And if it moves on the same pace after it is passed B , that it did in coming to it, the space AB described in a certain time must necessarily be equal to the space Bc , which the body left to itself, or not impelled by the centripetal force, would describe in an equal time. From hence it follows that the triangle ASB is equal to the triangle BSc : for their bases AB and Bc are equal, and as they are both terminated at the same point S their perpendicular heights are the same. Eucl. b. I. prop. 38. Now the centripetal force, which impelled the body, when it was at B , is supposed to act in the direction BS : and let the centripetal force in this direction be to the projectile force in the direction AZ as BG to Bc . The body impelled by both forces together is made to describe BC the diagonal of a parallelogram of which Bc is one side, and the other side or BG is in the direction BS . But when one side is BG and the diagonal is BC , the opposite side or cC is parallel to BG . Eucl. b. I. prop. 34. Therefore cC and BS , of which BG is a part are likewise parallel to each other. But BS is the base both of the triangle BSC and of the triangle BSc ; and these two triangles are both of them between the lines BS and Cc , which we have just proved to be parallel to each other. Therefore, since all triangles upon the same base and between the same parallels are equal, Eucl. b. I. prop. 37. the triangle BSC is equal to the triangle BSc . And since ASB has been likewise proved to be equal to BSc , it follows that ASB and BSC , the areas described in equal times round S , are equal to each other; because each of them is equal to BSc .

When the body is arrived at C , the projectile force, if it was left to itself, would carry it strait on from thence in the same direction, in which it was moving before: and, because all motion arising from the single action of one force is uniform, by proposition 11, the body, after it is
 passed

passed C, would take up just as much time to describe Cd equal to BC , as it took up before in describing BC . That is to say, BC and Cd the spaces described in equal times by the projectile motion are equal; because the body, when it is once put into motion by the projectile force, will continue always to move on the same pace, and consequently will describe equal spaces in equal times. But since BC and Cd the bases of the two triangles BSC and CSd are equal, and the height of these two triangles is the same, as they are both terminated at the same point S , it follows that the triangles are equal. Eucl. b. I. prop. 38. Now the time of the bodys motion from B to C being the interval between two strokes of the centripetal force; upon its arrival at C it will be impelled towards S ; as S by the supposition is the point to which this force tends. Let the centripetal force in the direction CS be to the projectile force in the direction Cd as CH to Cd : and the body acted upon by both forces together will describe the diagonal CD of the parallelogram whose two sides are CH and Cd , by proposition 16, and the area passed over by a radius drawn from the body to the center u will be CSD . Now the triangle CSD is equal to the triangle CSd . For, since $HCdD$ is a parallelogram, the two opposite sides are parallel; that is, Dd is parallel to HC ; and consequently Dd is parallel to SC , of which HC is a part. Therefore the triangles CSD and CSd , being between the two lines Dd and CS , or between the same parallels, and being likewise upon the same base SC , are equal. But BSC has likewise been proved to be equal to CSd . Therefore BSC the area swept by the radius, whilst the body is passing from B to C , is equal to CSD the area swept by the same radius whilst the body is passing from C to D . And consequently, since the body is just as long in passing from B to C as from C to D , the areas swept in equal times round S , the point where the centripetal force is placed, are equal to each other.

What has here been proved to be true of a body moving in a polygon, if the centripetal force acts at certain intervals, is equally true, when a body moves in a curve; since a curve may be considered as a polygon consisting of innumerable infinitely small sides. And if the intervals between the several impulses of the centripetal force are infinitely short, that is, if the centripetal force is continued, or acts constantly upon the revolving body in every instant of its motion, then the sides AB , BC , CD , DE , &c, must be infinitely short. Because these are the lines described by the revolving body between any two impulses of the centripetal force: which lines must necessarily be infinitely short, when these impulses are continued, or succeed so closely to each other that no time

passes between any one of them and the next that follows it. And when the sides are infinitely short, there will of course be innumerable such sides in the whole orbit described by the body.

But after we have shewn that the areas described by the radius in equal times round the point S , where the centripetal force acts, are equal; it follows that in double times a double area will be described, and in triple times a triple area, that is, that the areas described in unequal times are greater in proportion as the times of describing them are greater, or less as the times are less.

160. *When a body by a radius drawn to any immoveable point within its orbit describes equal areas in equal times round that point, or in unequal times areas proportional to the times, it is impelled towards that point by the centripetal force, which retains it in its orbit.*

This proposition is the converse of the foregoing one. In the last proposition we supposed that S , Plat. VI. fig. 4. was the point towards which the body is made to tend by the centripetal force: and from thence we proved that the body by a radius drawn to S must necessarily describe equal areas in equal times round S . But in this proposition we suppose that equal areas are described in equal times round S ; and from thence we are to prove that S is the point towards which the body is impelled by the centripetal force.

The use, which is made of this proposition, will help to explaine the meaning of it. If from observation it appears that the moon by a radius drawn to the earth describes equal areas in equal times; and if the earth is an immoveable point within the moons orbit; then, from this proposition, it will follow that the moon is retained in its orbit and kept from running off in a tangent by a centripetal force, which acts in the earth and constantly impells it thither.

In like manner, if it is found by observation, that all the planets by a radius drawn to the sun describe equal areas in equal times; and if the sun is an immoveable point within the respective orbits of the planets; then, from this proposition, we may shew that the planets are constantly urged towards the sun by a centripetal force; or that the force, which retains them in their orbits, acts in the sun and by constantly impelling the planets makes them tend thither.

If a body revolves in the orbit $m b n b$, Plat. V. fig. 10. and describes by a radius drawn to F equal areas in equal times: for instance, if it is just as long in moving from m to l as from r to s ; and $m F l$, which is the area passed over, whilst the body moves from m to l , is equal to $r F s$.
the

the area passed over, whilst it moves for an equal time from r to s ; then since we have seen by the last proposition that a centripetal force acting in F or constantly impelling the body thither will make the body move in such a manner, it is probable, that where there is such a motion as this it is owing to a centripetal force acting in the point F .

This we may demonstrate more exactly in the following manner. Let $AB, BC, CD, DE, \&c$, Plat. VI. fig. 4. be the lines which a revolving body describes in equal times, that is, suppose the body to be just as long in moving from A to B as from B to C , or from C to D , or from D to E . Suppose farther that the centripetal force acts at certain intervals of time, and that one stroke of it impells the body when at B , another when at C , a third when at D . And let S be the point round which equal areas are described in equal times. Then from the supposition it follows, that, since the body takes up an equal time to move from A to B , and from B to C ; the area swept by the radius whilst the body is moving from A to B , must be equal to the area swept by it, whilst the body is moving from B to C ; that is, the triangular space ASB by the supposition is equal to the triangular space BSC . Now as only the projectile force acts upon the body, whilst it moves from A to B , the line AB will be described with an uniform velocity, by the first law of motion, proposition 11. And if no centripetal force was to act upon the body, when it arrives at C , it would continue to move on uniformly as before in the same right line, and in an equal time would describe the line Bc equal to the line AB . From the equality of the two lines AB and Bc , which are the bases of the triangles ASB and BSc , it follows, that these two triangles, being both of the same height, are equal to each other. Eucl. b. I. prop. 38. But ASB and BSC are equal by the supposition. Therefore BSc and BSC are equal. Now BSC and BSc because they are upon the same base SB , and are equal to each other, must be between the same parallels. Eucl. b. I. prop. 39. But they are between the lines Cc and SB ; and consequently Cc and SB are parallel to each other. When the body arrived at B , if it had been left to itself, the motion communicated by the projectile force would have carried it on uniformly from B to c . But the centripetal force impelled it at B and made it move in the line BC instead of Bc . Since therefore the body when at B , was acted upon by two forces, the projectile force in the line Bc and the centripetal force, and these two forces together made it describe BC , it follows that BC is the diagonal of a parallelogram, of which Bc is one side, and the centripetal force acts at B in the direction of the other side. The question then is, in what direction this other
side.

side of the parallelogram lies at B ; for when this is determined the direction of the centripetal force will be known. Now in a parallelogram, whose diagonal is BC and one of whose sides is Bc, Cc must be another : and as the opposite sides of a parallelogram are always parallel to each other, the direction of the centripetal force, when the body was at B, must be parallel to cC. But, from the description of equal areas round the point S, we have just now proved that cC and BS are parallel to each other. Therefore from the description of equal areas round S it follows that the centripetal force, when the body is at B acts in the direction BS or makes the body tend towards S.

When the body is moving in the direction BC and arrives at C ; it would, if no new force was to act upon it, go on uniformly from C to d in the same right line, in which it was moving before. But at C another impulse of the centripetal force changes its direction and makes it move in the line CD. Therefore CD is the diagonal of a parallelogram, and since Cd is one of its sides, dD must be another. And, when the body was at C the centripetal force impelled it in a direction parallel to this other side or to dD. But, because equal areas are described in equal times round S, CS is parallel to dD ; therefore the line CS is parallel to dD, and consequently, the centripetal force impelled the body, when at C in the direction CS or made it tend towards S. The same may be shewn in every other point of its orbit. Therefore the centripetal force tends to that point round which the body by a radius drawn thither describes equal areas in equal times. In what manner it follows from the description of equal areas in equal times round S, that dD is parallel to CS has been shewn already.

But to make it the more intelligible and to fix it the better in the readers memory ; we will go over the proof of it again in a different manner. BC and CD are described by the body in equal times, and therefore BSC and CSD the areas described in the mean time by a radius drawn to S are equal by the supposition. But since CD is the diagonal, and Cd is one side of the parallelogram CdDH, it follows, by proposition 16, that the body describes this diagonal CD by the centripetal and projectile forces together, in the same time that it would describe Cd by the projectile force alone. But since BC and CD are described in equal times, and likewise CD and Cd, it follows that BC and Cd would be described in equal times, if the body was left to be carryed on by the projectile force only : and as the velocity arising from the action of this one force is uniform, the spaces described by such a velocity in equal times must be equal. Therefore BC is equal to Cd. From hence it follows that, since BC
and

and Cd the two bases of the triangles BSC and CSd are equal and since these triangles have the same height, their areas are equal, or BSC is equal to CSd . But BSC is likewise equal to CSD by the supposition. Therefore CSD is equal to CSd . Now two triangles such as CSD and CSd upon the same base SC and equal to each other are between the same parallels, Euc. b. I. prop. 39. But CSD and CSd are between the lines dD and CS . Therefore dD and CS are parallel. And when the body was at C the centripetal force impelled it in a line parallel to dD , that is in the line CS towards the point S , round which it describes equal areas in equal lines.

What has thus been proved in this part of the orbit $ABCD$, might be shewn to be equally true in every other part of the orbit. And though $ABCD$ is a polygon and not a curve; yet in a curve likewise, if equal areas are described in equal times by a radius drawn to S , S is the point in which the centripetal force acts. Because a curve is only a polygon of innumerable infinitely short sides. The line here described, as has been observed already under the two foregoing propositions, is a polygon and not a curve for no other reason, but because we supposed the several strokes of the centripetal force to be repeated separately from each other and after certain intervals of time: so that, if the body begins to move from A , there should be time enough allowed for it to arrive at B before any impulse of that force urged it towards S , and after this stroke there should be an interval of time long enough for the body to move from B to C before it is impelled by a second: and the third stroke should not act upon it till it had described CD . Now if these intervals are infinitely short, that is, if the action of the centripetal force is continued, which is the case when the body revolves in a curve; these impulses will be infinitely near to each other: and consequently the spaces AB , BC , CD , &c. which are described between any one impulse and the next will be infinitely short. And since in the time of a whole revolution there must be innumerable strokes or impulses of the centripetal force, if they are thus repeated without any interval between them; it follows that when these sides AB , BC , CD , &c. are infinitely short, there will be innumerable of them, and the polygon will be changed into a curve.

161. *The velocity of a body, which is impelled towards an immoveable center, and revolves in spaces where it meets with no resistance, is in different parts of the orbit inversely as a perpendicular let fall from that center upon a tangent to that part.*

Let

Let $mbnb$, Plat.V. fig. 10. represent the orbit, which any planet describes round the sun at rest in the point F . Then if the planet is constantly attracted by the sun or impelled towards it by the force of gravity, and moves in a vacuum so as to meet with no resistance; the velocity of the planet in any one part of its orbit as m may be thus compared with its velocity in any other part as r . To the point m draw the tangent tmt , and to the point r draw the tangent trt , and from F draw Fm perpendicular to tmt , and Fe perpendicular to trt . The velocities at these two points m and r will be to each other as these perpendiculars Fm and Fe inversely: by which we mean that the velocity of the planet at m is to its velocity at r as Fm to Fe inverted, or as Fe to Fm . That is to say the velocity of it at m is as much less than at r as Fm is greater than Fe , or as Fe is less than Fm .

This follows from proposition 159. For since a radius drawn from the body to the point F describes equal areas in equal times round F , mFl and rFs being supposed to be areas described in equal times are two equal triangles. Therefore as much higher as mFl is than rFs , so much broader rFs is than mFl . For if rFs had not in breadth what it wants in height it could not be equal to mFl . Now the height of mFl means its perpendicular height or Fm , and its breadth is its base ml . So likewise the height of rFs is its perpendicular height Fe , and its breadth is its base rs . Therefore, because the triangles are equal, ml is as much less than rs as Fm is greater than Fe . Or the bases ml and rs are to each other as Fm to Fe inverted, that is as Fe to Fm . But the velocity of the body either at m or at r is proportional to the space described by it in equal times. And since ml and rs are the arcs described by the body, whilst the radius drawn to the center F describes equal areas; these arcs or the bases ml and rs are the spaces described by the body in equal times; and consequently the velocities at m and r must be to each other in the same proportion as these bases ml and rs . But the bases are to each other inversely as the perpendiculars let fall from F to the tangents at the points m and r . Therefore the velocities of the body at m and r are likewise inversely as those perpendiculars.

Here it will not be improper to observe that, if a body revolves in an elliptical orbit, such as $mbnb$, and the centripetal force impells it towards one of the focuses F , the tangent is perpendicular to the line in which this force acts, when the body is either at the greatest distance m , or at the least distance n . Therefore at these two distances, the perpendiculars are the distances themselves: and since the velocity in every point of the orbit is inversely as the perpendicular to a tangent at that point, the respective

respective velocities of the body at the greatest and least distances will be as Fm to Fn inversely, or inversely as the distances of the revolving body from F .

162. *In circles, and in all curves which have the same curvature with circles, the nascent or evanescent subtense of the angle of contact is as the square of the conterminous arc.*

To the point A Plat. VI. fig. 5. in the semicircle ADC draw the tangent AB . Then the angle BAD , which this tangent makes with the curve at the point A , is called the angle of contact. Draw BD and likewise bd parallel to AC the diameter. Either of these lines BD or bd is called the subtense of the angle of contact; for both of them, though at different distances from A , subtend the angle at A . AD an arc, which begins at the point of contact A and is terminated at BD , is called the conterminous arc to the subtense BD . And Ad is the conterminous arc to the subtense bd . The right line AD , which is the chord of the arc AD , is called the conterminous chord to the subtense BD . And the right line Ad is called the conterminous chord to the subtense bd . From D draw the line DC ; and from d draw the line dC . If these two lines DC and dC are conceived to turn round upon the point C as upon a center, so that the two points D and d , and with them the two subtenses BD and bd , may approach towards A ; then it is evident that as these subtenses BD and bd come nearer to A , they will be shorter, and will, as they keep moving on, continue to decrease till at last they become points, coincide with A , and so vanish. The ultimate ratio of these subtenses is their last proportion, or that proportion, which they bear to each other when they are evanescent, that is, when they are the nearest possible to A , or at the instant when they vanish.

The proposition affirms that, whatever may be the proportion of BD and bd to each other at any other time, yet ultimately, at the instant they vanish, or when they are the nearest possible to A , BD is to bd as the square of the arc AD is to the square of the arc Ad . For, from what has been said, it appears that this is what we mean by affirming that the evanescent subtenses of the angle of contact are ultimately as the squares of the conterminous arcs.

In order to prove this we must first shew that the subtenses BD and bd , not only when they vanish, but at all times, and in all states or conditions, are to each other as the square of the chord or right line AD to the square of the chord or right line Ad . Let ED be drawn parallel to AB , and ed parallel to Ab . Then, because AB is a tangent at the point A and consequently is perpendicular to the diameter AC , Eucl. b. III.

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prop.

prop. 18. ED which is parallel to AB must likewise be perpendicular to AC. And the same may be shewn to be true of ed for the same reason. Now the angle ADC, and likewise the angle $A d C$, are right ones: because they are both in a semicircle. Eucl. b. III. prop. 31. Therefore in the right-angled triangle ADC the line DE is drawn from the right angle at D perpendicular to AC the base, and consequently AED one of the smaller triangles into which this line divides ADC is similar to ADC. Eucl. b. VI. prop. 8. So that AE the shortest side of the triangle AED bears the same proportion to AD its longest side, that the same AD, which is the shortest side of the triangle ADE, bears to AC its longest side. Eucl. b. VI. prop. 4. That is to say, AE is to AD, as AD to AC. Or, AE, AD, and AC, are in continued geometrical progression. Therefore AE multiplied into AC is always equal to the square of AD. Eucl. b. VI. prop. 17. But AE and BD, being parallel to each other, by the construction of the figure, are the opposite sides of a parallelogram and consequently are equal to each other, Eucl. b. I. prop. 34. Therefore BD multiplied into AC is equal to AE multiplied into AC, or, by what has just been proved, to the square of AD. That is, the subtense of the angle of contact multiplied into the diameter is always equal to the square of the conterminous chord. Now, if two quantities are always equal to each other, when one of the quantities decreases, or is by any means made less, the other must necessarily decrease in the same proportion. For if one of them was to decrease in a greater proportion than the other does, then that quantity which decreases the fastest would become less than the other; whereas they are supposed to be always equal. But BD multiplied into AC is equal to the square of AD: and for the same reason bd multiplied into AC is equal to the square of Ad . That is, when the subtense is BD, the product of it multiplied into the diameter AC is equal to the square of the conterminous chord AD. And when the subtense is nearer to A, and consequently is less than BD, as suppose it equal to bd , still bd the subtense multiplied into AC the diameter is equal to the square of the conterminous chord Ad . Therefore the product arising from BD multiplied into AC always decreases in the same proportion with the square of AD. But in all multiplication, where the multiplicand is given, the product will be proportional to the multiplier. And in the product arising either from BD multiplied into AC, or from bd multiplied into AC, or from any subtense whatever multiplied into the diameter, this diameter, which is the multiplicand, is a given quantity, and consequently the product will be proportional to the subtense. Thus BD multiplied into AC, is to bd multiplied

Fig. 2.

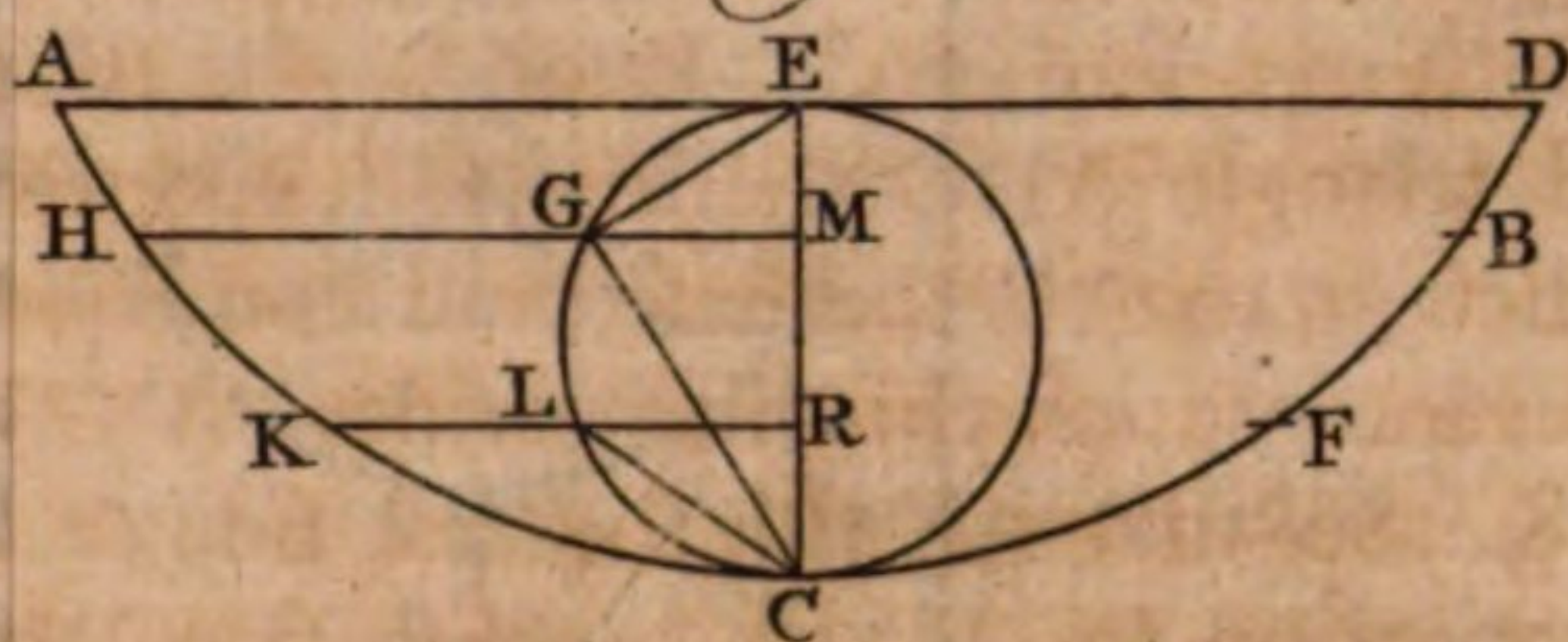


Fig. 3.

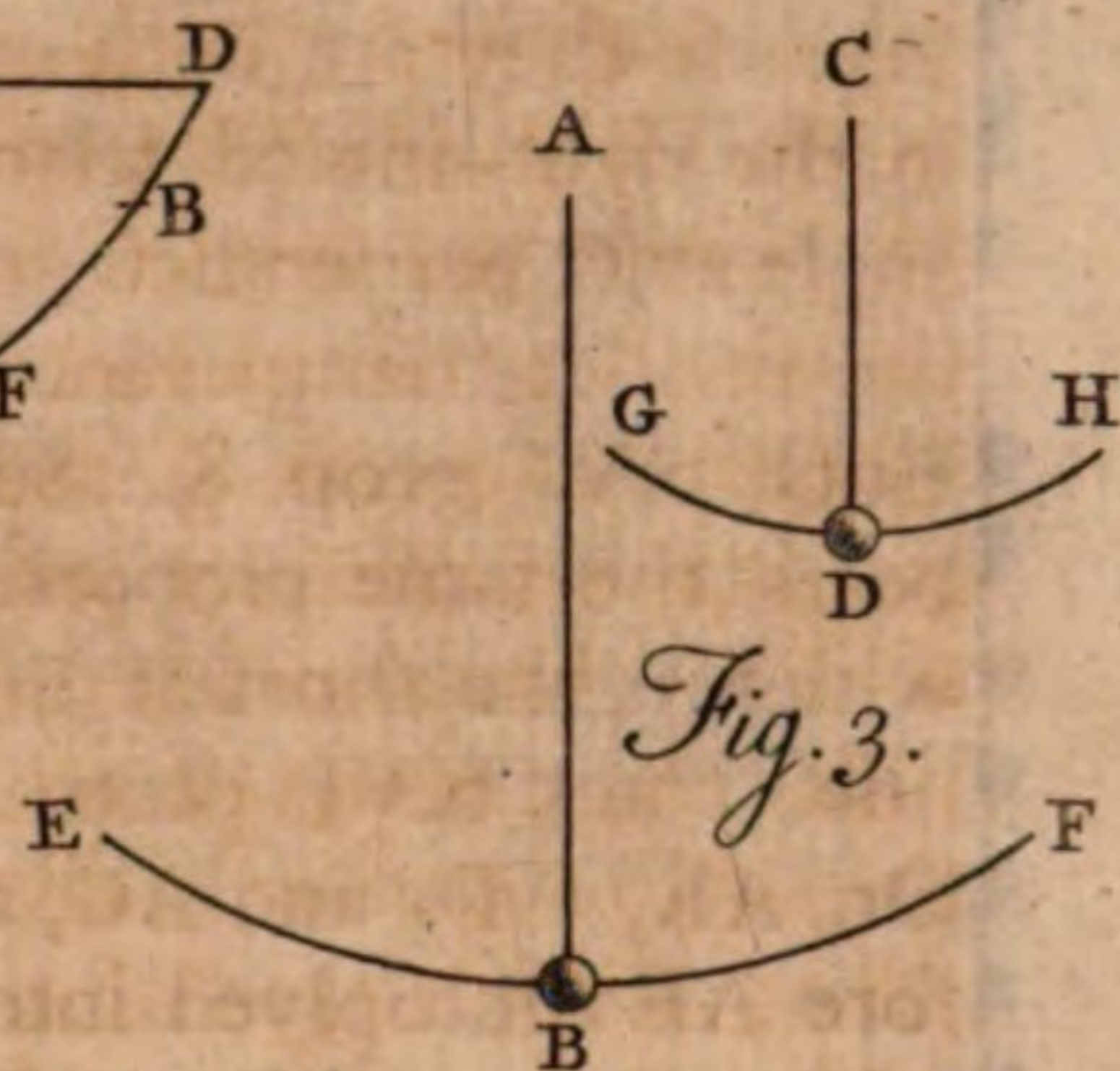


Fig. 5.

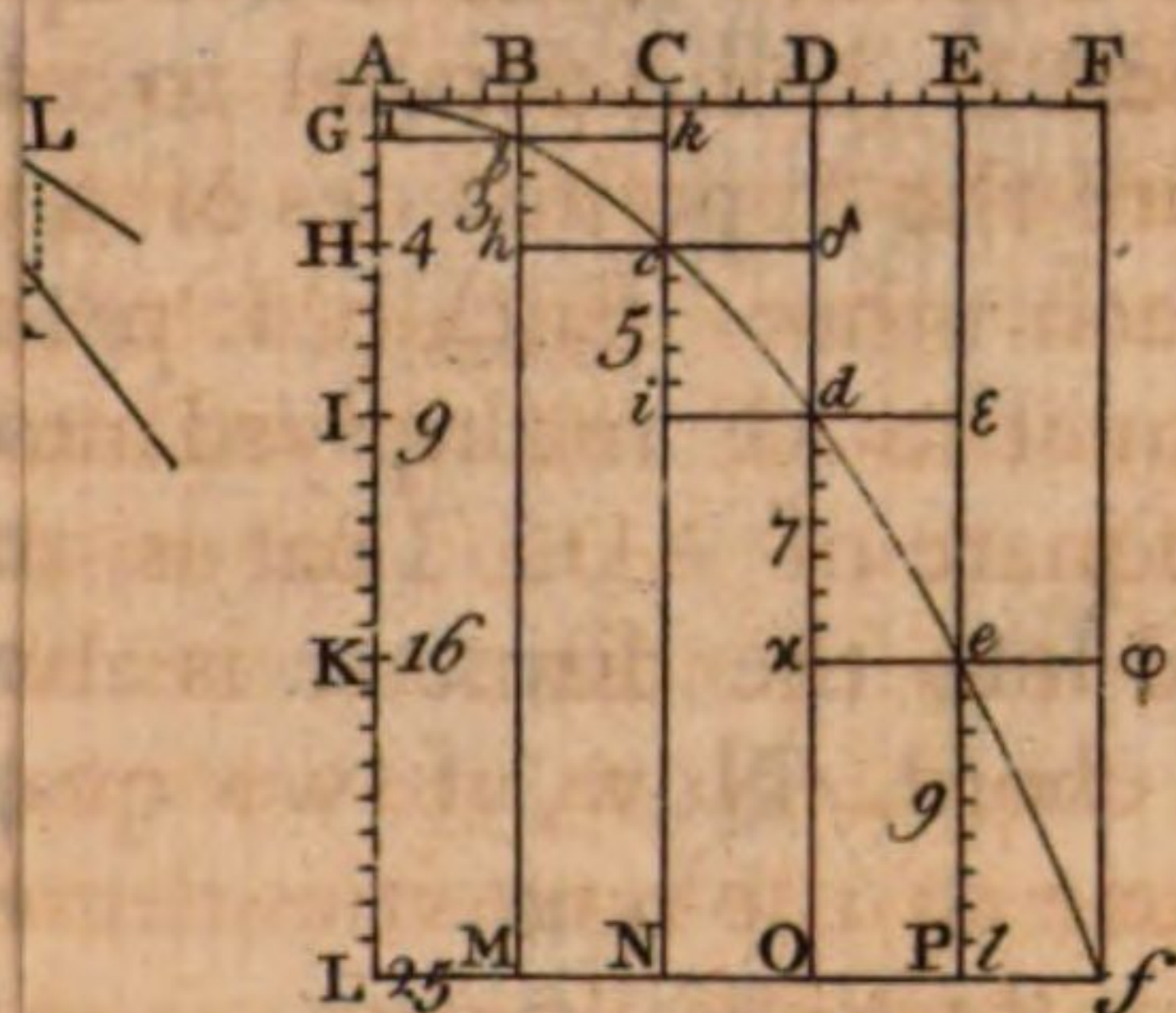


Fig. 6.

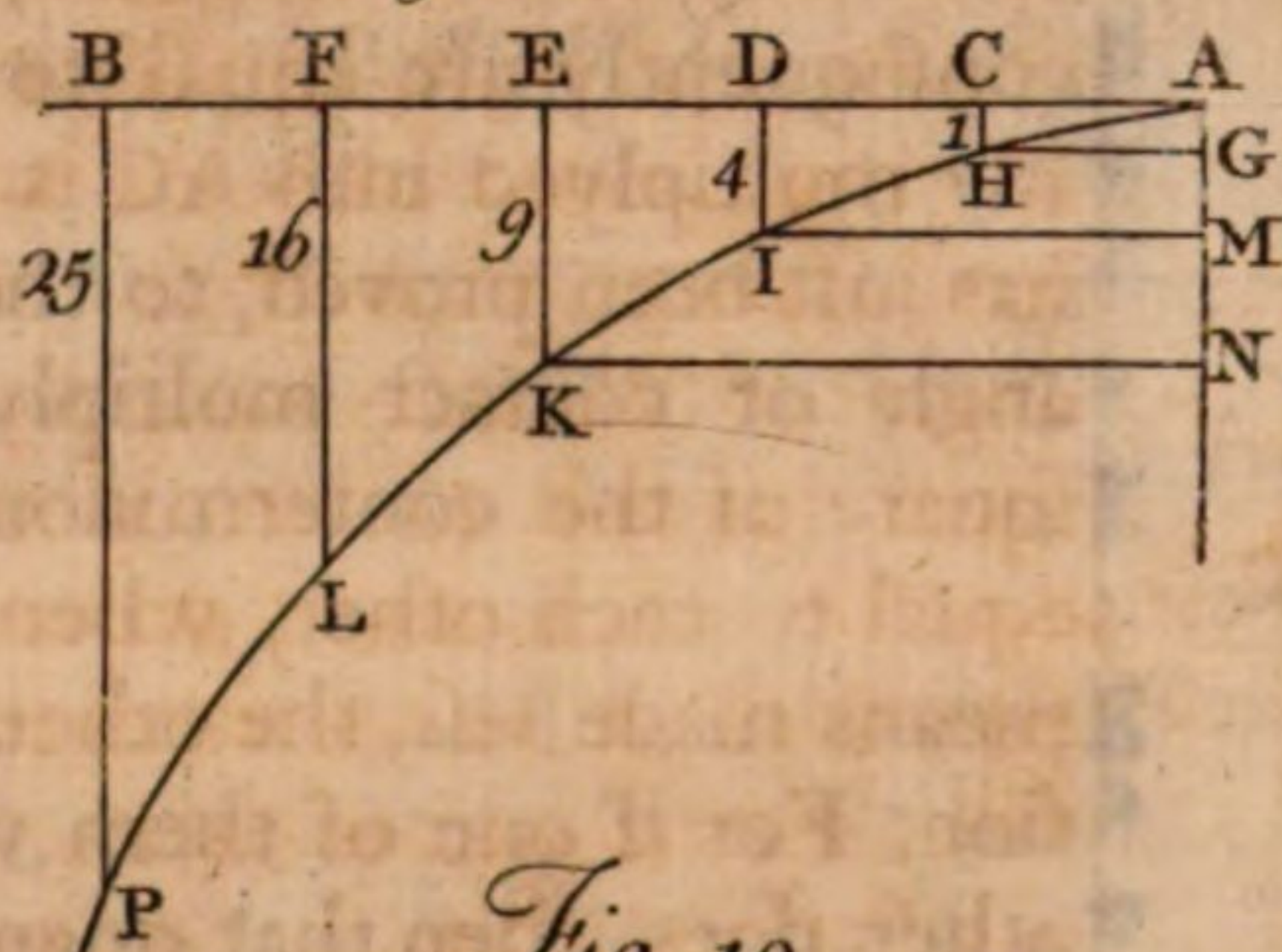


Fig. 10.

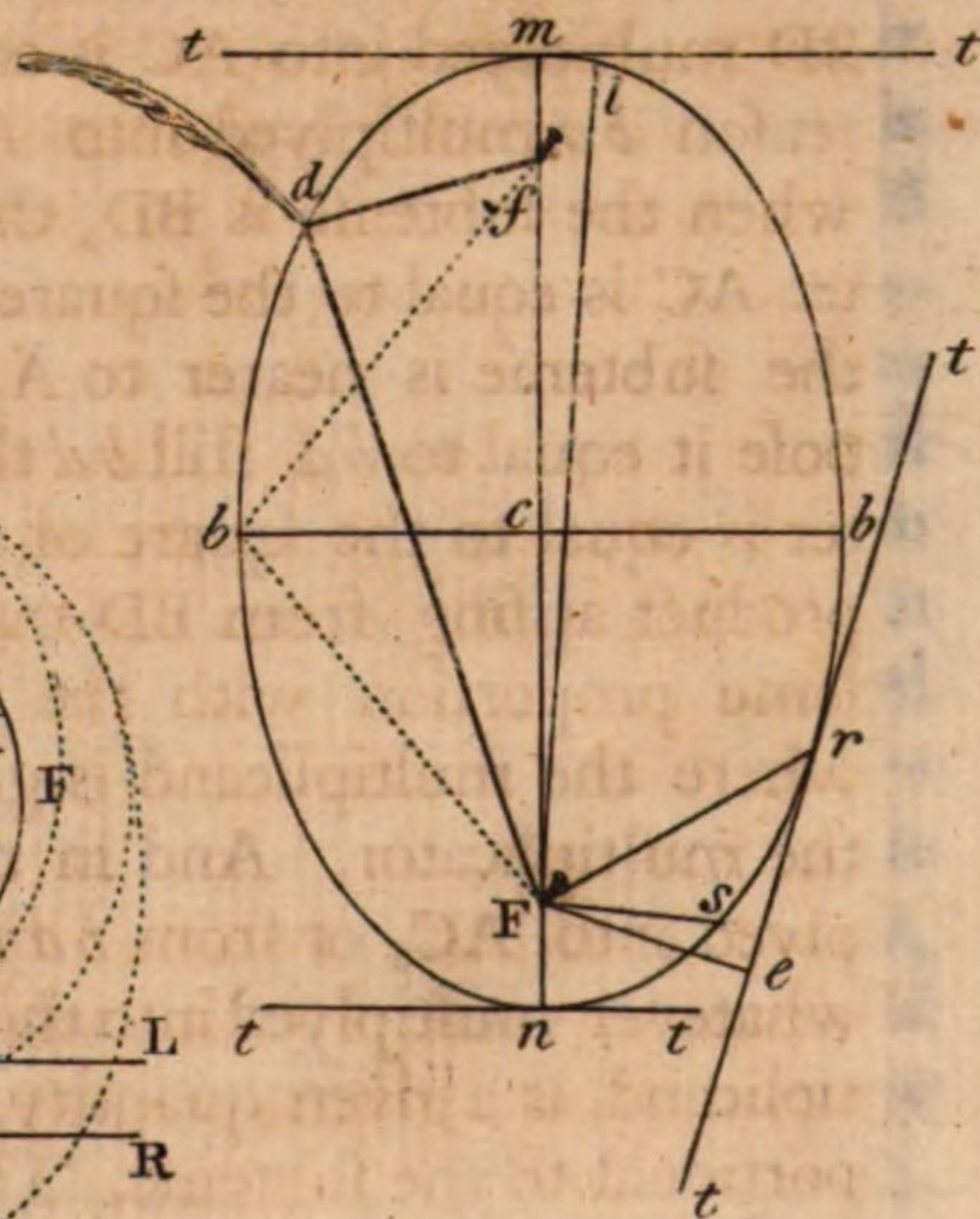
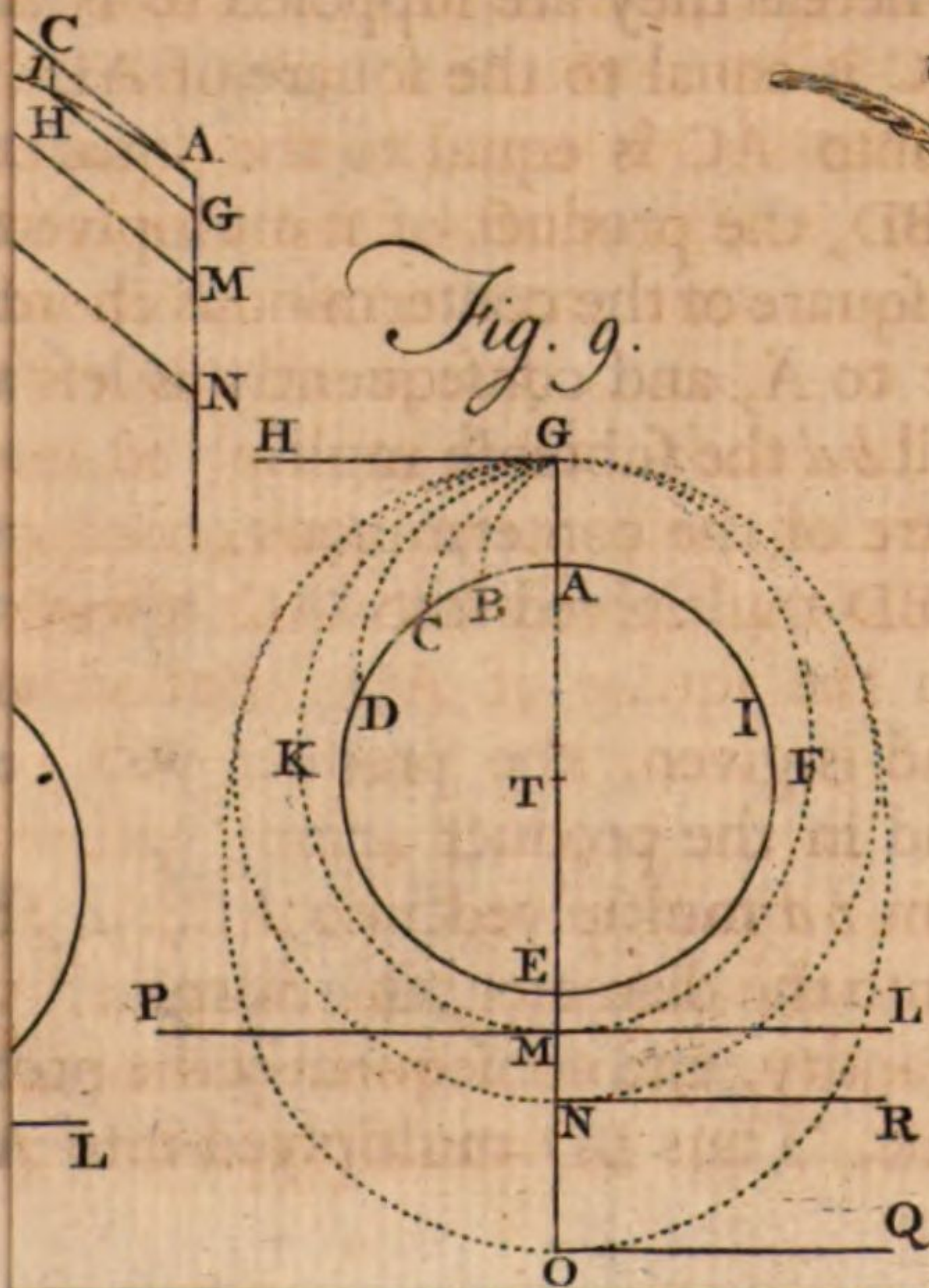


Fig. 9.



tiplied into AC , as BD to bd . But these products of the diameter multiplied into the subtenses are proportional to the squares of the conterminous chords. BD multiplied into AC bears the same proportion to bd multiplied into AC , that the square of AD bears to the square of Ad . Therefore, because AC is a given quantity, BD is to bd as the square of AD to the square of Ad . That is, the subtenses are proportional to the squares of the conterminous chords.

But when BD and bd are evanescent, that is, when they are the nearest possible to A , or at the instant when they vanish, the arcs AD and Ad are so small as not to differ from right lines or from their respective chords. Therefore whatever is the proportion of the chords the same is ultimately the proportion of the conterminous arcs. And since the subtenses of the angle of contact are universally as the squares of the conterminous chords; it follows, that, when they are evanescent they are, ultimately, as the squares of the conterminous arcs. In the construction of this figure, the subtenses BD and bd were supposed to be drawn parallel to the diameter AC , and the foregoing demonstration depends upon this supposition. But yet, though the subtenses should not be parallel to the diameter, we may still demonstrate that they are ultimately as the squares of the conterminous arcs, provided they are only parallel to each other.

By what has been already proved mn and fg , Plat. VI. fig. 3. being subtenses of the angle of contact at B drawn parallel to the diameter BD are ultimately as the squares of the centerminous arcs Bn and Bg . Draw og and an two subtenses, which are parallel to each other, but not to the diameter; these subtenses are likewise ultimately as the squares of their conterminous arcs. For since Ba is a tangent at the point B , and is therefore perpendicular to BD : Eucl. b. III. prop. 18. and since mn and fg are parallel to BD , it follows that mn and fg are likewise perpendicular to BD , and consequently the angles ofg and amn are both of them right ones and equal to each other. But because og and an are parallel to each other, by the construction, the angle fog is equal to the angle man , for one is the external angle and the other is the internal opposite angle on the same side made by the line Ba crossing the two parallel lines og and an . Eucl. b. I. prop. 29. And since in the triangle fgo there are two angles respectively equal to two angles in the triangle mna , and all the three angles of each are only equal to two right ones, it follows, that the third angle at g in one of them is equal to the third angle at n in the other. Eucl. b. I. prop. 32. coroll. 2. But similar or equiangular triangles have their sides proportional. Eucl. b. VI. prop. 4. There-

Therefore an is to og as mn is to fg . And we have already proved that mn is ultimately to fg as the squares of the conterminous arcs. From whence it follows that an is likewise ultimately to og as the squares of the conterminous arcs Bn and Bg .

Or lastly, suppose an and og were both of them directed to C the center of the circle. In this case they would each of them be a semidiameter continued from g and n respectively to the tangent Ba and would neither be parallel to each other nor to BD in any but their evanescent state. But when they are the nearest possible to B , though they are directed to C , they will be parallel to each other: for when an and og vanish at B , they must be parallel to each other; because then they both coincide with the same point B in the same right line BC , and therefore, by the foregoing demonstration, they will be ultimately as the squares of the conterminous arcs.

When D and d coincide with A , Plat.VI. fig. 5. suppose them to set out from A and to move back again, the lines DC and dC turning still upon the point C as they did before, only the contrary way. In this case, as D and d move away from A , the subtenses, which were nothing when they coincided with A , will begin to exist, and the first proportion which they bear to each other, when they thus begin, is their prime ratio in their nascent state. Now the reader will easily conceive that they begin, when D and d are thus moving from A , in the same proportion that they before ended, when D and d were considered as moving towards A . Therefore the nascent as well as the evanescent subtense of the angle of contact is as the square of the conterminous arc.

This proposition is true of other curves as well as circles, provided they are such that a circle may be drawn of the same curvature. Let Bg and Bn , Plat.VI. fig. 6. be nascent arcs in an ellipsis. The points g and n in the nascent arcs Bg and Bn are the nearest possible to B ; and therefore they are not only arcs in the ellipsis, but are likewise arcs in a circle which has the same curvature that the ellipsis has at B . But, since sn and rg , or the nascent subtenses in any circle, have been proved to be as the squares of the nascent arcs Bn and Bg , it follows that the same subtenses sn and rg , when those arcs are in an ellipsis, are likewise in the same proportion.

163. *The nascent or evanescent subtense of the angle of contact in circles is equal to the square of the conterminous arc divided by the diameter.*

BD , Plat.VI. fig. 5. is equal to the square of AD divided by AC ; if BD and AD are in their nascent state, that is, if the point D is supposed to be the nearest possible to A . We have already seen in the foregoing proposition,

proposition, that universally BD the subtense, AD the conterminous chord, and AC the diameter, are in continued geometrical progression; and that, when D is taken the nearest possible to A, the chord and arc AD do not differ from one another. Therefore the nascent subtense BD, the nascent arc AD, and the diameter AC are in continued geometrical progression. And consequently BD multiplied into AC is equal to the square of AD, Eucl. b. VI. prop. 17. that is, the product $BD \times AC$, where BD is the multiplicator and AC the multiplicand, $= AD^2$. But whenever two quantities are equal, if they are divided by equal divisors the quotients will be equal. Therefore if $BD \times AC$ and AD^2 are each of them divided by AC the quotients will be equal. But if $BD \times AC$ is divided by AC the quotient is BD: because when any product is divided by the multiplicand the quotient is the multiplicator. Therefore BD is equal to the square of AD divided by AC, that is $BD = \frac{AD^2}{AC}$.

164. *When bodies describe circles, and the centripetal forces tend to the centers of those circles; these forces in different circles, and likewise the centrifugal forces, are to one another as the squares of the arcs that the bodies describe in the same time, divided by the semidiameters of the circles.*

Let there be two bodies revolving round the center C; one in the circle BnD, Plat. VI. fig. 3. and the other in the circle R/E: and suppose that the centripetal force, which retains them in their respective orbits, impells them towards C. One of these bodies setting out from B would by the first law of motion go on in the right line Ba, if there was no centripetal force which acted upon it, and which by impelling it constantly towards C made it describe the curve Bn. When the body has moved in this curve through the arc Bg, or from B to g, the space through which it has been drawn from the tangent Ba towards the center C is fg: and consequently, since it is the centripetal force, which has thus drawn it, this force is proportional to the space which it has made the body describe, or proportional to fg. In like manner the body which revolves in the circular orbit R/E describes the arc R/ and does not run out from R in the tangent Rs, as it would have done by the action of the projectile force alone. And the centripetal force which has drawn it from this tangent must be proportional to the effect produced by it. But the effect produced by it, whilst the body is moving from R to l, is to impell or draw the body from the tangent through the space bl. Therefore the centripetal force is proportional to bl. Now if Bg and R/ are the

the arcs which these two bodies describe in equal times in their respective orbits, fg and hl , or the centripetal forces, are to each other as the square of Bg divided by BC is to the square of Rl divided by RC .

For suppose Bg and Rl to be the least arcs possible, or nascent arcs, then fg is equal to the square of Bg divided by BD ; and hl is equal to the square of Rl divided by RE . That is, $fg = \frac{Bg^2}{BD}$ and $hl = \frac{Rl^2}{RE}$, by proposition 163. Therefore fg is to hl as $\frac{Bg^2}{BD}$ is to $\frac{Rl^2}{RE}$. For since the first term or fg has been proved to be equal to the third or to $\frac{Bg^2}{BD}$ and the second term or hl has likewise been proved to be equal to the fourth or to $\frac{Rl^2}{RE}$, it follows that the first must bear the same proportion to the second that the third bears to the fourth. Now the proportion, which $\frac{Bg^2}{BD}$ and $\frac{Rl^2}{RE}$ bear to each other, will not be altered by changing the two divisors BD and RE for two other divisors that are in the same proportion. But BC , which is half BD , and RC , which is half RE , are in the same proportion to each other with BD and RE . Therefore $\frac{Bg^2}{BD}$ is to $\frac{Rl^2}{RE}$ as $\frac{Bg^2}{BC}$ is to $\frac{Rl^2}{RC}$. And consequently fg and hl are to each other as $\frac{Bg^2}{BC}$ to $\frac{Rl^2}{RC}$. That is the nascent subtenses fg and hl , and consequently the centripetal forces, by which the revolving bodies are impelled towards C , are as the squares of the arcs Bg and Rl divided respectively by the semi-diameters BC and RC .

This will be equally true though Bg and Rl are not nascent arcs or the smallest arcs possible, provided they are arcs, which the bodies describe in the same time, that is, provided one of the bodies describes the arc Bg , whilst the other is describing the arc Rl . For whatever are the velocities with which one body sets out from B and the other from R , the arcs which they describe will begin in the same proportion with those velocities; the nascent arc at B will be just as much greater or less than that at R , as the velocity of the body which sets out from B is greater or less than the velocity of the other body which sets out from R . But the arc, which one body describes in any given time, will likewise be to the arc, which the other describes in an equal time as the velocity of the former body is to the velocity of the latter. Therefore the arcs described in a given time are to one another as the nascent arcs; and

and consequently the squares of the arcs described in a given time are as the squares of the nascent arcs. But the centripetal forces have been proved to be as the squares of the nascent arcs divided by the semidiameters. Therefore the centripetal forces are likewise as the squares of the arcs described in a given time divided by the semidiameters.

We may apply this reasoning to another case by observing farther that the two curves BnD and R/E are similar and consequently BD is to RE , or BC to RC , as the whole curve BnD is to the whole curve R/E . Upon which account, we may change the two divisors BC and RC in the fractional quantities $\frac{Bg^a}{BC}$ and $\frac{R/l^a}{RC}$ for two others BnD and R/E without changing the proportion of the fractions. That is, $\frac{Bg^a}{BnD}$ bears the same proportion to $\frac{R/l^a}{R/E}$ that $\frac{Bg^a}{BC}$ bears to $\frac{R/l^a}{RC}$. But the centripetal forces, by which the two bodies are retained in their respective orbits, are to each other as $\frac{Bg^a}{BC}$ to $\frac{R/l^a}{RC}$ and consequently are likewise to each other as $\frac{Bg}{BnD}$ to $\frac{R/l}{R/E}$. That is the centripetal forces are as the squares of the arcs described in the same time divided respectively by the whole curves.

What has here been said concerning bodies that revolve in circles, may be applied to bodies that revolve in similar elliptical orbits. For if they describe equal areas in equal times round one of the focusses, that is, by proposition 160, if the centripetal force tends to one of the focusses, then in parts of their orbits, which are similar and likewise similarly placed in respect of that focus, the centripetal force will be as the square of the arc described in a given time divided by the distance from the focus. Suppose that one body revolves in the ellipsis BT , Plat. VI. fig. 6. and another in the ellipsis RP , and that each of them describes equal areas in equal times round F . Then, by proposition 160, the centripetal force tends to the point F , that is, the force, which retains each body in its orbit, impells it constantly towards this focus. Now if Bg and R/l are the smallest arcs possible, and are described in the same time, the spaces, through which the bodies are respectively impelled towards F , whilst one of them is describing the arc Bg and the other the arc R/l , are rg and hl . From whence it follows that these subtenses rg and hl are to each other as the centripetal forces, which make the bodies tend to F . And if B and R , have the same curvature or are similar places of these elliptical orbits, and are likewise situated in the same manner in respect of F , as they will be for instance when B is the greatest distance from F in one orbit, and R is the greatest distance from F in the other; then rg is to hl , as the square of the

the arc Bg divided by BF is to the square of the arc Rl divided by RF. That is, fg is to hl as $\frac{Bg^2}{BF}$ to $\frac{Rl^2}{RF}$. For from what has just been proved concerning bodies revolving in circular orbits, it follows that as B and R are similar parts of similar orbits and are likewise similarly situated in respect of F, rg is to hl as the square of Bg divided by the whole curve BgT to the square of Rl divided by the whole curve RlP. That is rg is to hl as $\frac{Bg^2}{BgT}$ is to $\frac{Rl^2}{RlP}$. But BF is to RF as BgT to RlP; because the curves are similar. Therefore in the fractional quantities $\frac{Bg^2}{BgT}$ and $\frac{Rl^2}{RlP}$ the two divisors BgT and RlP may be changed into BF and RF without affecting the proportion of the fractions. And consequently rg is to hl as $\frac{Bg^2}{BF}$ is to $\frac{Rl^2}{RF}$. That is, the centripetal forces are as the squares of the arcs described in the same time divided by the distances from F.

What has here been proved of the centripetal force need not be particularly shewn to be true likewise of the centrifugal force; since it cannot possibly be otherwise: because, by proposition 156, these two forces are always equal to one another.

165. *When bodies revolve in orbits of equal semidiameters, the central forces are proportional to the squares of the velocities with which the bodies move, or are as those squares directly.*

If two bodies revolve round different centers in equal orbits, or round the same center C, Plat. VI. fig. 3. in one and the same orbit, where BC is the common semidiameter of the orbit they describe; then, supposing their velocities to be unequal, the forces that impell them towards this center must be unequal too. A greater force will be requisite to retain that body in its orbit, which moves the faster of the two, than what will be sufficient to retain the other body, which moves slower, in the same or in an equal orbit. And supposing the velocity of one to be to the velocity of the other as 5 to 2, the centripetal forces must be as the squares of these velocities, that is, as the square of 5 to the square of 2, or as 25 to 4. And whatever may be the proportion of the velocities, with which the bodies revolve, the centripetal forces, when the semidiameters of the orbits are equal, must always be as the squares of the velocities.

The nascent arcs are proportional to the velocities, with which the bodies begin to move. Thus the nascent arc Bn, is to the nascent arc Bg, as the velocity of one body is to the velocity of the other. The centripetal forces, which have acted in the mean time upon these

two

two bodies to retain them in their orbit, are to each other as mn to fg , that is as the nascent subtenses of the angle of contact. But these subtenses in the same circle or in equal ones, by proposition 162, are to each other as the squares of the conterminous arcs Bn and Bg . And since these arcs are proportional to the velocities with which the bodies revolve; the squares of these arcs must be proportional to the squares of the velocities. Therefore the centripetal forces being as the nascent subtenses are likewise as the squares of the conterminous arcs, or as the squares of the velocities with which the bodies move.

We may likewise deduce this proposition from the foregoing one. For since the arcs described in equal times are in all cases as the velocity with which the body revolves, and since, by proposition 164, the centripetal force is as the square of the arc described in a given time divided by the semidiameter; it follows that the centripetal force is as the square of the velocity divided by the semidiameter. Thus if one body describes the arc Bn in the same time that another body describes the arc Bg , the centripetal forces, by which these two bodies are impelled to the center C , will be to each other, as the quotient arising from the square of Bn divided by BC is to the quotient arising from the square of Bg divided by BC , that is as $\frac{Bn^2}{BC}$ to $\frac{Bg^2}{BC}$. But in all division, where the divisor is given the quotients are as the dividends. Now BC , or the divisor is here a given quantity, because the bodies are supposed to revolve either in the same circle or in circles of equal semidiameters. Therefore the quotients, which are here expressed by the two fractions $\frac{Bn^2}{BC}$ and $\frac{Bg^2}{BC}$, are to each other as the dividends, or as Bn^2 to Bg^2 , which are the numerators of those fractions. And consequently the centripetal forces are to each other as the square of Bn to the square of Bg , that is, as the squares of the arcs described in a given time, or as the squares of the velocities with which the bodies revolve.

The reader must remember that when we have proved a proposition to be true of the centripetal force, it must necessarily be true of both the central forces, the centrifugal as well as the centripetal; because one of these forces is equal to the other.

166. *When bodies revolve with equal velocities in unequal circles, the central forces are inversely as the semidiameters of the circles.*

If one body revolves in the orbit BnD , Plat. VI. fig. 3. and another in the lesser orbit R/E the centripetal force, which retains the former body
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in the orbit BnD is to the centripetal force, which retains the latter in the orbit R/E , as BC to RC inversely, or as BC to RC inverted, that is as RC to BC . Or otherwise; when one body moves with the same velocity in the orbit BnD that another moves with in the orbit R/E ; the force which impells the former body towards C is as much less than that which impells the latter body thither, as BC is greater than RC , or as RC is less than BC .

Let Bg and $R/$ be the arcs, which the two bodies describe in equal times in their respective orbits. Then because they move with equal velocities, by the supposition, the spaces described in equal times, or Bg and $R/$, must be equal. And consequently the square of Bg is equal to the square of $R/$: that is, $Bg^2 = R/^2$. But, by proposition 164, the centripetal forces, by which these bodies are impelled towards C , are to each other as the squares of the arcs described in the same time divided by the semidiameters; or as the quotient arising from the square of Bg divided by BC to the quotient arising from the square of $R/$ divided by RC : that is, as $\frac{Bg^2}{BC}$ to $\frac{R/^2}{RC}$. But in all division, if the dividends are equal, the quotients are inversely as the divisor, less as the divisor is greater, and greater as the divisor is less. Therefore since Bg^2 and $R/^2$ are equal dividends, the quotients, which are expressed by the two fractions $\frac{Bg^2}{BC}$ and $\frac{R/^2}{RC}$, are as BC to RC inversely. And the centripetal forces, since they are as these quotients, are likewise as BC to RC inversely, that is, as the semidiameters inversely.

167. *When bodies revolve in circles and tend towards the center of these circles, the central forces are always as the squares of the velocities directly and as the semidiameters of the orbits inversely.*

If the semidiameters are equal and the velocities different, then the central forces are as the squares of the velocities directly, by proposition 165. And if the velocities are equal, but the semidiameters unequal, then the central forces are as the semidiameters inversely, by proposition 166. Therefore, where both the velocities and the semidiameters are different, the central forces will be as the squares of the velocities directly, and as the semidiameters inversely. So that this proposition is nothing more than the two foregoing ones joyned together.

Call the velocity with which one body revolves V , and the velocity with which any other body revolves v . Call the semidiameter of the orbit described by the former R , and the semidiameter of the orbit described

scribed by the latter r . By proposition 164, the central forces are as the squares of the arcs described in the same time divided by the semidiameters, and because the arcs described in the same time are as the velocities, the squares of those arcs will be as the squares of the velocities. Therefore the central forces are as the square of V divided by R to the square of v divided by r , or as $\frac{V^2}{R}$ to $\frac{v^2}{r}$. And since in all division the quotient is directly as the dividend and inversely as the divisor, these two quotients expressed by the two fractions $\frac{V^2}{R}$ and $\frac{v^2}{r}$ are as V^2 to v^2 directly and as R to r inversely. But V^2 and v^2 are the squares of the velocities, and R and r are the semidiameters of the respective orbits described by the bodies. Therefore the central forces are as the squares of the velocities directly and as the semidiameters inversely.

The 165, 166, and 167 propositions concerning the central forces of bodies revolving in circles have been all of them deduced from proposition 164. Where we proved that the central forces are universally as the square of the arc described in a given time divided by the semidiameter. Now this proposition 164 was shewn to be true of bodies revolving in similar curves, when they are in parts of these curves that are similar and also similarly placed in respect of the center towards which the bodies tend, provided we substitute the distances from that center for the semidiameter. In particular it was shewn that at B and R , Plat. VI. fig. 6. which are similar points of two similar elliptical orbits, and are similarly placed in respect of F the focus, towards which the revolving bodies tend, the central forces are to each other as the square of the arc Bg divided by BF , to the square of the arc Rl divided by RF : that is, as the squares of the arcs described in the same time divided by the distances from the focus F , towards which they are impelled. From hence therefore it follows, that the 165, 166, and 167 propositions are applicable to bodies that revolve in elliptical orbits. And if two bodies were to revolve in the orbit BgT with unequal velocities the centripetal forces at B would be as the squares of those velocities. If one body revolved in the orbit BgT , and the other in a similar orbit RlP with the same velocity, then the centripetal forces at B and R would be as BF to RF inversely. And universally in similar parts that are similarly placed in respect of the point F , towards which the revolving bodies are impelled, the centripetal force is as the square of the velocity directly and as the distance from F inversely.

168. *When bodies revolve in equal circles, the central forces are inversely as the squares of their periodical times.*

If two bodies were to revolve either in the same circle BnD , Plat. VI. fig. 3. round the same center C , or in equal circles round different centers, and each body was to take up a different time to describe its respective orbit, as suppose the periodical time of one of them to be 10 days and of the other 6 days: the squares of their periodical times are then to each other as 100 to 36. And these squares inverted are to each other as 36 to 100. The centripetal forces, which impell them towards the center of their orbits, will be in this proportion. And, whatever may be the proportion of the periodical times of the revolving bodies, the centripetal forces, and consequently the centrifugal forces likewise, will always be as the squares of these times inverted.

The orbits, or the spaces which the bodies have to describe in the time of a revolution, are equal by the supposition. Therefore the faster any body moves the sooner it will go round in its orbit, and the slower it moves the longer it will be in going round. That is, since the spaces to be described by each body are equal, the times taken up to describe those spaces, or the periodical times, will be inversely as the bodys velocity, by proposition 9. But, because the periodical times are inversely as the velocities, therefore the squares of the periodical times are inversely as the squares of the velocities: or, which is the same thing, the squares of the velocities are inversely as the squares of the periodical times. Now by proposition 165, when the orbits are equal, the central forces are as the squares of the velocities. And by what has just been proved the squares of the velocities are inversely as the squares of the periodical times. Therefore the central forces are inversely as the squares of the periodical times.

169. *When bodies describe unequal circles, if their periodical times are equal, the central forces are proportional to the semidiameters of their respective orbits, or are as those semidiameters directly.*

If one body revolves in the orbit BnD and any other body revolves in an equal time in the orbit R/E ; then their orbits are unequal and their periodical times are the same. In this case the centripetal force by which the former body is retained in the orbit BnD is as much greater than that which retains the latter in the orbit R/E , as BC the semidiameter of the former bodys orbit is greater than RC the semidiameter of the latters orbit.

Each

Each body describes its respective orbit in the same or an equal time, by the supposition. And, since each orbit contains 360 degrees, each body describes 360 degrees in its respective orbit in the same time. But if, whilst one is describing 360 degrees, the other describes just as many; then whilst one is describing 180 degrees or half its orbit, the other must likewise describe 180 degrees or half its orbit; whilst one is describing 90 degrees or a quarter of its orbit, the other must likewise describe 90 degrees or a quarter of its orbit; whilst one is describing 60 degrees the other must describe 60; and universally, when the periodical times are equal, each body in any given part of its periodical time describes an equal number of degrees in its respective orbit. But arcs, that contain an equal number of degrees, bear the same proportion to their respective orbits, or are similar arcs: as for instance if Bn and Rl contain each of them 60 degrees, then the arcs Bn and Rl are similar, that is, Bn bears the same proportion to the whole orbit BnD , that Rl bears to the whole orbit RlE . Therefore, when the whole periodical times of the revolving bodies are equal, the arcs described by them in any equal parts of those times are similar. But, by proposition 164, the central forces are as the squares of the arcs described in the same time divided by the semidiameters; and consequently, when the periodical times are equal, the central forces are as the squares of similar arcs divided by the semidiameters. Thus if two bodies revolve round C one in the orbit BnD the other in the orbit RlE in equal times; any arcs as Bn and Rl , which they describe in the same time in their respective orbits, will be similar: and the centripetal force by which one of the bodies tends to C will be to the centripetal force by which the other tends thither, as the square of Bn divided by BC to the square of Rl divided by RC ; that is, as $\frac{Bn^2}{BC}$ to $\frac{Rl^2}{RC}$. But similar arcs are in the same proportion to each other that the whole orbits are, or that the semidiameters of the orbits are. Thus Bn is to Rl as BC to RC : and consequently the square of Bn is to the square of Rl as the square of BC to the square of RC : that is Bn^2 is to Rl^2 as BC^2 to RC^2 . Therefore in the two fractions $\frac{Bn^2}{BC}$ and $\frac{Rl^2}{RC}$, BC^2 may be substituted for Bn^2 and RC^2 for Rl^2 without changing the proportion which those fractions bear to each other, that is $\frac{Bn^2}{BC}$ is to $\frac{Rl^2}{RC}$ as $\frac{BC^2}{BC}$ is to $\frac{RC^2}{RC}$. But when the square of any quantity is divided by the quantity itself, the quotient is likewise the same quantity. Therefore $\frac{BC^2}{BC}$ or the

the square of BC divided by BC is equal to BC, and likewise $\frac{RC^2}{RC}$ is for the same reason equal to RC. From whence it follows that $\frac{Bn^2}{BC}$ is to $\frac{R^2}{RC}$ as BC to RC. But the centripetal forces are to each other as $\frac{Bn^2}{BC}$ to $\frac{R^2}{RC}$. Therefore they are likewise to each other as BC to RC, or as the semidiameters of the orbits in which the bodies revolve.

170. *When bodies revolve in circles and are impelled towards the centers of those circles, the central forces are always as the semidiameters of their orbits directly and as the squares of their periodical times inversely.*

If the periodical times are equal, and the semidiameters unequal, the central forces are as the semidiameters directly, by proposition 169. And if the semidiameters are equal, and the periodical times unequal, the central forces are inversely as the squares of the periodical times by proposition 168. Therefore where both the semidiameters and the periodical times are unequal, the central forces will be in a ratio compounded of both, or as the semidiameters directly and as the squares of the periodical times inversely.

Call the semidiameter of one bodys orbit R, and the semidiameter of any other bodys orbit r. Call the periodical time of the former T and the periodical time of the latter t. The squares of the periodical times will be $T \times T$ and $t \times t$, that is T^2 and t^2 , or TT and tt . For the squares of T and t are expressed in all these three different ways. Now since in all division the quotient is directly as the dividend and inversely as the divisor, and since the central forces are as the semidiameters directly and as the squares of the periodical times inversely, they will be as R divided by TT to r divided by tt , or as $\frac{R}{TT}$ to $\frac{r}{tt}$.

The 168, 169, and 170 propositions are deduced as we have seen from propositions 164 and 165, which have already been shewn to be applicable not only to bodies revolving in circular orbits, but also to bodies revolving in similar elliptical ones, when they are in parts of these orbits, that are similar and also similarly placed in respect of the center towards which the bodies are impelled; provided we substitute the distances from that center for the semidiameters. Therefore with these restrictions the 168, 169, and 170 propositions are true of bodies revolving in elliptical orbits. The central

central forces, when the periodical times are equal, will be as the distances from the focus towards which the bodies are impelled. When the distances from that focus are equal, they will be inversely as the squares of the periodical times. And when neither the periodical times nor distances are equal, they will be in a ratio compounded of both, that is, as the distances directly and the squares of the periodical times inversely; or calling the distances of two bodies, that revolve round the same point, D and d and their periodical times T and t , the central forces will be as D to d directly and as TT to tt inversely, that is, by what has been shewn, as $\frac{D}{TT}$ to $\frac{d}{tt}$.

171. *When bodies revolve round the same center, if the squares of their periodical times are to each other as the cubes of their distances from that center, the central forces of those bodies are inversely as the squares of their distances.*

The planets revolve round the sun and the squares of their periodical times are found by observation to be to each other, as the cubes of their distances. For instance, The square of Jupiters periodical time bears the same proportion to the square of Mercurys periodical time, that the cube of Jupiters distance from the sun bears to the cube of Mercurys distance. And the same analogy, or proportion, holds good in all the other planets that revolve round the sun and are impelled towards it. And from hence we are to prove that the force, with which the sun acts upon them to retain them in their respective orbits, is at different distances from the sun inversely as the squares of those distances.

Call the distance of Jupiter from the sun D and the distance of Mercury d . Call the periodical time of Jupiter T , and the periodical time of Mercury t . Then, by what is supposed in the proposition, TT or the square of Jupiters periodical time, bears the same proportion to tt or to the square of Mercurys periodical time, that $D \times D \times D$ or D^c or DDD or the cube of Jupiters distance bears to $d \times d \times d$ or d^c or to ddd , or to the cube of Mercurys distance. Now by what was shewn in the last proposition the centripetal force that retains Jupiter in its orbit is to the centripetal force that retains Mercury in its orbit as $\frac{D}{TT}$ to $\frac{d}{tt}$. But DDD bears the same proportion to ddd that TT bears to tt , by the supposition. Therefore in the two fractions $\frac{D}{TT}$ and $\frac{d}{tt}$ instead of TT and tt , we may substitute DDD and ddd . And because these quantities DDD and ddd bear the same proportion to each other with TT and tt , the proportion of the
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two fractions will not be altered by this change, but $\frac{D}{DDD}$ and $\frac{d}{ddd}$ will be to each other as $\frac{D}{TT}$ and $\frac{d}{tt}$. But the centripetal force of Jupiter is to that of Mercury as $\frac{D}{TT}$ to $\frac{d}{tt}$. Therefore, since the squares of their periodical times are as the cubes of their distances, the centripetal force of Jupiter is likewise to that of Mercury as $\frac{D}{DDD}$ to $\frac{d}{ddd}$. Now neither of these fractions are in their least terms. The way of reducing them to their least terms is this: let D the numerator of $\frac{D}{DDD}$ be divided by D , that is, by itself, and the quotient arising from thence will be 1; for the quotient arising from any quantity divided by itself is always 1: and let the denominator DDD , or the cube of D , be likewise divided by D , and the quotient will be DD ; for when the cube of any quantity is divided by that quantity the quotient will be its square. And taking 1 the first quotient for a new numerator and DD the other quotient for a new denominator we shall have a new fraction $\frac{1}{DD}$ equal to $\frac{D}{DDD}$. So likewise if d the numerator of the fraction $\frac{d}{ddd}$ is divided by itself we shall have 1 for a new numerator, and if ddd the denominator is divided by the same quantity d we shall have dd for a new denominator. And when the numerator and denominator of the fraction $\frac{d}{ddd}$ are thus reduced to their least terms by dividing both of them by the same quantity d , the fraction is changed into $\frac{1}{dd}$, which is equal to $\frac{d}{ddd}$. But, since the centripetal forces are as $\frac{D}{DDD}$ to $\frac{d}{ddd}$, and since $\frac{D}{DDD}$ is equal to $\frac{1}{DD}$ and $\frac{d}{ddd}$ is equal to $\frac{1}{dd}$; it follows that the centripetal force of Jupiter is to that of Mercury as $\frac{1}{DD}$ to $\frac{1}{dd}$. But all fractions express a quotient arising from a division of their numerator by their denominator. And in all division, where the dividend is given, the quotient is inversely as the divisor. Therefore all fractions, whose numerators are equal must be inversely as their denominators. From hence it follows, that the two fractions $\frac{1}{DD}$ and $\frac{1}{dd}$, whose numerators are each of them 1 and consequently equal, must be to each other inversely as DD to dd . But DD and dd are the squares of the distances. There-

Therefore these fractions, and consequently the centripetal forces, which are as these fractions, are to each other inversely as the squares of the distances.

From this proposition it appears that the force, which impells the planets towards the sun, and by impelling them prevents them from running out in right lines and retains them in their respective orbits, may be the force of gravity. Since this force at different distances from the sun is varied in the same proportion that the force of gravity would be. It has been shewn, in proposition 23, that the weight of bodies or the force of gravity that acts upon them at different distances from the earths center is inversely as the squares of those distances. From whence we may conclude that, if heavy bodies, such as the planets are, tend towards the sun, in the same manner that other heavy bodies tend towards the earth; the force of gravity, that acts upon the planets and impells them towards the sun, would at different distances from the sun be inversely as the squares of those distances. And since it is found by observation that the squares of the periodical times of the planets are as the cubes of their distances, it follows from this proposition, that the force, which retains them in their respective orbits, is likewise at different distances from the sun inversely as the squares of those distances: and consequently this force may be the force of gravity; since at different distances from the central body both these forces are found to be varied by the same law.

172. *If a body, that revolves in an ellipsis, describes equal areas in equal times by a radius drawn to one of the focuses; the centripetal force, which retains this body in its orbit, is, at different distances from the focus, inversely as the squares of the distances.*

If the moon revolves in the elliptical orbit ADPG. Plat. VI. fig. 1. and a radius or line drawn from the moon to the earth at S in one of the focuses describes equal areas in equal times as the moon goes round in its orbit; then, by proposition 160, the centripetal force constantly impells the moon towards the earth: and in this proposition we are to shew that in this case the centripetal force, that retains the moon in its orbit, must at different distances from the earth be inversely as the squares of the distances. In like manner we may apply this proposition to any of the planets: for as they revolve in elliptical orbits having the sun in one of their focuses, and describe by a radius drawn to the sun equal areas in equal times, this proposition affirms that the centripetal force, which tends towards the sun and prevents them from running out in rectilinear tangents, is, at different distances from the sun, inversely as the squares of those distances.

In order to prove this we will shew, that at A and P, or at the greatest and least distances of the revolving body from S the central body, the centrifugal force is inversely as the squares of the distances : and from hence it will follow that the centripetal force likewise, which is equal to the other, must be varied in the same proportion. It was proved, in proposition 165, that when bodies revolve in equal circles, that is, in circles which in equal parts have the same curvature, the centrifugal force is as the square of the velocity with which the body moves. Now at A and P the greatest and least distances from S the curvature of the ellipsis in equal parts is equal. Therefore the centrifugal force, when the body is at A, is to the centrifugal force of it when at P, as the square of its velocity at A to the square of its velocity at P. But, by proposition 161, the velocity of the revolving body, when it is at A, is to its velocity, when it is at P, as the distances inversely. From whence it follows that the square of its velocity at A is to the square of its velocity at P as the squares of the distances inversely. But the centrifugal forces are as the squares of the velocities. Therefore the centrifugal forces at A and at P are likewise as the squares of the distances inversely. And since the centripetal force is equal to the centrifugal force, it must be varied by the same law ; and consequently at A and at P the centripetal force must be inversely as the squares of the bodys distances from the point S, towards which it is impelled.

The application of proposition 167 to the case of bodies moving in elliptical orbits may occasion a difficulty here, which we should take care to clear up. It is there asserted that the central forces, of which the centrifugal force is one, are in elliptical orbits directly as the square of the revolving bodys velocity and inversely as its distance from the focus, towards which it tends, that is, as the square of the velocity divided by the distance. Now here without any regard to the distance from S we say, that the centrifugal force is as the square of the velocity. But the reader will remember that proposition 167 relates to bodies that are in such parts of their orbits as are similarly placed in respect of the focus. Whereas we are now speaking of the same body in different parts of the same orbit A and P, which, because one is the greatest and the other the least distance from S, are not similarly placed in respect of S. Therefore what was said in proposition 167 about the centrifugal force cannot be so applyed here as to be made the foundation of any objection.

By the foregoing proposition we prove that the centripetal force, by which the planets that move round the sun are retained in their respective orbits, is, at different distances from the sun, inversely as the squares of
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of those distances. For the squares of the periodical times of the planets that move round the sun are found by observation to be to each other respectively as the cubes of their distances. Thus the square of Mercurys periodical time bears the same proportion to the square of Jupiters; that the cube of Mercurys distance bears to the cube of Jupiters. And in the foregoing proposition we determined the law by which the centripetal force is varied, wherever this analogy or proportion obtains. But that proposition will never demonstrate that the centripetal force, by which the moon is retained in the orbit that it describes round the earth, is, at different distances from the earth, inversely as the squares of those distances: because this analogy or proportion cannot be applied where there is only one body revolving round the same center, which is the case of the moon revolving round the earth. For the nature of the analogy requires that there should be the squares of two different periodical times, and the cubes of the distances of two different bodies to compare with one another. And since we cannot say the square of the moons periodical time is to the square of the periodical time of some other body revolving round the earth, as the cube of the moons distance is to the cube of that other bodys distance, because there is no other body besides the moon, that does revolve round the earth; it follows that we cannot apply that analogy to the case of the moon, but must make use of some other argument to prove that the centripetal force, which retains it in its orbit, is, at different distances from the earth, inversely as the squares of those distances. The proposition now before us supplies us with such an argument. For since it is found, by observation, that if the moons motion was not to be disturbed by the sun, it would describe an immoveable ellipsis such as ADPG round the earth at S, or in one of the focuses, it follows that the centripetal force, by which it is impelled towards the earth, is, at different distances, inversely as the squares of those distances. And, since it has been proved that the force of gravity decreases and encreases by the same law at different distances from the earths center, it follows farther that the force, which impells the moon towards the earth and by so impelling it retains it in its orbit, may, as far as yet appears, be the force of gravity or the same force that makes heavy bodies descend.

173. *When a body moves in an elliptical orbit and the centripetal force impells it towards one of the focuses, the greatest distance of the revolving body from that focus is called the higher apsis, and the least distance is called the lower apsis.*

Thus Plat. VI. fig. 1. A is the higher apsis and P is the lower.

174. *A line drawn from one apsis to the other is called the line of the apses.*

Such is the line ASP, which is the longer axis of the orbit described by the revolving body.

175. *If, in the departure of the revolving body from the focus where the centripetal force acts, this force decreases faster than the squares of the distances from that focus encrease; or if in the approach of it to the focus, the centripetal force encreases faster than the squares of the distances decrease; the orbit will be a moveable ellipsis, and the line of the apses will turn round upon the focus in the same direction in which the body revolves.*

For the better understanding this proposition, we will first explaine what is meant by a centripetal force that in the departure of a body from the focus, towards which that force impells it, decreases faster than the squares of the distances encrease, and will shew that a force, which decreases too fast in the departure of the revolving body, will likewise encrease too fast in the approach of it to the point S or focus, towards which that body is impelled. Whilst a body moves round in the orbit ADPG Plat.VI. fig. 1. it is sometimes nearer to S and sometimes more remote from it: suppose four different distances of the body in four different parts of the orbit, that are to each other in the proportion of the following numbers, 1, 2, 3, 4, the squares of these numbers are 1, 4, 9, 16, and the reciprocals of these squares are $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. And any centripetal force, which at the different distances 1, 2, 3, 4, is inversely or reciprocally as the squares of these distances, decreases in the same proportion with these fractions $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, which are the reciprocals of the squares of the distances 1, 2, 3, 4. The force of gravity in departing from the earths center decreases in this proportion. So that if we call the force 1 which impells the moon towards the earth, when its distance from the earth is called 1, then, if the moon was at the distance 2 from the earth or at twice the former distance, the force of gravity, that impells it towards the earth, would be less than before; nor does it decrease only as the distance itself, but as the square of the distance encreases: thus at the distance 2 the force of gravity is not $\frac{1}{2}$ of what it is at the distance 1, but only $\frac{1}{4}$. So likewise at the distance 3 this force is not $\frac{1}{3}$ of what it is at the distance 1, but only $\frac{1}{9}$. And at the distance 4, the gravity, is reduced to $\frac{1}{16}$ of what it is at the distance 1. But if the force of gravity in the departure of the moon or any other heavy body from the earths center, decreases in the proportion of $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, whilst the distances of the moon encrease in the proportion of 1, 2, 3, 4; it is evident that in the approach

proach of the moon towards the earth again, whilst the distances decrease in the proportion of 4, 3, 2, 1, the force of gravity must necessarily encrease in the proportion of $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$. A force of this sort, which at different distances from the center is inversely as the squares of those distances, will, as has been said before, make a revolving body describe round any point S, towards which that body is impelled, an ellipsis ADPG having S in one of its focuses.

The different distances of the revolving body from S, as we just supposed them, are 1, 2, 3, 4. The cubes of these distances are 1, 8, 27, 64. The reciprocals of these cubes are $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$. And a force, which in the departure of the body from S at the distances 1, 2, 3, 4, decreases in the proportion of $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, is as the reciprocals of the cubes of the distances, or decreases as the cubes of the distances encrease. Now it is plain that the fractions $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, decrease faster than those mentioned in the last paragraph, that is, faster than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Therefore a force, which in the departure of the revolving body decreases as the cubes of the distances encrease, decreases faster than the force of gravity, which decreases only as the squares of the distances encrease. The same may likewise be said of any force, which, though at the distances 1, 2, 3, 4, it does not decrease so fast as $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, yet decreases faster than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Now it is plain that a force, which at the distances 1, 2, 3, 4, decreases in the proportion of $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, in the departure of the revolving body from the center towards which it is impelled, will at the distances 4, 3, 2, 1, in the approach of the body encrease in the proportion of $\frac{1}{64}$, $\frac{1}{27}$, $\frac{1}{8}$, $\frac{1}{1}$. But these fractions increase faster than $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$, that is, such a force will encrease faster than the squares of the distances decrease, or a centripetal force which decreases too fast in the departure of the revolving body will likewise encrease too fast in the approach of it.

But before we go on to shew that such a force as this will make the body describe a moveable ellipsis, it will be necessary to explaine what is meant by such an ellipsis. In order to do this we must observe that the line of the apses always passes through the greatest and least distances. Now suppose A, Plat. VI. fig. 7. to be the greatest distance of the revolving body from the focus T, B to be the least distance, and ATB the line of the apses. Let the body set out from A, in the direction AH, and by some centripetal force, which urges it towards T, let it be drawn from the tangent AH and be made to move in an elliptical orbit round T. This orbit would be an immoveable one, if the body setting out from A or the greatest distance was to come to the least distance by moving to B, that is, by going half round the focus T. But suppose that the body was

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not to come to its least distance till it had gone more than half round T, but that after it had passed by B it was still to approach nearer to T, till it came to D, so that D should be the least distance when the body set out from A. ATB passing through A the higher apsis, through T the focus, and through B, that was the lower apsis of the orbit in which the body began to move, was the line of the apses: but after the body has arrived at its least distance, then, because D is this least distance, the line of the apses, which always passes through the least distance, must have changed place, and instead of being ATB as it was at first, is now got into the situation DTC. That is, during the motion of the body in its orbit from the higher to the lower apsis, the line of the apses has turned round upon the point T in the same direction, in which the body was moving. In like manner if the body sets out from D the lower apsis, and in going half round the focus T comes to the greatest distance at C, then the orbit itself would be immoveable during this part of the bodys revolution. But if it must go more than half round T before it arrives at its greatest distance, as suppose this greatest distance to be at G, then, whilst the body is moving from the lower apsis to the higher, this higher apsis shifts its place and moves on before the body, that is, the line of the apses, and consequently the whole orbit along with this line, does likewise in this part of the bodys revolution turn a little way round upon the point T in the same direction in which the body itself is moving. So that this part of the motion is performed in part of an ellipsis whose higher apsis is at G and its lower apsis about F.

Having thus explained, what is meant by the terms of the proposition, we may now go on to shew the truth of it by proving that if a centripetal force in the departure of a revolving body from the center, towards which it is impelled, decreases faster than the squares of the distances encrease, or in the approach of the body towards that center encreases faster than the squares of the distances decrease, the orbit described in this case may be considered as a moveable ellipsis, the line of whose apses keeps turning round upon the focus, towards which the body is impelled, in the same direction in which the body itself is moving.

We have seen that if a body sets out from A to move round the focus S in the ellipsis ADPG, Plat. VI. fig. 1. then in some part of its motion between A and P, suppose at D, it has such a velocity as, if the centripetal and projectile forces were perpendicular to each other, would make the body move in a circle round S at the distance DS. And we have seen besides that the centripetal force, which makes a body revolve in such an orbit, must, at different distances from S, be inversely

ly as the squares of those distances. Now if the centripetal force, in the approach of the body towards S, encreases faster than the squares of the distances decrease, this force will be every where greater in proportion than such a force as would make the body describe the orbit ADPG. Therefore though the bodys velocity when it is at D, would be sufficient to make it describe a circle round S, if the centripetal force encreased only as the squares of the distances decrease, yet since the centripetal force does by the supposition encrease faster and so is every where greater in proportion, the velocity, that is requisite to make the body describe a circle round S, must be greater than what it has at D. Because the greater the centripetal force is so much greater in proportion must be the velocity of the body in order to make it revolve in a circle. From hence then it follows that the body must move or descend farther than D before it can acquire such a velocity. And the farther the body must move from A before it acquires such a velocity as would be requisite to make it revolve in a circular orbit round S, so as to keep always at the same distance from S, the farther it must necessarily move from A, before it can have acquired such a velocity as will be sufficient to make it begin to depart from S. Now if the centripetal force had encreased as the squares of the distances decrease, the body in moving half round S, or in descending from A to P, would have acquired velocity enough to make it begin to depart from S, or P would have been its least distance. Therefore if the centripetal force encreases faster, the body must move more than half round S or must descend farther than from A to P before it will have acquired the requisite velocity, that is, before it will begin to depart from S or be arrived at the least distance. And since when the centripetal force encreases faster than the squares of the distances decrease, the body setting out from A must move more than half round S before it comes to its least distance or lower apsis, it follows from what has been said already in explaining the proposition, that the lower apsis must have moved forwards, or that the line of the apses must have turned round upon the focus, towards which the centripetal force is directed, whilst the body has been moving from the higher apsis to the lower: and consequently the orbit described will be such an one as is represented in fig. 7. where the body setting out from A, which is the higher apsis, does not arrive at the lower apsis till it has got to D, which is at the distance of more than half the orbit from A.

In like manner, fig. 1. if the body sets out from P, and the centripetal force decreases as the squares of the distances encrease, the velocity becomes such, when the body arrives at G, as is requisite to make it describe

scribe a circle round S at the distance SG so as neither to depart farther from S nor approach nearer to it. But if the centripetal force decreases faster than this, or is every where less in proportion, then the body must ascend farther than from P to G , before its velocity, which was so great at P as to make it depart from S , is far enough diminished to suffer the body to keep revolving round S so as to keep always at the same distance from it. Now if the body must ascend farther than G before its velocity is sufficiently diminished to suffer it to revolve round S so as always to keep the same distance from it, then it must necessarily ascend farther than A , before this velocity will be small enough to permit it to approach towards S , that is, the body must move farther than from P to A or more than half round, before it will arrive at its greatest distance or higher apsis. And consequently the orbit will be such an one as is represented fig. 7. where if B is the lower apsis from whence the body begins to move the higher will not be at A , but at C , or if D is the lower apsis the higher will not be at C but at G ; so that the orbit revolves round T in the same direction that the body itself revolves.

Or otherwise. If a body that is impelled towards S , Plat. VI. fig. 1. and revolves round it in an elliptical orbit having S in one of its focuses; the centripetal and projectile forces make a right angle with each other at A , or when the body is at the higher apsis or greatest distance from S . And this angle is again a right one, when the body is at P the lower apsis or least distance. Whilst the body is descending from A towards P this angle diminishes till it becomes the least of all at D , and then encreases again till it becomes a right one at P . In like manner, whilst the body ascends from P towards A , this angle encreases till it becomes the greatest of all at G , after which it decreases till it is again reduced to a right one at A . This has been proved, in proposition 152, and, in proposition 172, we have shewn farther, that when a body moves in this manner the centripetal force at different distances from S is inversely as the squares of those distances.

But if in the approach of the body towards S the centripetal force encreases faster than in this proportion, that is, if it is every where greater in proportion, the body will be more urged towards S ; and consequently the angle made by the projectile and centripetal forces will be every where less. Now if the centripetal force encreased as the squares of the distances decrease, this angle would be a right one when the body had descended from A to P ; and when this angle in the descent of the body is a right one, the body is then at its lower apsis or least distance from S and not before. But if the centripetal force encreases faster than in this proportion

proportion, this angle is every where less. Therefore if the centripetal force encreases faster than in this proportion, the angle at P will be less than a right one; and consequently the body must move farther than P, or more than half round S, before this angle becomes a right one, that is, before it arrives at its least distance from S, or at the lower apsis. And because the longer axis of the elliptical orbit or the line of the apses always passes through the least distance, if the least distance has shifted place and gone forwards from P, the line of the apses must have turned round upon the focus T, towards which the body is impelled, in the same direction in which the body itself is moving.

In like manner, if we consider P as the least distance and the body as setting out from thence; since the centripetal force, whilst the body departs from S, decreases faster than the squares of the distances encrease, that is, since this force is every where less in proportion, the body will be every where less urged towards S, and consequently the angle made by the projectile and centripetal forces will be every where greater. Now if the centripetal force decreased only as the squares of the distances encrease, this angle would be a right one, when the body had ascended from P to A, and when this angle in the ascent of the body is a right one, the body is then at its higher apsis or greatest distance from S and not before. But if the centripetal force decreases faster than in this proportion, this angle is every where greater. Therefore, if the centripetal force decreases faster than in this proportion, the angle at A will be greater than a right one; and consequently the body must move farther than A or more than half round S before this angle becomes a right one, that is, before it arrives at its greatest distance. And because the longer axis of the elliptical orbit, or the line of the apses, always passes through the greatest distance, if the greatest distance has shifted place and gone forwards from A, the line of the apses must have turned round, as before, upon the focus T, towards which the body is impelled, in the same direction in which the body itself is moving.

We shall have occasion to apply this proposition to the moons motion. For though the centripetal force, which impells the moon towards the earth, would make it describe an immoveable ellipsis having the earth in one of its focuses, if there was no other force but this to act upon it: yet we shall see hereafter, that the moons motion is disturbed by the action of the sun upon it. For as the moon gravitates towards or is attracted by the earth, and by this force is retained in its orbit; so likewise it gravitates towards or is attracted by the sun, and this force disturbs its motion round the earth. In particular it will appear in astro-

nomy that though the force of gravity, by which the moon is impelled towards the earth, is, at different distances, inversely as the squares of those distances, and consequently would make the moon revolve round the earth in an immoveable ellipsis: yet at the new and full of the moon the force with which the sun acts upon it causes its gravity towards the earth to decrease in the departure of the moon, and to encrease in the approach of it, faster than in the inverse proportion of the squares of the distances. From whence it will follow that, at the new and full, the moons orbit will be a moveable ellipsis, such an one as is represented, Plat. VI. fig. 7. the longer axis of which turns round upon T, or upon the focus where the earth is placed, in the same direction in which the moon moves.

176. *If in the departure of the revolving body from the focus, where the centripetal force acts, this force decreases slower than the squares of the distances from that focus encrease; or if in the approach of it to the focus the centripetal force encreases slower than the squares of the distances decrease, the orbit will be a moveable ellipsis, and the line of the apsides will turn round upon the focus in the contrary direction to that in which the body revolves.*

We will make use of the same method in this proposition that we did in the last. First, by a centripetal force, which is inversely as the squares of the distances, we mean such an one as decreases in the same proportion that the squares of the distances encrease, or encreases in the same proportion that those squares decrease. Suppose for instance that the moon in revolving round the earth is sometimes nearer to it and sometimes farther from it. Then if at any distance, which we will call 1, the force of gravity is likewise called 1, or $\frac{1}{1}$; at the distance 2 the force of gravity would be $\frac{1}{4}$; at the distance 3 it would be $\frac{1}{9}$; and at the distance 4 it would be $\frac{1}{16}$. For these distances encrease in the proportion of 1, 2, 3, 4, and the squares of these distances are 1, 4, 9, 16. Therefore any force, such as that of gravity, which is inversely or reciprocally as the squares of these distances, will be as the reciprocals of 1, 4, 9, 16, that is, as $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$. In like manner when the moon is approaching towards the earth and the distances decrease in the proportion of 4, 3, 2, 1, the squares of these distances are 16, 9, 4, 1, the reciprocals of these squares are $\frac{1}{16}, \frac{1}{9}, \frac{1}{4}, \frac{1}{1}$. Therefore the force of gravity, which is inversely or reciprocally as the squares of the distances, at the several distances 4, 3, 2, 1, encreases in the proportion of $\frac{1}{16}, \frac{1}{9}, \frac{1}{4}, \frac{1}{1}$. But suppose in the departure of the moon from the earth the centripetal force was inversely, not as the squares of the

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the distances, but as the distances themselves. For instance, suppose at the several distances 1, 2, 3, 4, the force which impells the moon towards the earth was not as $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, or as the reciprocals of the squares of the distances, but as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or as the reciprocals of the distances themselves. I imagine it is evident that such a force as this decreases slower than the force of gravity does. Because the fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, decrease slower than $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$. But the same force, which in the departure of the moon from the earth decreases in the proportion of $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, will in the approach of it encrease in the proportion of $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}$. Now as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, decrease slower than $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$. So likewise $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}$, encrease slower than $\frac{1}{16}, \frac{1}{9}, \frac{1}{4}, \frac{1}{1}$. Therefore a centripetal force, which is inversely as the distances, decreases slower in the departure of the revolving body and encreases slower in the approach of it than the force of gravity, or than any other force which is inversely as the squares of the distances. And so likewise by the same way of reasoning it may be shewn that any force which decreases too slowly in the departure of such a body will encrease too slowly in the approach of it.

Having premised this we are next to consider what is meant by an ellipsis that revolves upon one of its focuses in the contrary direction to that in which the body itself moves. Let the body set out from A, Plat. VI. fig. 8, that is, let the projectile force act in the direction AH, whilst the centripetal force constantly impells it towards T. A is the higher apsis; or greatest distance, and if the body must move half round T, or from A to B, before it comes to the lower apsis or least distance, then the orbit would be an immoveable ellipsis having T in one of its focuses. But if the body setting out from A comes to the lower apsis or least distance at D, or at any other point before it arrives at B, that is, before it has gone half round T; then this least distance or lower apsis must have shifted place so as to meet the body. For since the longer axis or line of the apses always passes through the greatest and least distances, and since, when the body set out from A, this point A was the greatest distance, B, or the opposite point, through which the line ATB passes, must have been the least distance. But the body is at the least distance at D before it arrives at B. Therefore this least distance or lower apsis must have shifted from B to D, whilst the body was moving downwards from A; that is, the line of the apses must have turned round upon the point T in the contrary direction to that in which the body moves, so as to meet the body before it has got half round. Or if we consider the body as setting out at first from B the lower apsis; then if it moves half round T before it comes to the higher apsis or greatest distance, and A is this

higher apsis; the orbit is then an immoveable ellipsis. But if, when it sets out from B, it goes on in the curve BC instead of BA, and is at its greatest distance, when it arrives at C, or before it has gone half round the focus T, then because the line of the apses was in the position ATB, when the body was at the least distance from T, and is in the situation CTD, when it arrives at its greatest distance, it is evident that this line of the apses must have turned round upon the point T so as to meet the body, that is, it must have moved in a direction contrary to that in which the body is moving. We have thus considered the nature of such a moveable ellipsis as is described in the proposition, by supposing the body to set off from A the greatest distance and to arrive at its least distance when it comes to D: and then by supposing it to set off from its least distance at B and to arrive at its greatest distance when it comes to C. The reader will perceive that such a motion as is here supposed is not a continued one: for if D is the least distance, it cannot move on and set off from B, so as that B shall be the least distance too; because this would be making two least distances. But what has here been said will serve to explain the motion of a body in such an ellipsis, when it is a continued one. For if it sets out from A, and arrives at its least distance at D before it has gone half round T, and after this, instead of going on in the orbit DC so as to move half round T before it comes at its greatest distance, if it moves along in the curve DG, and arrives at its greatest distance before it has gone half round T, the orbit is a moveable ellipsis, and the longer axis of it, or line of the apses, turns round upon the focus towards which the body is impelled, in a contrary direction to that in which the body itself is moving.

We are now to prove, if a body moves in an elliptical orbit, and the centripetal force encreases and decreases slower than in the inverse proportion of the squares of the distances from the focus towards which the body is impelled, that then this ellipsis will be a moveable one, and the longer axis or line of the apses will move the contrary way to the body.

If a body sets out from A, Plat. VI. fig. 1. and moves in an elliptical orbit by the action of a centripetal force which constantly impells it towards S; and if this centripetal force, as the revolving body approaches S, encreases in the same proportion that the squares of the distances from S decrease; then at D this body will have such a velocity as is requisite to make it move round S in a circle having DS for its semidiameter. But such a centripetal force is every where greater in proportion than one which encreases slower than the squares of the distances decrease. And
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the greater the centripetal force is, so much greater velocity is required to make a body revolve in a circle. Therefore though the body must descend from A to D in order to acquire a sufficient velocity to make it move in a circle round S, if the centripetal force encreases as fast as the squares of the distances decrease; yet if the centripetal force encreases slower and so is every where less in proportion, it need not descend so far as D in order to acquire such a velocity. Now the velocity, that would cause the body to move in a circle, is one that balances the centripetal force in such a manner as to keep the body constantly at the same distance from the center. And the velocity which a body revolving in an elliptical orbit has at its least distance is such an one as is sufficient to make the body depart farther from the center, towards which it is impelled. But when the centripetal force encreases slower than the squares of the distances decrease, the revolving body will acquire a sufficient velocity to make it move in a circle before it arrives at D; and consequently it need not move so far in order to acquire a sufficient velocity to make it depart from the center S, as it must have done if the centripetal force had encreased as fast as the squares of the distances decrease. But if the centripetal force had encreased as fast as the squares of the distances decrease, the velocity acquired by descending from A to P or by moving half round S would have been sufficient to make the body begin to depart from S, and P would have been the least distance. Therefore if the centripetal force encreases slower than the squares of the distances decrease, the body will acquire a sufficient velocity to make it begin to depart from S before it has descended so far as from A to P, that is, the body will be at its least distance from S before it has gone half round: and consequently the orbit described by the body must be such an one as is represented fig. 8. where the body setting out from A does not move half round or to B in order to come to its least distance, but is at this least distance or at the lower apsis when it has descended from A to D.

In like manner if the body sets out from P, fig. 1. and the centripetal force, whilst it departs from S, decreases as fast as the squares of the distances encrease, the body, when it is at G, will have such a velocity as in proper circumstances would make it describe a circle round S always keeping at the same distance GS from the point S. And when the body arrives at A having gone half round S its velocity would be so far diminished as to suffer it again to approach towards S. But if the centripetal force decreases slower, then it must be every where greater in proportion, and consequently the velocity requisite to make the body move in a circle so as to keep always at the same distance from S must
be

be greater than what it has at G. And consequently this centripetal force, which is every where greater in proportion than one that decreases as fast as the squares of the distances encrease, will cause the body to begin to approach towards S before the velocity is so small as it is at A. But the body is at its greatest distance from S, when it begins to approach it. Therefore a centripetal force, which decreases slower than the squares of the distances encrease, will cause that the revolving body setting out from P shall arrive at its greatest distance or higher apsis before it arrives at the opposite point A, that is, before it has gone half round S: and consequently the orbit described by a body, which is acted upon by such a centripetal force, will be such an one as is represented fig. 8. where the body setting out from D the lower apsis does not go half round T or to C in order to come to the higher, but moves in such a curve as DG, which is part of an elliptical orbit, whose lower apsis is about F and its higher lies nearer to D than the point C does.

Or otherwise. When a body revolves in the ellipsis ADPG, Plat. VI. fig. 1. the angle that the projectile force makes with the centripetal force at A is a right one. This angle decreases till the body arrives at D; where it is the least of all, and then encreases till it becomes a right one again at P. And as the body goes on from P and ascends towards A, this angle keeps encreasing till it becomes the greatest of all at G, and then decreases so as to be a right one when the body is returned to A. Now the less a body is urged towards S by the centripetal force in any part of its orbit, so much greater is the angle which the centripetal and projectile forces make with each other in that part. For instance, at the point C the centripetal force urges the body towards S in the line CS, whilst the projectile force impells it in the tangent Cc; and the less the body is urged in the direction CS so much farther Cc will recede from CS, or so much greater will be the angle made by the lines CS and Cc. This angle is a right one, when the body is at A, and is a right one again, when the body has gone half round S or is come to P the opposite point to A, provided the centripetal force encreases as fast as the squares of the distances decrease, by proposition 152. But if the centripetal force encreases slower, then it is every where less in proportion, and consequently, since the body is every where less urged towards S, the angle made by the centripetal and projectile forces will be every where greater as the body descends, and therefore will be a right one before the body arrives at P or before it has gone half round S. But, when this angle is a right one, the body is then at its least distance or at its lower apsis. Therefore if the centripetal force encreases slower than

PLATE VI.

Fig. 2.

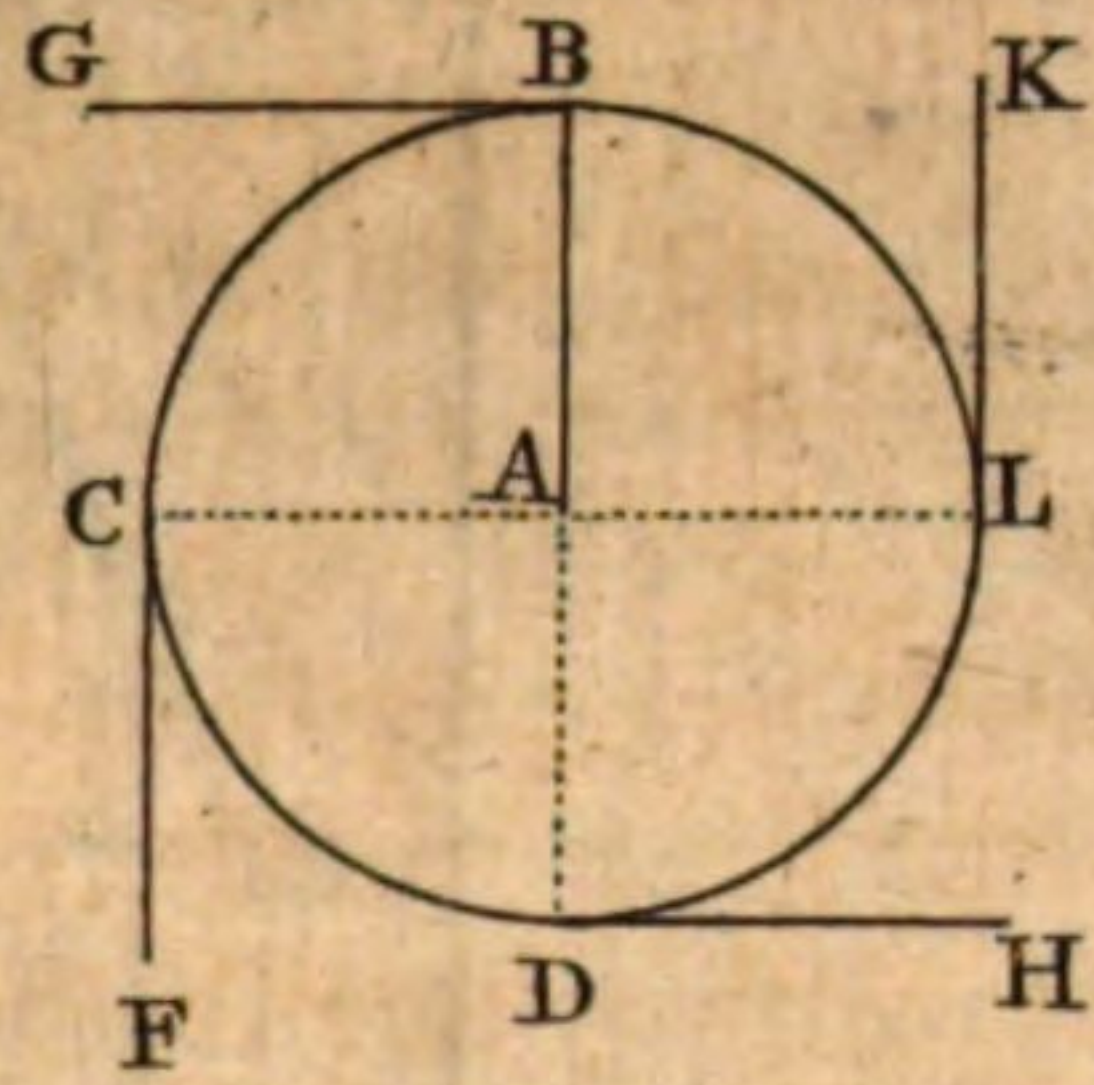


Fig. 3.

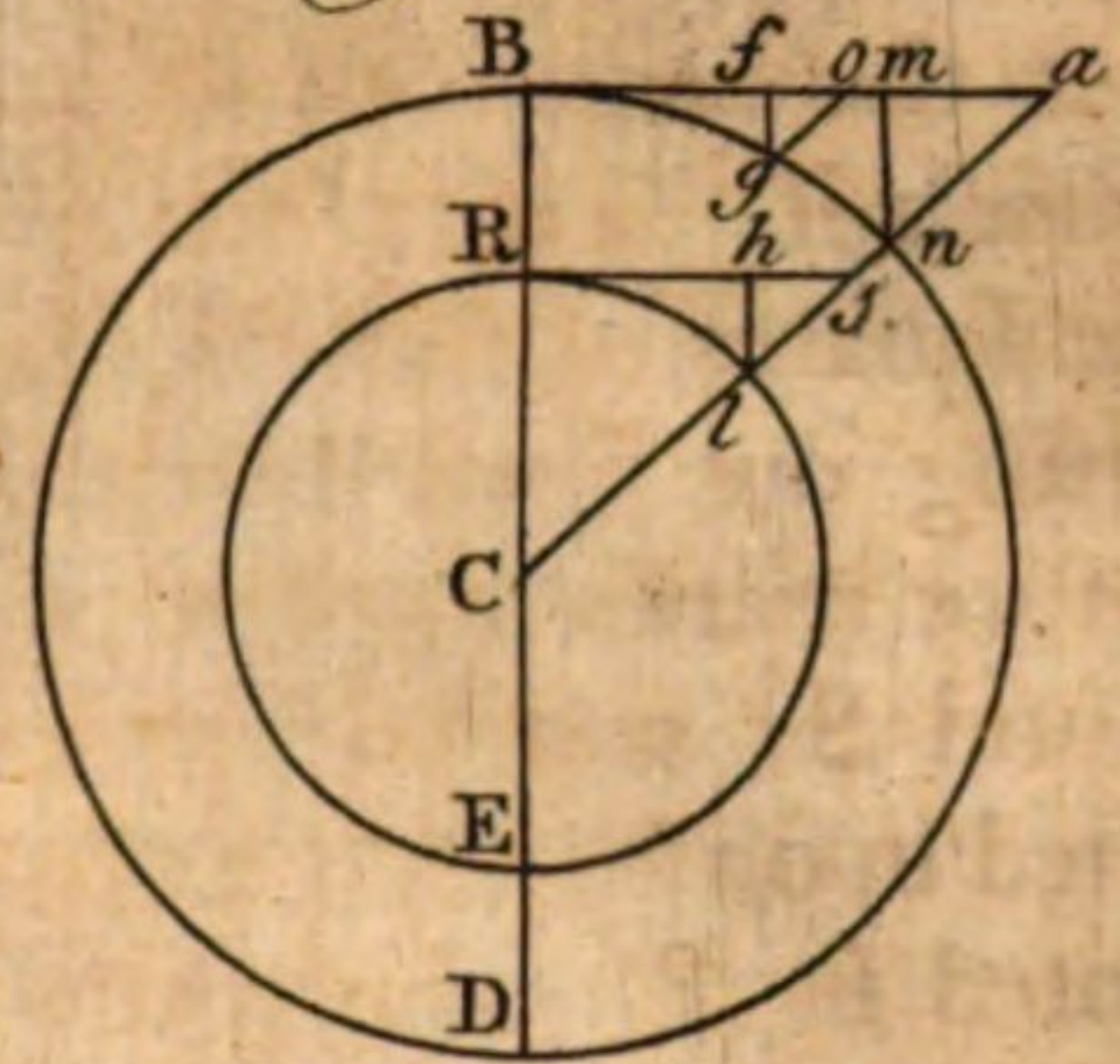


Fig. 5.

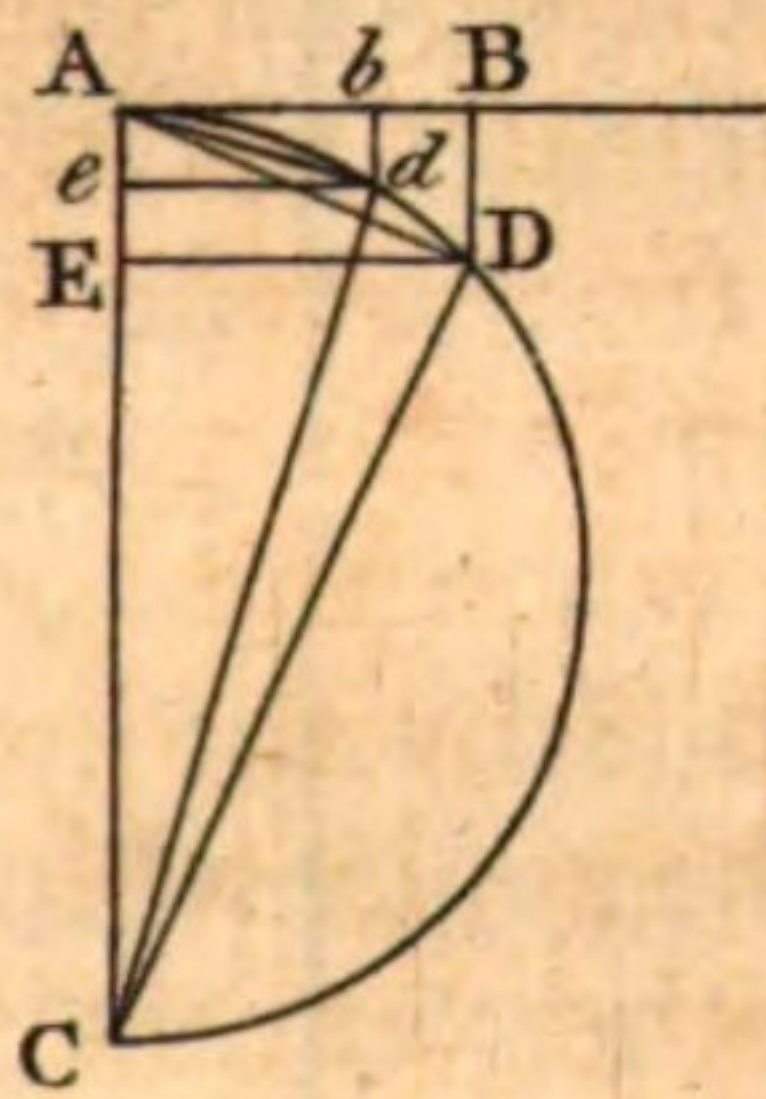


Fig. 6.

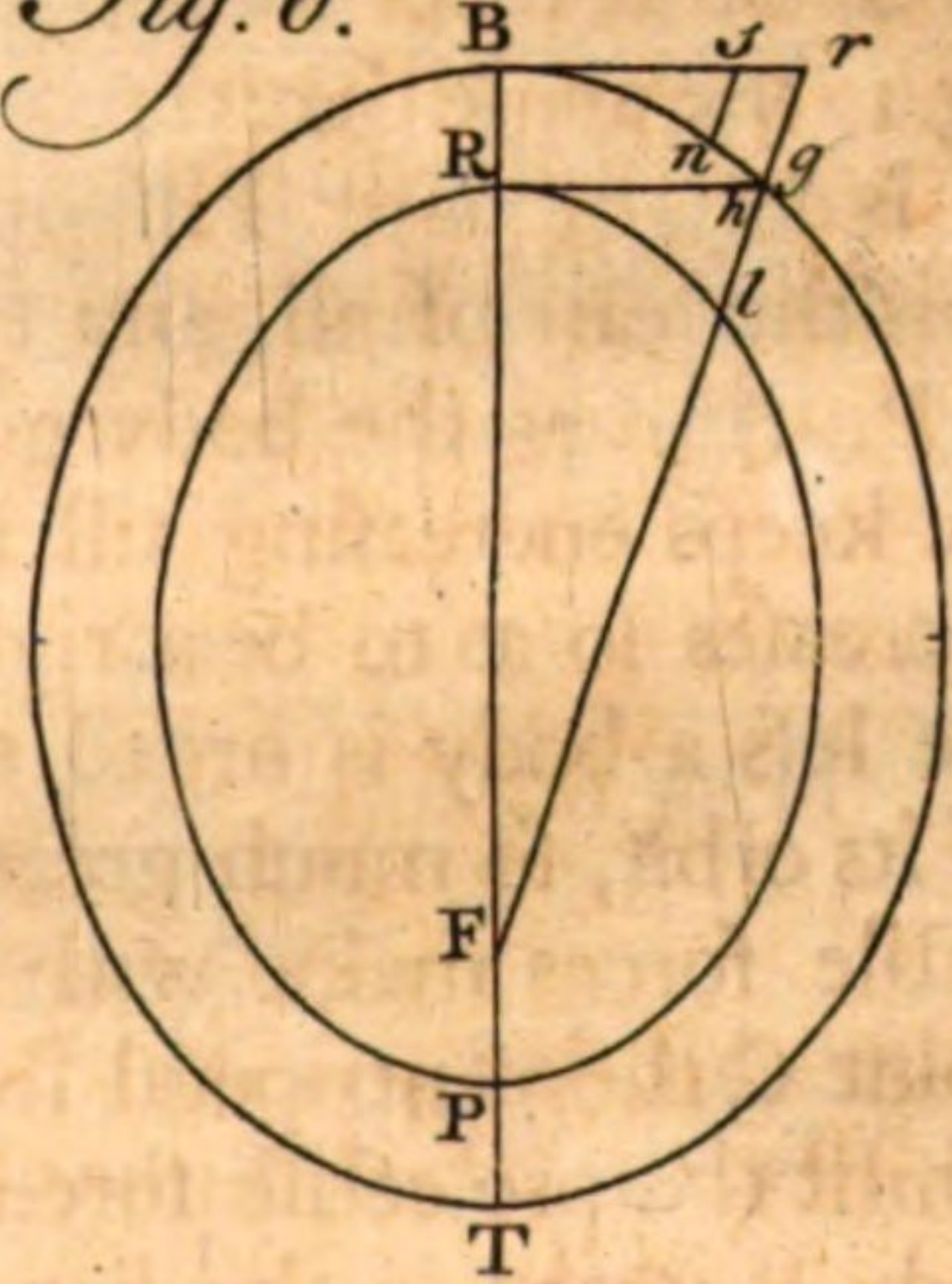
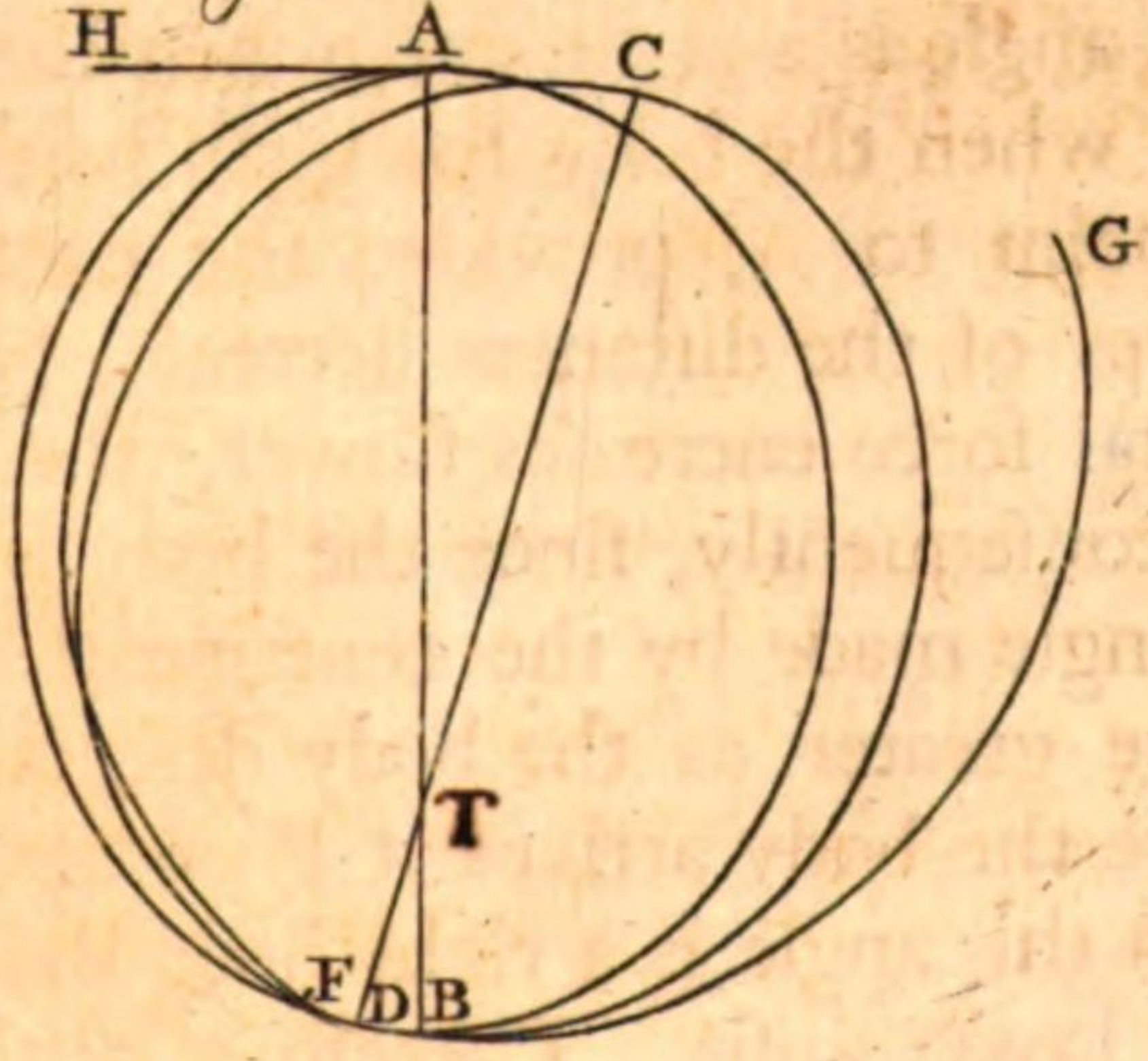


Fig. 7.



Fig. 8.



than the squares of the distances decrease, the body setting out from its higher apsis will arrive at its lower before it gets half round S; and consequently the line of the apses must have turned round upon S in a contrary direction to that in which the body is moving, so that the lower apsis may meet the body.

In like manner, when the body is at P the lower 'apsis, and begins to ascend towards A, the angle made by the centripetal and projectile forces, which is a right one at P, will be a right one again when the body arrives at A or at the point opposite to P; provided the centripetal force decreases as fast as the squares of the distances encrease. But a centripetal force, which decreases slower than this, will as the body ascends from P be every where greater, and consequently the body will be every where in its ascent more urged towards S. For which reason the angle made by the centripetal and projectile forces will be every where less. Therefore it will be reduced to a right one before the body arrives at A, that is, before it has moved from P to the opposite point or half round S. But if the body setting out from P the lower apsis gets to the higher apsis before it has moved half round S, this higher apsis must have come to meet the body, that is, the line of the apses must have turned upon the point S in a contrary direction to the motion of the body.

This proposition we shall likewise apply in astronomy to the moons motion. For at the quarters, the disturbing force of the sun causes the centripetal force, which impells the moon towards the earth, to decrease and encrease slower than in the inverse proportion of the squares of the moons distances from the earth. Upon which account the moons orbit at the time of the quarters is to be considered as a moveable ellipsis, whose longer axis turns upon the earth, placed in one of its focuses, in a direction contrary to that in which the moon itself moves.

I hope that what has been said about the motion of bodies both in immoveable and moveable elliptical orbits will be useful to some of my readers, and that they, who are masters of a better method of proving these propositions, will excuse the many errors and want of exactness in these tedious and prolix attempts to explaine what I have found to be very little understood by such persons as are unacquainted with the deeper parts of mathematics.

The end of Mechanics.

A SYSTEM OF NATURAL PHILOSOPHY.

OPTICS.

CHAP. I.

Definitions and lemmas.

THE cause and nature of vision are properly the subject of that part of philosophy, which is called by the name of optics. But as light is the cause of vision the word *optics* is commonly used in a more extensive sense; and every thing in philosophy is looked upon as a part of optics, which relates to the nature and qualities of light.

If we use the word *optics*, in the stricter sense of it, for the theory of vision; the science of optics is divided into two parts: one part is called dioptrics, and the other catoptrics. The laws of refraction, and the effects, which the refraction of light has in vision, are the subject of dioptrics. The laws of reflection, and the effects, which the reflexion of light has in vision are the subject of catoptrics.

Thus, when the rays of light pass through a convex lens or burning glass, they are refracted or bent from a straight line. And it is a part of dioptrics to shew, both in what manner they are bent, and likewise what effect this bending of them will have, if we look at an object through such a lens. When the rays of light come straight forwards from any object that we are looking at, and are not refracted at all, till they come to the eye; yet as they pass into the eye, they are refracted; and vision itself depends upon this refraction. Therefore, as the rays of light are refracted by the eye in order to make us see objects, we find that in treatises of dioptrics the structure of the eye, the effects that it has upon light, and the whole theory of plane vision, or vision with the naked eye, are usually explained.

When the rays of light strike upon a polished surface of glass or metal they are thrown back from it or reflected. And it is the business of catoptrics

catoptrics to shew both in what manner this reflexion is made and what effect it will have upon an eye that looks in the glass.

These few instances may serve to bring the reader acquainted with the meaning of the words *dioptrics* and *catoptrics*. The distinction itself will be of no great use to us. For it is scarce worth our while to separate these different branches of optics from each other, and to keep the dioptrical part quite distinct from the catoptrical. Indeed there are many propositions in optics where both parts are mixed; and many that cannot properly be reduced to either. And therefore in the following treatise I shall not attend much to the division of this science into these two branches but shall rather chuse to follow such a method as the nature of the subject leads me to.

Before we enter upon the science itself, there are several terms to be explained; such as the reader may be supposed unacquainted with, as they are not much used in common life and are many of them almost peculiar to optics. There are likewise some lemmas to be demonstrated, that is, some propositions are to be proved, which are previous to the science now before us, and will clear the way to the demonstration of those, which follow.

1. *A ray of light is the least particle of light, that can either be intercepted alone, whilst all the rest are suffered to pass, or that can be let pass alone, whilst all the rest are intercepted.*

In the first parts of optics we shall not keep exactly to this definition of a ray of light, but shall use the word *ray* in a larger sense for a single particle of light as it comes from the sun. And however small a particle of light is in that state, in which it is emitted by the sun; yet, when we come in the latter part of optics to examine more accurately into the nature of light, we shall find that these particles are not the least possible, not the least, that can be intercepted alone, whilst all the rest are suffered to pass, or that can be let pass alone, whilst all the rest are intercepted. For a ray, as it comes from the sun, will be found to be made up of many other smaller rays, which are in their nature and quality quite different from each other. Every ray, in the sense we shall use this word at first, is properly speaking a bundle of seven smaller rays. But, till we come to shew by what means these smaller parts may be separated from each other, we shall consider this bundle as if it was but one single ray, or as if it was the smallest part into which light can be separated.

2. *Every thing is called a medium in optics, that is transparent, or that affords a passage for the rays of light to go through it.*

Air, water, glass, a diamond, or even a vacuum are any of them a medium; for they are all transparent or give light a free passage through them.

3. *The inflexion of a ray of light consists in its being bent from a strait course, whilst it is passing on in one and the same medium.*

If a beam of light or number of rays is let into a dark room through a hole in the window-shutter, each ray, of which the beam consists, will move strait forwards or in a right line. But if the edge of a knife be held near the beam, the rays that pass nearest to the knife are bent from their strait course towards it. If HIF, Plat.VII. fig. 1. is the edge of a knife, near to which a beam of light is to pass consisting of several rays, some of which describe the lines AB, EF, and would, if they were left to themselves, go on in strait lines BK, FL, these rays suffer an inflexion at the edge of the knife; that is, they are bent towards it, and are made to describe the lines BD, FG. But in this bending the rays still pass on, and are not thrown back again or reflected. Nor are they thus bent by passing out of one medium into another; for as they were moving in the air, whilst they described the lines AB, EF, so likewise they continue to move on in the air, that is, in the same medium, when they are bent from their strait course into the lines BD, FG.

4. *The refraction of a ray of light consists in its being bent from a strait course, whilst it is passing out of one medium into another.*

If a ray of light describes the line AB, Plat.VII. fig. 2. whilst it is moving in the air, and at B passes out of the air, which is one medium, into a piece of glass XYZ, which is a different medium; it will not go strait forwards in the glass and describe the line BD, but will be bent in its passage at B, and be made to describe the line BC. This bending of the ray in its passage out of one medium into another is the refraction of it.

5. *The reflexion of a ray of light consists in its being so much bent from a strait course as to be turned back again into the same medium or same part of the medium that it was moving in before.*

If ABDE, Plat.VII. fig. 3. is a thick piece of glass and the air is behind the lower surface of it DCE, a ray of light, which described the line AC, instead of passing into the air, when it comes to C, may be bent back again into the glass or the same medium it was in before, and be made to describe the line CB. This bending of the rays course is called the reflexion of it.

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But as this is a less usual sort of reflexion, we may instance in another. If DCE is the surface of a common looking-glass, and a ray of light describes the line AC in the air, and strikes upon the glass at C, it is bent in its course at C so much as to throw it back again into the same part of the air that it was moving in before, that is, into the air, which lies before the glass. This bending of the ray is the reflexion of it.

By this time we must have seen how the inflexion of light differs both from the refraction and the reflexion of it. When a ray is inflected, its course is bent, and yet the ray continues all the while in one and the same medium. When it is refracted, its course is bent by passing out of one medium into another. When it is inflected, it is bent but a little, and passes on from the part of the medium that lies before the inflecting substance to the part that lies behind it. When it is reflected, it is bent so much as to be thrown back again into the same part of the medium, that it was moving in before.

In what refraction differs from reflexion is plane likewise from the foregoing definitions. For in refraction the ray passes out of one medium into another, whereas in reflexion it is thrown back again and continues in one and the same medium.

6. *The angle of incidence is that, which is contained between the line described by the incident ray and a line at the point of incidence drawn perpendicular to the surface, on which the ray strikes.*

When the ray describes the line AB, Plat.VII. fig. 2. and falls on the refracting surface XBZ; at B the point of incidence, or point where the ray falls, draw BH perpendicular to XZ: then the angle ABH contained between AB and BH is the angle of incidence. Or if the ray falls in the direction AC on the reflecting surface DE, Plat.VII. fig. 3. then drawing OC, at the point of incidence, perpendicular to DE, the angle ACO, which is contained between AC and CO, is the angle of incidence.

7. *The refracted angle is that, which is contained between the line described by the refracted ray, and a line perpendicular to the refracting surface at the point where the ray passes through that surface.*

When a ray of light that has described AB, Plat.VII. fig. 2. passes out of air into glass through the surface XZ, and in passing is refracted, so that, instead of going straight forwards in the line BD, it is bent into and made to describe the line BC; if the perpendicular HB is continued to G; the angle CBG is contained between BC the line described by the refracted ray and BG a perpendicular to the refracting surface at the point B. And this is called the refracted angle.

8. *The angle of refraction is that, which is contained between the direction of an incident ray, and the direction of the same ray after it is refracted.*

AB, Plat.VII. fig. 2. is the line which the incident ray describes; and BD, or the line AB continued beyond the refracting surface, is in the same direction in which the incident ray moved. BC is the direction of the refracted ray. The angle DBC, which is contained between these two lines, is the angle of refraction.

9. *The angle of reflexion is that, which is contained between the line described by a reflected ray and a line perpendicular to the reflecting surface at the point of reflexion.*

If a ray, which at its incidence described the line AC, Plat.VII. fig. 3: is reflected from the point C into the line CB, and OC is drawn at this point C perpendicular to DE the reflecting surface; the angle OCB, which is contained between these two lines CB and CO, is the angle of reflexion.

10. *Rays of light, that come from one and the same point, and, as they go farther from this point, spread and separate farther from each other, are called diverging rays.*

If rays of light flow or are propagated from B, Plat.VII. fig. 4. in the directions BD, BA, BC, BE, these rays are diverging ones. They come from one and the same point B, and as they go farther from this point they spread and separate farther from each other. For it is evident, that these rays are farther asunder when they are at D and E, than when at F and G.

11. *Rays of light are called converging ones, when they tend or are directed to one and the same point, and, as they approach this point, come nearer to each other.*

If rays are propagated in the lines DB, AB, CB, EB, Plat.VII. fig. 4: so as to be all of them directed to B; these rays converge. They are directed or tend to one and the same point B; and, as they approach this point, they come nearer to each other. For it is evident, that they are closer together when at F and G, than when at D and A.

12. *Rays of light are parallel, when the lines, that they describe, are parallel.*

13. *The point from whence diverging rays are propagated is called a radiant.*

Thus

Thus B, Plat. VII. fig. 4. is the radiant from which the rays that describe BD, BA, BC, and BE are propagated.

14. *A focus is that point, to which converging rays tend.*

Thus if converging rays describe DB, AB, CB, and EB, Plat. VII. fig. 4 so that all of them tend or are directed to B; then B is the focus of those rays.

From what has been said, it appears that, if rays, which diverge from B in the directions BD, BA, BC, and BE, are by any means turned back again in the lines they were describing, they then become converging ones. And consequently B, which is considered as the radiant, when the rays diverge, becomes the focus, when they are made to converge.

15. *Rays, which converge to a focus, when they have met at it and pass on from thence, become diverging, and the focus is changed into a radiant.*

If two rays describe the lines AC, BC, Plat. VII. fig. 5. and so converge to the focus C; when they have met at C, they must pass on from thence straight forwards in the same lines that they were describing before, that is, in the lines CE and BD. But rays that describe CE and BD diverge from C as from a radiant. Therefore the same point, to which the rays tended at first, is the point from whence they are propagated afterwards; or C, which is considered as the focus of the rays, when they are on one side of it, and are coming to it, is to be considered as the radiant, when they are on the other side, and are going from it.

16. *That point from whence refracted or reflected rays appear to diverge is called an imaginary radiant.*

If rays diverge from R Plat. VII. fig. 6. in the lines RA, RB, then R is the radiant. But if those rays by passing through the surface Ss are refracted into the lines AD and BG, instead of going straight forwards in the lines AC, BE; then the rays after refraction diverge from one another just as much as they would have done if they had come straight from r : and as they appear to have been propagated from r or go on as if they had come from thence, this point r is likewise a radiant. But because the rays did not come from r , but are only made by the refraction to proceed as if they had come from thence; r though a radiant is not a real one, but, to distinguish it from a real one, is called an imaginary one.

In:

In like manner, if rays diverge from a radiant A Plat. VII. fig. 7. in the lines Ab , Ad , Ae , and fall upon the surface of a looking-glass be , from whence they are reflected in the lines bf , dg , eh ; these rays after reflexion proceed as if they had come from the point a . This point a , which is the radiant from whence the reflected rays appear to diverge, is an imaginary radiant.

The same point, which is here called an imaginary radiant, is sometimes called a virtual focus. The reason of this latter name will be explained hereafter.

17. *When converging rays are turned out of the way by being either refracted or reflected, the point, towards which they tended before refraction or reflexion, is called an imaginary focus.*

It is called a focus, because the rays are directed towards it; but then it is only an imaginary one, because the refraction or reflexion, which they suffer, prevents them from meeting there. If two rays describe the lines TM , WN , Plat. VII. fig. 6, so as to converge towards f ; and if, before they arrive at f , they pass through the surface Ss and instead of going straight on in their first course or in the lines Mf , Nf , are refracted into the lines MF , NF , then these rays, though they tended towards f , will never meet there; and f , which was their focus before refraction, is an imaginary focus.

In like manner if rays, that converge, describe the lines fb , gd , he , Plat. VII. fig. 7. so as to tend towards the point a , this point is their focus. But if, before they arrive there, they fall upon the surface of a looking-glass be and are reflected from thence in the lines bA , dA , eA , then they will meet at A , which is therefore their real focus, and the point a , towards which they were directed before this reflexion, is their imaginary focus.

18. *A lens is a thin round piece of polished glass, which has either both its sides spherical, or else one of them spherical and the other plane.*

Spherical surfaces are of two sorts; either convex or concave. From whence it is evident, that, as a thin piece of glass has two sides, by multiplying the number of sides into the different sorts of surfaces, which such a glass may have, there may be six different kinds of glasses. A glass may have both sides convex; or one convex and the other plane; or both sides concave; or one side concave and the other plane; or one side convex and the other concave; or both sides plane. But the last sort of glass, that has both sides plane, is not called a lens; so that there are only five sorts of lenses.

A glass, which is convex on both sides is called a convex lens. The name *lens* is given it, because it is in the shape of a lentil. And from this one glass, which is properly called a lens, the name has been transferred to other glasses having either one or both sides spherical, though they are not of the same shape. If a convex lens was to be cut into two equal parts by a plane passing through the middle of it, the section, or place where it is cut, would appear like *abcd*, Plat. VII. fig. 8. A glass which is convex on one side and plane on the other, is a plano-convex lens. A section of it would be like *ef*. A glass, which is concave on both sides, is called a concave lens; and *gb* is a section of it. A glass, which is concave on one side and plane on the other, is a plano-concave lens, and *ik* is a section of such a lens. A glass, which is convex on one side and concave on the other, is called a meniscus or moon-shaped lens, because a section of it *lm* is in the form of a moon or crescent.

19. *The axis of a lens is a right line, which is conceived as passing through the substance of it, and is perpendicular to both its surfaces.*

If any of these sections of the several sorts of lenses between A and B Plat. VII. fig. 8. was to turn round upon the line *cd*, which is drawn through the substance of each of them, and is perpendicular to both the surfaces; the section in such a revolution would generate the lens, of which it is a section. Thus, for instance, if *ef* was to turn round upon *cd*; *ecf* which is a segment of a circle would, as it turned round, describe a segment of a sphere; and the side towards *c* would be convex. The right line *adb* would in the mean time describe a plane surface. Therefore both together would describe or generate a solid, which would be convex on one side and plane on the other: and such a solid is a plano-convex lens. Because the lens, considered as a mathematical solid, is generated by a revolution of a section of it upon the line *cd*; therefore this line is called the axis of the lens.

20. *The poles or vertexes of a lens are the two extremities of its axis.*

In each of the lenses Plat. VII. fig. 8. *c* and *d* are the two poles or vertexes.

21. *Diverging rays, when they fall upon a lens, form a cone, the apex of which is the radiant, and the base is the lens.*

Let *k* Plat. VII. fig. 9. represent a radiant, and *ab* a convex lens: then if *ka*, *ke*, *ko*, *kb*, are the lines, which rays of light diverging from *k* and falling upon the lens describe, this whole diverging beam and the lens together are a cone, the apex of which is the radiant *k*, and the base is the lens *ab*.

Indeed this figure is not a cone: for upon a plane surface such as the paper is, the entire cone could not be well represented so as to make it answer the uses we shall have of it. But akb is a section of this cone made by a plane passing quite through it from the apex k to the base ab . And if the lens is conceived to turn round its axis cd , and the other part of the figure ak and bk to turn round at the same time upon the line kc ; in such a revolution $acbd$ would generate a circular lens for the base of the cone, and akb would generate the cone itself. That is, the whole figure kab revolving in this manner would describe the space which is taken up by the lens itself and by rays diverging in all directions from k and falling upon the lens.

In like manner if B , Plat. VII. fig. 10. is a radiant, and rays diverge from it in the directions BR , BD , BS ; these rays and the lens together form a cone, the apex of which is B , and the base RS . Or, if C is a radiant, and the rays, which diverge from it describe the lines CR , CD , CS ; these rays and the lens form an oblique cone RCS , the apex of which is C , and the base RS .

22. *In every beam of light the middle ray is called the axis.*

In a diverging beam, or cone of diverging rays, such as CR , CD , CS . Plat. VII. fig. 10. or BR , BD , BS , the middle ray, or that which describes CD in one of these cones, or BD in the other, is the axis of the cone, and is therefore called the axis of the beam. And from hence in a beam of parallel light the middle ray is likewise called the axis of it. Thus if rays were proceeding in the parallel lines bR , $\propto D$, bS , the middle ray, or that which describes $\propto D$, is the axis. And in the parallel beam aR , $\propto D$, aS , the middle ray $\propto D$ is the axis.

23. *Rays are said to fall directly upon a lens, if their axis coincides with the axis of the lens: but if their axis is oblique to that of the lens, they are said to fall obliquely.*

The diverging beam BR , BS , BD , Plat. VII. fig. 10. falls directly upon the lens RS . For though all the rays which compose that beam are not perpendicular to the lens, yet the middle ray or axis BD is. Therefore since the axis of the lens is likewise perpendicular to its surface, the middle ray, as it passes through the lens, describes this axis, that is, the axis of the beam coincides with the axis of the lens.

But CR , CS , CD , is an oblique beam, for the middle ray or axis CD is oblique to the surface of the lens, and consequently is oblique likewise to a line drawn perpendicular to this surface at D , which line is the axis of the lens.

24. *When a beam of light passes through a lens, if the refraction of the lens makes the rays converge, these refracted rays form a cone, whose base is the lens and whose apex is the focus.*

When the rays of light, which at their incidence describe ka, ke, kc, ko, kb , Plat. VII. fig. 9. have passed through the convex lens ab , and are made to converge in the lines af, nf, df, rf, bf , so as to meet in the focus f ; all these refracted rays taken together make up a cone, having the lens ab for its base, and the focus f for its vertex. The whole cone is not indeed expressed in this figure; for ab is not the whole lens or base, but only a section of it; and consequently the figure represents only such a section of the converging rays, or refracted cone, as would be produced by a plane passing through the middle of its base and through the middle of the cone itself so as to split it quite up to its apex.

Or, if the incident rays are not diverging but parallel; as they would be, if they described the lines bR, xD, bS , Plat. VII. fig. 10; then likewise the refracted ones being made to converge form a cone of the same sort having the lens RS for its base, and the focus B for its apex. The rays aR, xD, aS , which are parallel at their incidence and fall obliquely, form an oblique cone RAS after they are refracted.

25. *When a beam of light is made to converge by being refracted, the incident rays and refracted cone taken together is called a pencil of rays.*

The incident beam akb , Plat. VII. fig. 9. and the refracted one afb taken together make a pencil. So likewise, when the incident rays are parallel, and describe the lines bR, xD, bS , Plat. X. fig. 10. they and the refracted cone RBS taken together make a pencil. This is called a pencil, because, if B the focus is received upon a paper, it will paint the image or figure of that point from whence the rays came which are united in that focus. Thus if there were several beams of light, the rays of one beam, describing aR, xD, aS , the rays of another describing bR, xD, bS , and the rays of a third cR, yD, cS , these rays, when refracted, would form three cones RAS, RBS, RCS . And these cones with their respective incident beams make three pencils. Now supposing these three beams to have proceeded from a cross at a great distance from the lens, one from the highest point of it, another from the middle, and the third from the lowest point; the focuses of these pencils, received upon a paper at ABC would paint the image of these three points in the cross. And if the rays proceeding from each point

of the cross were refracted in the same manner, they would all of them together paint the entire image of the cross ABC. The manner how this effect is produced will be explained in its proper place. It is here mentioned only to shew the reason why the incident rays of any beam and their refracted cone taken together should be called a pencil.

26. *The sines of small angles are nearly proportional to the angles themselves.*

Upon C Plat. VII. fig. 11. as a center describe the circle ABED. If the angle ECG is double the angle ECF; then the arc EG, which subtends the former, will be double the arc EF, which subtends the latter. And in all cases whatever, whether the angles are great or small, the arcs that subtend them are greater or less in the same proportion that the angles are; a double angle is subtended by a double arc, a triple angle by a triple arc, and so on. But the sines of angles, when they are great ones, are not proportional to the angles. For though a double angle is subtended by a double arc, yet it has not a double sine. In the angle ECG from G the extremity of one leg draw GK perpendicular to the other, and GK, which is the sine of this angle, is plainly not double Fp, which is the sine of the angle ECF; notwithstanding the angle ECG, of which GK is the sine, is double the angle ECF, of which Fp is the sine. For if FI is drawn parallel to EC, then Fp and KH are equal, because they are opposite sides of a parallelogram Euc. b. I. prop. 34. and it is evident that KH is more than half KG, or that KG is not double KH, and consequently is not double Fp. This is the case in great angles: but in small ones it is otherwise. For if the angle ECO, is double ECL and consequently the arc EO double the arc EL, the sine On will likewise be very nearly double to the sine Lm; in the same manner a triple angle would have nearly a triple sine, and so on; the sines encreasing, whilst the angles are small, in a proportion so near that of the angles themselves, that it may be looked upon as the same proportion without any sensible error.

It is plain that if the arcs and sines coincided, they must always encrease in the same proportion, because they would always encrease together and always be equal to each other. Now in great angles the sines and arcs differ very much or are far from coinciding; as GK differs much from GE, and Fp from FE. But then, since Fp is nearer coinciding with FE than GK is with GE, it follows that Fp will differ less from FE than GK does from GE, and consequently that Fp does not bear the same proportion to FE, that GK bears to GE. Therefore

fore neither does Fp bear the same proportion to GK , that FE bears to GE ; Euc. book V. prop. 16. or the sines are not in the same proportion with the arcs, and consequently are not in the same proportion with the angles, which those arcs subtend. But in small angles the sines and arcs nearly coincide; and consequently differ little from each other. As On and OE are nearly equal, and so are Lm and LE . Therefore On and OE will encrease in nearly the same proportion, and likewise Lm and LE : and On bears nearly the same proportion to OE that Lm does to LE , or, On is to Lm as OE to LE nearly; Euc. b. V. prop. 16. that is, the sines are nearly proportional to the arcs and consequently to the angles, which those arcs subtend.

The angles we shall have occasion to speak of in optics are commonly very small ones, and therefore we may for the future consider the sines as the measure of them, or may look upon the sines as proportional to the angles.

27. *The divergency of rays is measured by the angle contained between the lines, which the rays describe.*

The divergency of rays is considered as quantity, or as being capable of greater and less. Rays have a greater divergency as the lines of their direction contain a greater angle; and a less divergency as those lines contain a less angle. The rays which describe the lines BD and BE Plat. VII. fig. 4. diverge just as much more than those, which describe BA and BC , as the angle DBE is greater than the angle ABC .

The reader must observe that the divergency of rays is measured by the angle, which the lines of their direction contain, and not by the distance between those lines. For though the rays, that describe BD , and BE are at a greater distance from each other, when they are at D and E , than when at F and G , yet their divergency at D and E is no greater than at F and G : because the angle DBE , and FBG is the same. So likewise Plat. VIII. fig. 1. the rays, which come from the radiant D and describe DA and DB , diverge more than those, which come from the radiant E and describe EA and EB , though the distance between these rays when they are at A and B is the same. For the angle ADB is greater than the angle AEB ; and these are the angles, which measure their divergency.

28. *If the distance between diverging rays, that come from different radiants, is the same; then the distances of the radiants are inversely as the divergency of the rays, which come from them.*

If there are two different radiants D and E Plat. VIII. fig. 1. and the rays diverging from D describe the lines D A, D B, and those from E describe the lines E A, E B, so that the distance between the former rays, or AB, is the same as the distance between the latter, then those rays, which come from the nearer radiant D have a greater divergency than those, which come from the remoter radiant E. And we are to shew that CD, or the distance of D, is just as much less than CE, or than the distance of E, as the angle ADB, or the divergency of the rays proceeding from D, is greater than the angle AEB, or, than the divergency of the rays proceeding from E. That is, just in the same proportion as the distance CD is less than the distance CE the divergency of the rays proceeding from D is greater than the divergency of the rays proceeding from E; or the distance of the radiants are inversely as the divergency of the rays.

Now ADC is half ADB, and likewise AEC is half AEB. And what can be proved of half the angles of divergency, may be inferred to be true of the whole angles. I say therefore, that ADC is as much greater than AEC as CD is less than CE, or that, ADC is to AEC as CD to CE inversely, or as CD to CE inverted; that is, upon the whole ADC is to AEC as CE to CD: for CE and CD are CD and CE inverted.

At E draw EF so as to make the angle FEC equal to the angle ADC. Then it is plain that ADC is just as much greater than AEC, as FEC is greater than AEC: because ADC and FEC are equal by the construction. But if these angles are small they are very nearly in the proportion of the subtenses FC and AC, by proposition 26; that is FEC or ADC is just as much greater than AEC as FC is greater than AC. We are therefore to enquire how much FC is greater than AC. Now ADC is equal to FEC by the construction, and ADC is the external angle and FEC is the internal opposite angle on the same side made by the line CE crossing the two lines AD and FE. Therefore AD and FE are parallel, Euc. b. I. prop. 28. And because AD and FE are parallel, therefore CAD is equal to CFE, or the external angle made by a line CF crossing these two parallel ones is equal to the internal opposite angle on the same side. Euc. b. I. prop. 29. Therefore there are two angles in the triangle FEC respectively equal to two angles in the triangle ADC; and consequently the third angle in one must be equal to the third angle in the other. Euc. b. I. prop. 32. corol. 2. That is; these two triangles FEC and ADC are equiangular, from whence it follows that their sides are proportional. Euc. b. VI. prop. 4. FC to AC as EC to DC. We find therefore that EC is just as much greater than DC as FC is greater than AC; or as the angle FEC or its equal the angle ADC is greater than the angle AEC; or as half the
divergency

divergency of rays that proceed from D is greater than half the divergency of rays that proceed from E. But EC, which is proportional to half the divergency of rays from D, is the distance of the radiant E, and DC, which is proportional to half the divergency of rays from E, is the distance of the radiant D. Therefore these distances of the radiants are as half the divergency of the rays proceeding from them inverted.

But since half the angles of divergency are inversely as the distances of the radiants, therefore twice those halves or the whole angles of divergency must be in the same proportion: twice ADC or ADB must be as much greater than twice AEC or than AEB, as ADC is greater than AEC. But ADC is to AEC as the distances of D and E inversely. Therefore ADB and AEB are likewise as the distances of D and E inversely.

29. *If the distance between converging rays, that tend to different focuses, is the same; then the distances of the focuses are inversely as the convergency of the rays, which tend to them.*

If AD, BD, Plat. VIII. fig. 1. are the lines described by rays converging to the focus D, and AE, BE the lines described by other rays converging to the remoter focus E; so that the distance AB between the former rays and the latter is the same; it is evident, that the rays, which tend to D the nearer focus, converge more than those, which tend to E the remoter focus, or that the angle ADC is greater than the angle AEB. These angles, or the convergency of the rays whose directions contain these angles, are inversely as the distances of the focuses D and E, or as those distances inverted. That is the angle ADB, or the convergency of rays tending to D, is as much greater than the angle AEB, or than the convergency of rays tending to E, as EC the distance of the latter focus is greater than DC the distance of the former focus.

That the angle ADB bears the same proportion to the angle AEB, that EC bears to DC has been already proved in the last proposition; and this is all that is necessary to prove the truth of this.

30. *If rays proceed from a radiant at an infinite distance, their divergency is nothing, and the rays are considered as parallel ones.*

Let F, Plat. VIII. fig. 2. be a radiant from whence rays diverge in the directions FDB, FCA. The distance between these rays at D and C is DC, and at B and A is BA. But BA is greater than DC; therefore the rays, the farther they go from F, are farther asunder from one another; and consequently are diverging ones. This is the case when the radiant F is at a finite distance. But if the distance AF of the radiant F is infinite, then DC will be equal to BA, or however the difference between them will be

be infinitely small; so that when the rays are at B and A the distance between them will be no greater, than when they were at D and C; or these rays proceed in lines which may be looked upon as parallel ones.

Draw AB perpendicular to FA, and CE parallel and equal to AB, and complete the parallelogram ACEB by drawing EB. The triangles FBA and BDE are similar. For the angle BAF is equal to the angle DEB; both because they are opposite angles in the parallelogram ACEB, Euc. b. 1. prop. 34. and because by the construction they are right ones. The angle BFA is likewise equal to the angle DBE: because FA and BE are parallel, and these two angles are the alternate angles made by the right line FB crossing these two parallel ones. Euc. b. 1. prop. 29. And consequently, since there are two angles in one of these triangles respectively equal to two angles in the other, the third angle FBA in one is equal to the third angle BDE in the other. Euc. b. 1. prop. 32. corol. 2. But equiangular triangles have their sides proportional. Therefore ED bears the same proportion to EB, that AB bears to AF. Euc. b. VI. prop. 4. Now if AF was infinite, the finite line AB would be so small in respect of AF as to bear no proportion to it; and consequently when AF is infinite ED will bear no proportion to EB. For as AB to AF, so is ED to EB. But EB is a finite line, whatever the length of AF is. And since ED is so small, when AF is infinite, as to bear no proportion to this finite line EB, it must be infinitely small. But ED is the difference between CD and AB. Therefore, when AF is at an infinite distance, the difference between CD and AB being infinitely small, these two lines may be considered as equal and consequently DB and CA may be considered as parallel. So that if F is a radiant at an infinite distance, the rays, that come from thence and describe lines such as DB, CA, are parallel to each other.

This proposition may likewise be deduced from prop. 28. For, since the divergency of rays is inversely as the distance of the radiant, it follows, that, when the distance of the radiant is infinitely great, the divergency of the rays must be infinitely small. And, when the divergency of rays is infinitely small they may be considered as parallel.

This is the case of rays that come from one and the same point in the sun. I say from one and the same point, because the whole body of the sun is not a single radiant but is made up of many; and the rays which come from its center are not parallel to those which come from its circumference. But all rays which come from the center of it are parallel to one another: because the distance of the radiant in this case, when compared with the distance between any two rays, may be taken for infinite.

finite. In like manner all the rays, which come from one and the same point either in the edge of the sun or in any part of its face are parallel for the same reason.

31. *If rays tend to a focus at an infinite distance, their convergency is nothing, and the rays are considered as parallel ones.*

Let F, Plat.VIII. fig. 2. be a focus to which rays tend in the directions AF and BF: then if the distance of F is infinite the lines BD and AC, which the rays describe are parallel to one another. This has already been proved in the foregoing proposition.

32. *The sides of plane triangles are to one another as the sines of the angles which they subtend.*

By plane triangles we mean triangles, whose sides are right lines. Thus the triangle ABC, Plat.VIII. fig. 3. is a plane one. Now in this triangle the side AB subtends or is opposite to the angle ACB: and the side AC subtends the angle ABC. Set one leg of the compasses on C and with the opening CB strike the arc BF, then continue AC to H and from B draw BE perpendicular to CH. This line BE is the sine of the angle BCE: and it is likewise the sine of the angle ACB. For ACB is the complement of BCE to two right ones; and the complement of any angle to two right ones has the same sine with the angle itself. Or otherwise. BE is the sine of ACB; because the sine of an angle is a line drawn from the extremity of one leg perpendicular to the other leg. Now B is the extremity of one leg, and no line can be drawn from B perpendicular to the other leg AC, unless AC is continued in the direction CH. Therefore a line drawn from B perpendicular to AC continued, that is, the line BE is the sine of the angle ACB. Then making B the center, with the same opening BC of the compasses as before, strike the arc CG and from C draw the line CD perpendicular to BD: this line CD is the sine of the angle ABC. Now these two sides AB and AC are to each other in the same proportion as the sines of the angles ACB and ABC, which these sides subtend. That is AB is to AC as BE to CD.

The triangle BAE is similar to the triangle CAD. For in the triangle BAE the angle at E is a right one, by the construction: and so is the angle at D in the triangle CAD. And the angle at A is common to both of them. Therefore there are two angles in one of these triangles respectively equal to two angles in the other: and consequently the third angle in one or EBA is equal to the third angle in the other or to DCA. Euc. b.1. prop. 32. corol. 2. From whence it follows that the sides of these

these triangles, which subtend the equal angles, are proportional, AB to AC as BE to CD. That is, AB and AC, two sides of ABC, are in the same proportion to one another with BE and CD the sines of the angles, which those sides subtend.

CHAP. II.

Of light, and the cause and laws of its refraction.

33. *The progress of light is not instantaneous, but the rays of it take up some time to pass from one place to another.*

THE Cartesians are of a different opinion and maintain that light is propagated from a luminous body to any distance however great in an instant. But the truth of our assertion is amply proved by observations, that have been made upon the satellites or moons that move round the planet jupiter. Let A be the sun, Plat.VIII. fig. 4. ORPS the earths orbit, F the planet jupiter, and NGH the orbit of its innermost moon or satellite. Now as jupiter is an opaque body it will cast a shadow opposite to the sun; and let FGH represent this shadow. If the satellite, whilst it is describing its orbit, passes through this shadow from G to H, it will be eclipsed as long as it continues in the shadow. For since it has no light of its own, and shines only by reflecting the suns light, when it is in any part between G and H the opaque body of jupiter prevents the sun from shining upon it: and therefore the satellite will receive no light, which it might reflect, and upon this account is eclipsed to a spectator in the earth which we will suppose to be at B. When the satellite comes to H it emerges out of the shadow, and will again become visible to a spectator at B. This innermost satellite is $42\frac{1}{2}$ hours in going round its orbit. That is, if it is at any time seen at H, in $42\frac{1}{2}$ hours it will have gone round and be returned to H again. Therefore if the earth was to be at rest at B, since the satellite emerges every time it comes to H, a spectator at B on the earth would see 30 emergences in the time of 30 revolutions or in $30 \times 42\frac{1}{2} = 1275$ hours time. And the 30th of these emergences would be seen at the end of these 30 revolutions or at the distance of 1275 hours from the first of them. But in 1275 hours the earth instead of standing still at B will be gotten in its orbit to C. And consequently the light from the satellite, after it emerges, has farther to go, before it gets to the earth, than it would have had, if the earth had continued at B. MC is the difference of these two distances. So that if the propagation of light is instantaneous, it will take up no time at all to describe MC; but

but a spectator at the earth will see the 30th emerfion, though the earth is at C, at the fame time that he would have feen it, if he and the earth had continued all the while at B. But in fact it is found that, when the earth is at C, this 30th emerfion will not be feen at the end of the 30 revolutions or 1275 hours after the firft; but fome time, as fuppofe 10 minutes, later. Therefore, fince this emerfion is feen 10 minutes later, when the earth is at C, than when it is at B, the firft ray of light which makes the fatellite vifible after its emerfion is 10 minutes longer in coming to C than it would be in coming to B. And confequently light is not propagated from any one place to any other in an inftant: for it takes up 10 minutes to defcribe the line MC, which is the difference between the two diftances of the earth from the fatellite at the two places B and C.

In like manner if the earth was at D, a longer time will be taken up for the fatellite to enter the fhadow at G 30 times, than would be, if the earth was nearer to it as at E. Or the laft light which comes from the fatellite when it enters the fhadow, will be longer in paffing to D than to E. For the fatellite does not difappear, or is not feen to enter the fhadow, or is not loft fight of, till this laft light is come to the fpectators eye. And confequently, fince this entrance happens later when the earth is at D than it does when the earth is at E, this laft light is longer in paffing to D than to E.

By computing from obfervations of this fort, it is found, that if tables of the times of entrance and emerfion of this or any other fatellite are calculated for the middle diftances of the earth from the fatellite, that is, if tables are made for finding the time that an emerfion will happen, when the earth is at O, or an entrance, when the earth is at P, thefe emerfions and entrances will happen about 7 minutes fooner than they ought to do by the tables, if the earth is at R, or at its leaft diftance from the fatellite, and about 7 minutes later than by the tables, if the earth is at S, or at its greateft diftance from the fatellite. Therefore thefe emerfions and entrances are feen at the earth about 2×7 or 14 minutes fooner, when it is at R, than when it is at S. Confequently a ray of light takes up about 14 minutes to defcribe the line RS; and about half 14 or 7 minutes to defcribe half RS or AS. But RS is a diameter of the earths orbit, and AS is a femidiameter of the earths orbit or the diftance of the earth from the fun. Therefore a ray of light is about 7 minutes in defcribing a femidiameter of the earths orbit or in coming from the fun to the earth.

The diftance of the earth from the fun is fo great that, if a cannon ball was to move at the rate of 500 feet in every fecond of time, it

E e

could

could not come from the sun to the earth in less than 25 years. This comparison sufficiently shews the incredible velocity of light. For at this rate the velocity of light will be above 10 million times greater than that of a cannon ball.

34. *The rays of light are propagated in right lines.*

This is plain from the shadows which opaque bodies cast. For if the rays of light did not describe strait lines, they would bend round any opaque body, and would prevent it from casting any shadow. It is plain likewise from lights finding no passage through bended tubes; and from the appearance of it as it passes through any little hole into a room filled with dust or smoke.

Since the rays of light are constantly propagated in right lines from luminous bodies; for the future, whenever we have occasion to represent a ray, we shall do it by a line reaching from the luminous body to the body illuminated; that is, we shall speak of the line, which a ray describes, as if that was the ray itself.

35. *Light is a body, and the several rays, or particles of it, are as well successive in the same line, as contemporary in different lines.*

We have seen already, in prop. 34. that light is propagated in right lines. Whereas if it was only an impulse of a luminous body upon some subtil fluid, it would pass through bended tubes as well as strait ones, and would bend round an opaque body so as to prevent it from casting a shadow; in the same manner that sound, which is owing to vibrations excited in a fluid, is propagated through bended tubes, and when any obstacle is placed between the sounding body and the ear, it goes round that obstacle and is heard on the other side. But since there is no middle opinion, and light is allowed to be either a real body or else an impulse upon some subtil fluid; having shewn that the latter of these opinions is false, we may conclude that the former is true.

That light consists of parts is evident; because in the same place you may stop that which comes one moment, and let that pass which comes presently after; and since that light, which is stopped, cannot be the same with that which is suffered to pass, there must be different parts. And as they come one after the other, they are successive. Again, you may stop the light in one place, and may at the same time let it pass in another; and because, as before, that which is stopped cannot be the same with that which passes, these are different parts; and since some parts are stopped at the same time that others pass, those parts, which are stopped, are contemporary with those, which are let pass.

36. *The rays of light are exceeding small.*

Since the velocity of a ray of light is above ten million times greater than that of a cannon ball; if a ray of light weighed only one grain it would have near 200 times the moment that a cannon ball of ten pound weight, has when it is first discharged from the cannon, and if it weighed only $\frac{1}{200}$ of a grain the moment of the ray would be equal to that of the ball. But if so small a particle as $\frac{1}{200}$ of a grain would have so great a moment, the rays of light must be inconceivably small; or else upon account of their great velocity their moment would be too great for so tender an organ, as the eye is, to bear the continual impulses of them.

If a hole is made with a pin in a piece of paper, a man looking through that hole sees all the objects that are before him, let them be ever so many. And since these objects and each point of them becomes visible by means of rays of light coming from thence to the eye; these rays must be extremely small to be able so many of them to pass through a pin-hole, without interrupting each others passage.

The reader will wonder the less at this extreme minuteness of the rays of light, when he recollects that matter is infinitely divisible, and that though we cannot find out means of separating it into very minute particles, yet still in its own nature it is separable without end. How far it is actually separated by a power infinitely superior to ours, by the power of the Author of nature, is seen in many instances, that we are well acquainted with, or however that we hear of very frequently. If a candle is lighted in the night-time at the top of any high building, it may be seen at the same instant any where at the distance of half a mile all round. Now in any one instant there is no sensible waste made of the candle and yet in one instant rays or particles of light enough must fly off from the candle to fill the whole space, in any part of which the light of it could be seen, and that space is a sphere whose semidiameter is half a mile. We have other instances of the minuteness of matter besides that of light. A piece of musk does not in any length of time lose its smell. And as there must be always effluvia fly off from it, whilst it retains its smell, these effluvia must be extremely small, because notwithstanding they are constantly thrown off for innumerable years, the weight of the musk is not sensibly diminished. Leeuwenhoek by the help of microscopes discovered what very minute particles of matter are capable of animal life. For the least sort of those animals, which he found in pepper-water, were 1000,000,000 times

times less than a grain of sand. We might have amused ourselves a little farther with speculations of this sort, and might have considered how small the entrails and particularly the heart of these animals must be; if the late discovery of the polype had not made us doubt whether entrails and a heart are necessary to the life of an animal.

37. *If a ray of light passes obliquely out of a rarer medium into a denser, it will be refracted towards a perpendicular.*

That light is attracted by other bodies, appears from what has been said in prop. 5. of mechanics: where we observed that, if the rays of it, come through the hole of a window-shutter into a dark chamber, and a knife is held near the hole with its edge towards the rays, those which pass very near it are bent down towards it. The force, which thus bends them, whatever it is or to whatever cause it is owing, we call attraction. It is a force that we are sure exists in nature, and if it is sufficient to explain the refraction of light, the rules of true philosophy will allow us to make use of it, and do not require us to look out for any other. But it is not iron only that attracts light in this manner. The edge of a thin plate of any other metal, or of a piece of broken glass, produces the same effect. Now this attraction will be so much stronger as there are more particles within any given space. Every particle attracts, and consequently the more there are of them the greater effect they must necessarily produce. From hence it follows that a ray of light is more attracted by a dense medium than it is by a rare one. Thus water attracts light more than air does, and glass more than water, and a diamond more than any of them.

Suppose then a ray of light to pass obliquely out of a rarer medium X Plat.VIII. fig. 5. into a denser one Z through the plane surface *ab*, and let *ro* be the incident ray. Thus for instance on the side X of the line *ab* let there be air, and on the side Z glass. The ray *ro*, when it gets into the glass, will not go straight forwards in the direction *og*, but will be refracted towards a perpendicular, that is, towards a line perpendicular to the surface *ab*: *ok* is such a line, and the ray instead of going on in the direction *og* will be bent into *os*, a line nearer to *ok*, or making a less angle with *ok* the perpendicular than *og* does.

The ray is more attracted by the glass than by the air: and for this reason it will be accelerated or made to move faster as it enters the glass. For though, when it gets into the glass, it is attracted equally all ways, and therefore will have no alteration made in its velocity by such attraction, since these equal and contrary forces will destroy one

ano-

another; yet as it is passing out of the air into the glass at o it is attracted more towards the glass Z than towards the air X . And because the ray is moving towards Z , at the same time that it is more attracted thither than it is the contrary way, this attraction conspires with the rays motion, and consequently will encrease its velocity. Therefore the ray moves faster, when it gets into the medium Z or the glass, than it did, whilst it was in the medium X or in the air. But the ray, whilst it was in the air, moved in the line ro , which is oblique to the attracting surface ab . And as the glass attracts the ray directly towards it, or in a line perpendicular to its surface, it follows that, if we resolve the oblique motion of the ray ro into two others, one of them as rc parallel to the surface ab and the other as rb or $co = rb$ perpendicular to it, the parallel motion rc of the ray is not affected by this attraction; for an attraction, which acts in the line rb or co , can neither accelerate nor retard the motion of a ray in the line rc . The change of velocity, which the ray suffers, is made in the perpendicular part of its motion co . Take therefore ok greater than co , and oi equal to bo or to rc . Then, since the rays parallel velocity is not altered by passing out of one medium into the other, as rc was its parallel velocity in the air, $oi = rc$ will be its parallel velocity in the glass. But the perpendicular velocity becomes greater in the glass than it was in the air; and as the line co represented this velocity, whilst the ray was in the air, let ok represent it, when in the glass; that is, let the ray be supposed to be accelerated in the proportion of ok to co , or let its velocity be as much greater in the glass than it was in the air as ok is greater than co . Now the ray, whilst it was in the air, described the line ro , and this oblique motion might be considered as if it was produced by the action of two forces, one in the line rc , the other in the line $rb = co$, which are the two sides of a parallelogram, whose diagonal is ro the direction of the ray; by prop. 20. of mechanics. One of these forces acts parallel to the surface ab and the other perpendicular to it. The parallel part of this motion is not altered by the passage of the ray out of one medium into the other. Therefore, when the ray enters the glass it is still acted upon by a force parallel to the surface ab and equal to rc . This force may then be represented by $oi = rc$. But the other part of the rays motion, which is in a line perpendicular to the surface ab , is accelerated, as the ray passes out of the air into the glass; so that if co was this part of the motion, whilst the ray was in the air, ok a line longer than co must represent it, after the passage of the ray into the glass. Therefore when the ray enters the glass it may be considered as acted upon by two forces.

forces oi and ob , and consequently will describe the diagonal os of a parallelogram, whose two sides are oi and ob ; by prop. 16. of mechanics. Now the parallelogram $oiks$ has one side oi equal to one side rc in the parallelogram $rchb$: but the side ok in the former is longer than the side co in the latter: and these sides ok and co are perpendicular to the surface ab . But in all parallelograms, where one side is given, the longer the other side is the less is the angle which the diagonal makes with it. Therefore, since rc and oi , are given, the diagonal os will make a less angle with the side ok than the diagonal ro does with the side co . But os is the direction of the refracted ray, and ob is the perpendicular; ro is the incident ray, and co the perpendicular. Therefore the refracted ray makes a less angle with the perpendicular, or is nearer to it, than the incident ray; that is, the ray by passing out of air into glass or out of a rarer medium, which attracts it less, into a denser, which attracts it more, is refracted towards a perpendicular.

This proposition is not only proved by reasoning in this manner from the cause of refraction, but is likewise confirmed by matter of fact. Let SB , Plat. VIII. fig. 6. the side of a vessel $SABD$ be perpendicular to the bottom BD : suppose the vessel to be empty and the sun to shine upon the side SB , so as to cast the shadow of this side upon the bottom; and let BD be the length of this shadow, that is, suppose the shadow to reach from B to D . Now this shadow is terminated by a ray IS , which passes by the edge of the vessel S : for all rays that pass above it fall upon the side AD ; and all that pass below it are intercepted by the side SB . Let the vessel then be filled with water up to the top of it SA ; and, though all things else continue as they were, the shadow will become shorter and will reach only from B to C . Therefore the ray IS , which went straight forwards from S to D and terminated the shadow at D , when the vessel was empty, describes the lines SC , when the vessel is full and terminates the shadow at C . Therefore this ray is refracted, when the vessel is full, by passing out of the air, where it described IS , into the water, where it describes SC . This refraction, which is made by the passage of the ray out of a rarer medium into a denser, is towards a perpendicular. For BS is perpendicular to the bottom of the vessel, and consequently BS continued, or SP , is perpendicular to the surface of the water. Therefore ISP is the angle of incidence. But BSD is equal to ISP , because they are vertical angles made by two right lines crossing each other; Euc. b. I. prop. 15. so that BSD as well as ISP is the angle, which the incident rays direction makes with the perpendicular. But BSC is the angle, which the refracted ray makes with the same perpendicular; and

BSC

BSC is less than BSD. Therefore the direction of the refracted ray is nearer to the perpendicular than the direction of the incident one.

Or otherwise. The length of the shadow BD, when the vessel is empty, subtends the angle BSD, which is equal to ISP or to the angle of incidence: and the length of the shadow BC, when the vessel is full of water and the ray passes out of the air into this water, subtends the angle BSC, which is the refracted angle. But the length of the shadow is less when the vessel is full. Therefore the refracted angle is less than the angle of incidence; or the refracted ray makes a less angle with the perpendicular than the incident one does.

38. *If a ray of light passes obliquely out of a denser medium into a rarer, it will be refracted from a perpendicular.*

If an oblique ray of light so , Plat. VIII. fig. 5. passes out of glass into air through the surface ab , that is, if there is a denser medium on the side Z of ab , and the ray passes out of it obliquely through the surface ab into a rarer medium on the other side X; the ray will not go straight forwards in the line ot , but will be refracted into the line or , so that *cor* the refracted angle, or angle, which the refracted ray or makes with the perpendicular oc , will be greater than sok the angle of incidence, or angle which the incident ray so made with the perpendicular ok .

The ray is more attracted by the glass than by the air, or more by the denser medium Z, whatever that medium is, than by the rarer one X. Upon this account when it comes to o , and is passing out of the glass into the air, it is not equally attracted all ways, for the air, into which it is passing, attracts it less than the glass, out of which it is passing. Therefore the ray will be drawn more backwards than it is forwards: and this difference of attraction will retard its motion, so as to make it move slower in the air than it did in the glass. But the oblique motion of the ray in the glass, or so may be resolved into two others, one of them sk parallel to the glass, the other ko perpendicular to it; by proposition 20. of mechanics. The parallel motion of the ray in the line sk can suffer no alteration as to the velocity from an attraction, which acts in the line ok , as the attraction does by which the ray is more drawn towards the glass Z than towards the air X. But this attraction will retard the perpendicular motion ko . From hence then it follows that taking ob equal to sk or io ; because sk was the parallel motion of the ray in the glass, ob will be its parallel motion when it enters the air at o : for the velocity of this motion undergoes no alteration, and therefore may always be represented by a line of the same length. But the perpendicular motion ko is retarded by the
rays

rays passing out of the glass into the air. Therefore, taking on equal to ko , if ko or on represented the perpendicular velocity before the ray was retarded, some line such as oc , which is shorter than on , must represent it afterwards. And, since the ray, when it is passing out of one medium into the other at o , may be considered as acted upon by two forces ob and oc , these forces together will make it describe or the diagonal of a parallelogram, of which ob and oc are the sides. The ray had indeed described the diagonal of a parallelogram, whilst it was in the glass, but then ks and ko were the sides. And since the side ob in one of these parallelograms is equal to the side ks in the other, but oc in the former is shorter than ko in the latter; it follows that or will make a greater angle with oc than so does with ko ; because in all parallelograms where one side is given, the shorter the other side is the greater is the angle which the diagonal makes with it. Now the angle cor , which or makes with oc , is that which the refracted ray makes with the perpendicular, or is the refracted angle; and the angle soh , which so makes with ko , is that which the incident ray makes with the perpendicular, or is the angle of incidence. Therefore, when a ray passes out of glass into air or out of a denser medium into a rarer, the refracted ray will be farther from the perpendicular than the incident one, that is, the refracted angle will be greater than the angle of incidence, or the ray will be refracted from a perpendicular.

We may apply this proposition to a common instance, which will both illustrate the meaning and prove the truth of it. If a vessel NBAL, Plat. VIII. fig. 7. is empty, and any small object, such as a shilling, is laid upon the bottom of it at C, when the spectators eye is at R, he will not be able to see the shilling. For this, like any other object, can only be seen by means of rays of light, which come from it to the eye. And since all rays describe right lines, when no particular cause prevents them, by prop. 34, and no right line can be drawn from C to R but what must pass through the side NA of the vessel, no ray can be propagated from C to the eye at R, because the side of the vessel is in the way and will intercept any ray that was coming from C towards R. But let the vessel be filled with water, and then, though the eye continues at R and the shilling at C, the shilling may be seen. Now it is evident, that no ray can come strait from C to R any more when the vessel is full of water than when it is empty; for the side AN will alike intercept any such ray in either case. Therefore the ray which makes C visible must have been bent or refracted in coming from C to R. This refraction must have happened, when the ray coming from C passed out of the water into the air:

air: for it will describe a right line in the water, and likewise a right line in the air: therefore no bending can happen in any part of the rays passage, but where the water and air are contiguous, or at N where the ray passes out of one medium into the other. It is farther evident, that the ray, which makes C visible to the eye at R, could not whilst it was in the water describe any line below CN, for all rays below this would be intercepted by the side of the vessel NA. Therefore C is made visible by some ray, which in the water describes CN, or some line above it; and either of these cases would equally answer our present purpose. Suppose it therefore to be the ray CN, which comes to the eye at R, then this ray instead of going straight forwards in the line ND, is bent at the point N, or refracted into the line NR: for it comes straight from C to N through the water in the line CN; and it likewise comes straight from N to the eye at R through the air; therefore NR is the line, which the ray so refracted describes; and the object being seen in the direction, which the ray has when it comes to the eye, will appear as if it was at L. That is, the object placed at C, will appear when the vessel is full, just as the same object would, if it was placed at L, when the vessel is empty. But if a ray describes CN, whilst it is in the water, and, as it passes from thence into the air, is refracted into NR, this refraction is from a perpendicular. For, let NP be drawn perpendicular to the surface of the water, then since CN or CD, which is CN continued, is the direction of the incident ray, when it falls upon the surface NB and is about to pass out of the water into the air, DNP is the angle of incidence; and since NR is the refracted ray, PNR is the refracted angle. But PNR is bigger than PND, that is, the refracted angle is bigger than the angle of incidence, or the refracted ray makes a greater angle with the perpendicular than the incident one does. Therefore the ray is refracted from a perpendicular.

39. *All refraction is reciprocal.*

If a ray of light is refracted in passing out of one medium into another, then, if that ray was to return back again, it would be refracted just as much the contrary way. If the ray *ro*, Plat. VIII. fig. 5. in passing out of air into water is refracted into *os*, then supposing this ray to return to the point *o* in the direction *so*, in passing out of the water back again into the air it will be refracted into *or*.

This may be expressed in another manner, which is this: if the refracted ray *os* becomes the incident ray, then the incident ray *or* will become the refracted one. That is, if a ray falling upon the surface *ab* de-

F f

scribed

scribed the line ro , and the line os is found, which that ray will describe, when it is refracted; then supposing a ray to fall in the direction so upon the surface ab the line, which the refracted ray will describe, may be known, for it will be or .

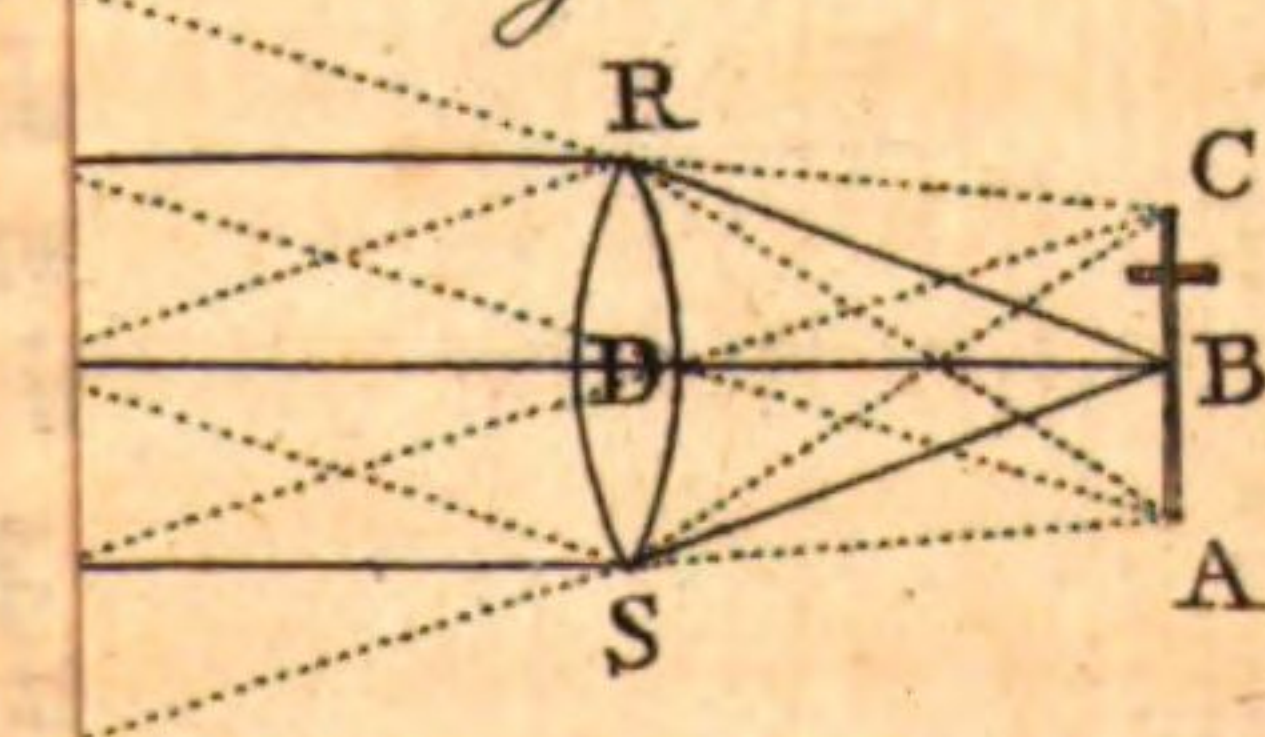
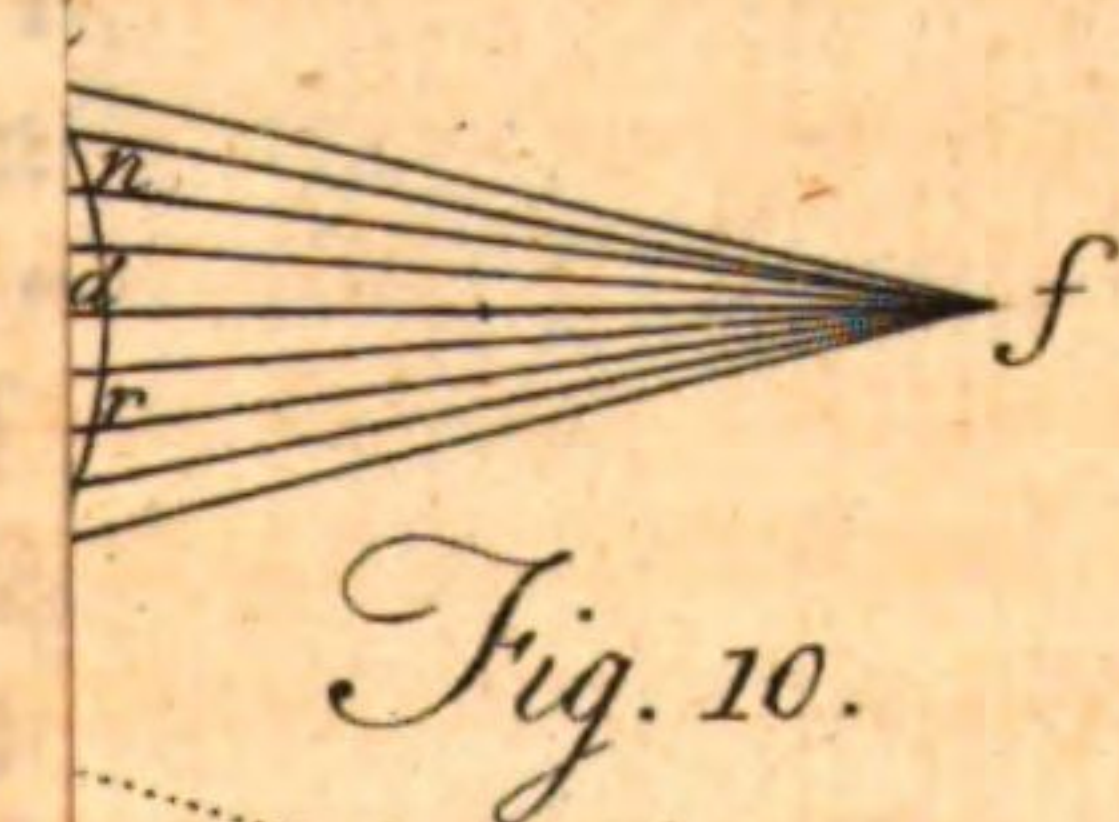
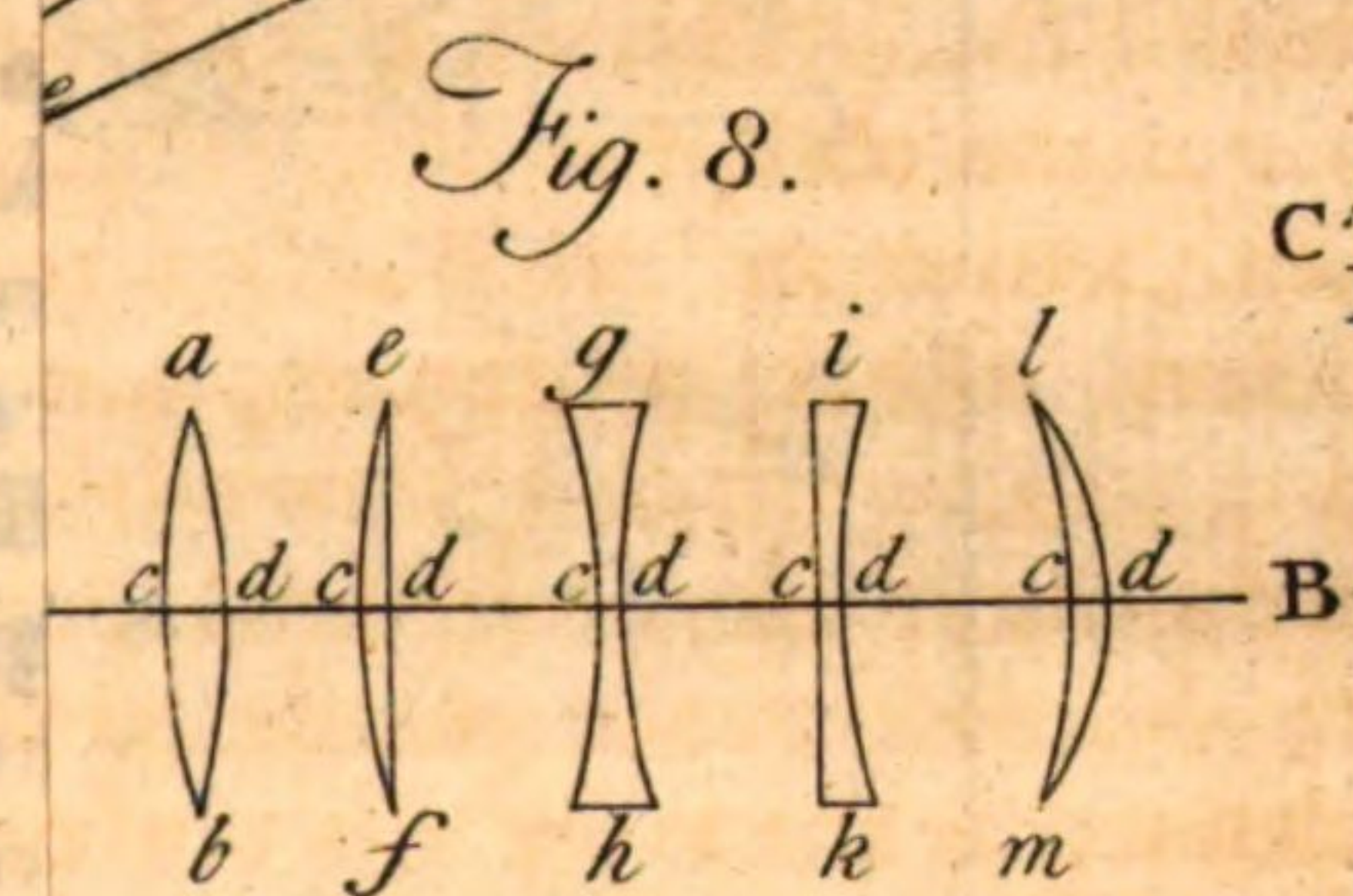
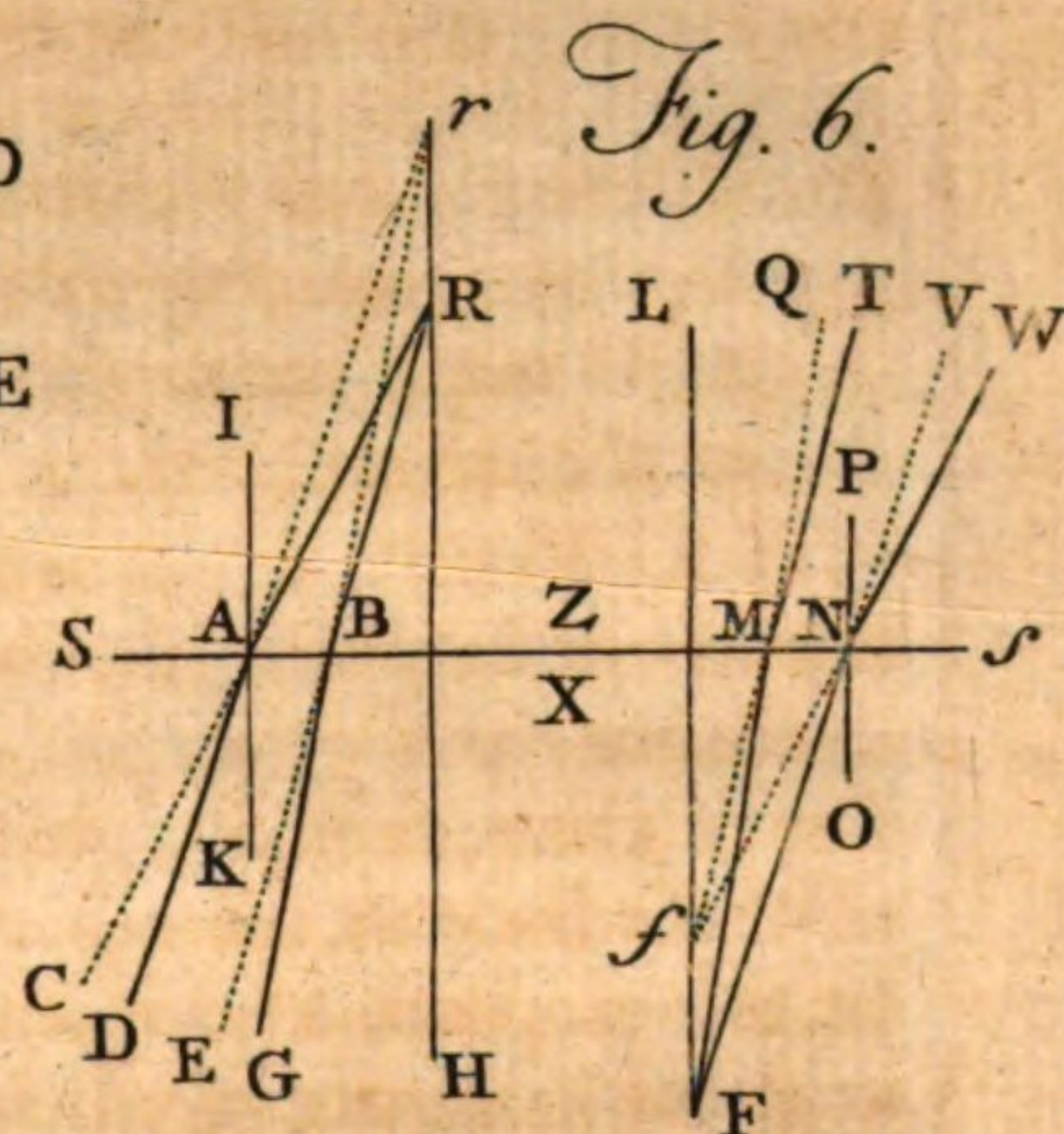
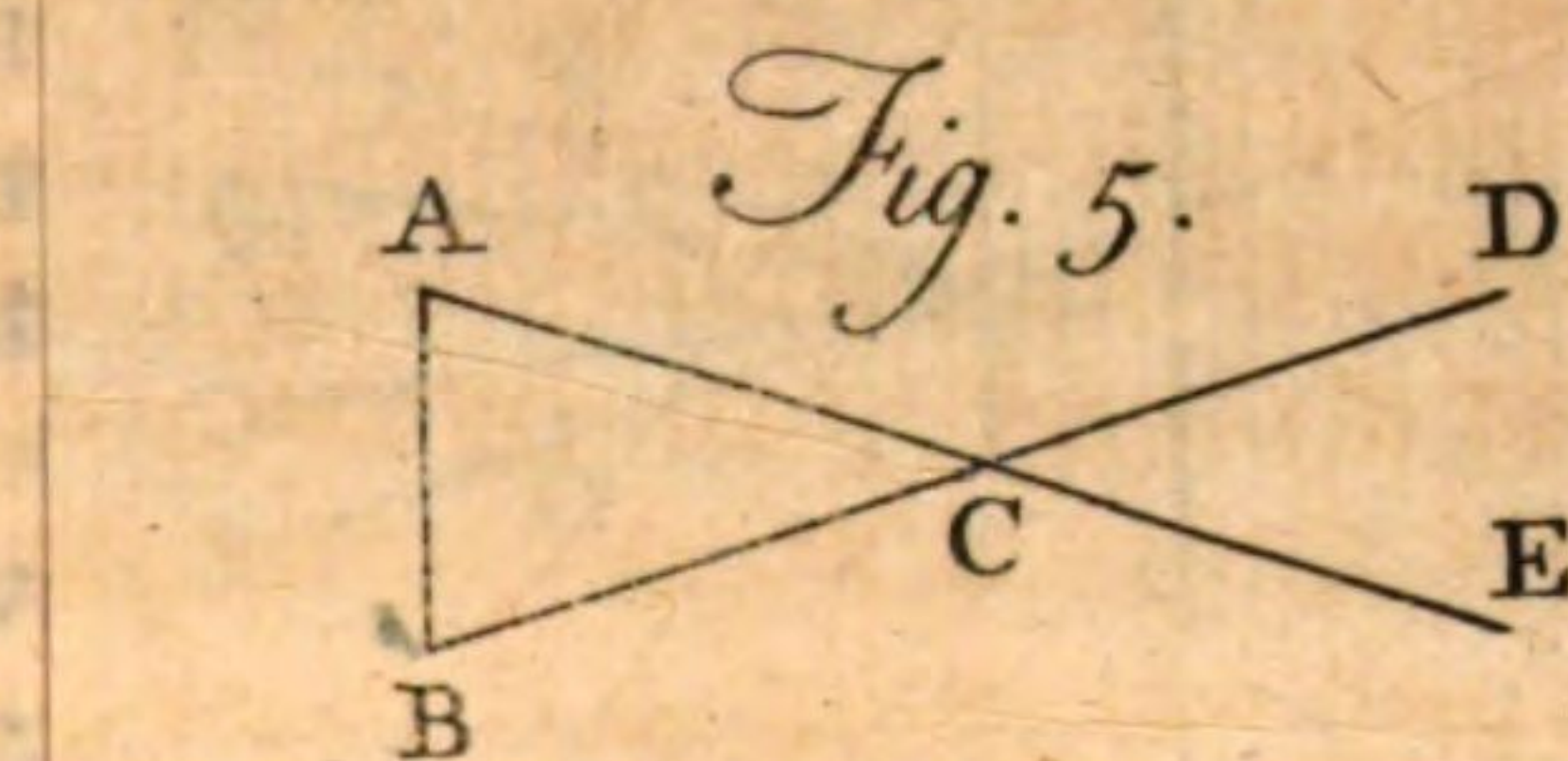
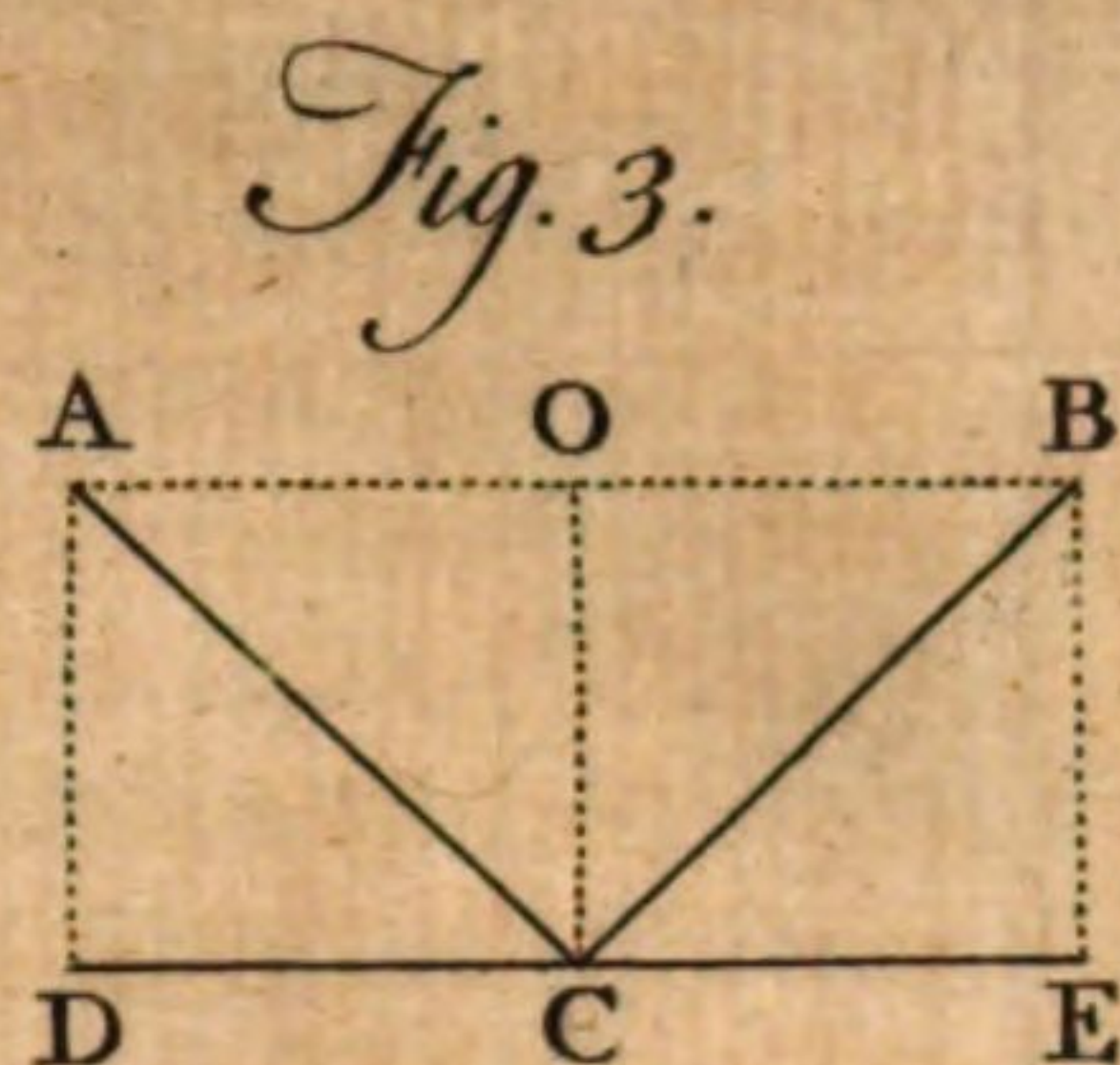
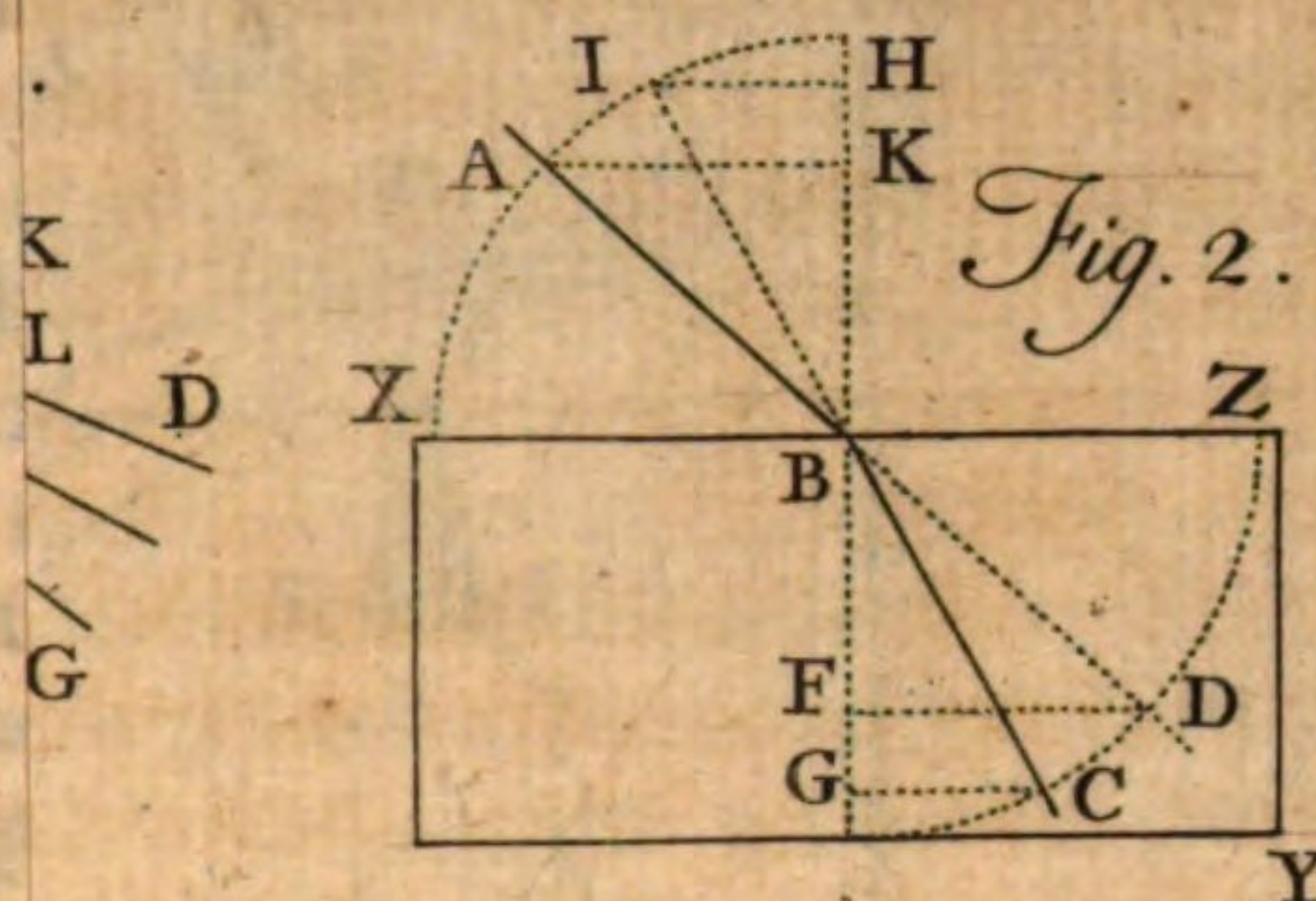
The ray ro is refracted into os , as it passes out of one of these mediums into the other; because it is accelerated and in the latter medium Z moves faster than it did in the former medium X . But when this or any other ray describes the line so and passes back again out of the medium X into the medium Z , it will be just as much retarded as it was accelerated before, and consequently will be just as much refracted the contrary way, and will therefore be bent into or .

40. *Where the two mediums are the same, the sine of incidence will always bear the same proportion to the sine of the refracted angle, when the ray passes out of one medium into the other; whatever may be the obliquity of the incident ray.*

If X , Plat. II. fig. 5. is air, and Z is glass, and a ray of light ro passes out of one into the other at the point o ; then supposing that the angle of incidence roc is 50 degrees, and that rc the sine of incidence is to df the sine of the refracted angle as 3 to 2. This which is the proportion of one of these sines to the other at the obliquity of 50 degrees, will likewise be the proportion between them at any other obliquity. That is, if the angle of incidence roc , instead of being 50 degrees, was 70 or 20 degrees, still rc would be to df as 3 to 2.

The oblique motion of the ray ro may be resolved into two others, such as rc parallel to the surface ab and rb or co perpendicular to it. This perpendicular motion co is accelerated as the ray passes out of the air into the glass. Take therefore od equal to co and continue the refracted ray of to s , so that, if ks is drawn perpendicular to ok , ks may be equal to rc , and draw df the sine of the refracted angle kof . Now the rays perpendicular velocity whilst it was in the air being co , its perpendicular velocity when it is in the glass will be ok , by proposition 37; that is, the ray is accelerated in the proportion of ok to od . But oks and odf are equiangular triangles: for the angle at k in one is equal to the angle at d in the other, because they are both of them right ones; and the angle at o is common to both; therefore the third angle in one is equal to the third angle in the other. Euc. b. I. prop. 32. corol 2. From whence it follows, that the sides about the equal angles are proportional ks to df , as ok to od . But ks is the sine of incidence, and df is the sine of the refracted angle; and ok is the rays velocity in the glass, and od its velocity in the air. There-

PLATE VII.



Therefore the sine of the angle made with the perpendicular in the air, or $ks=rc$, is to the sine of the angle made with the perpendicular in the glass, or to df , as the rays velocity in the glass is to the rays velocity in the air, or as ok to od . That is, the sines of incidence and of the refracted angle are as the rays velocity inversely. But, since the sine of incidence is to the sine of the refracted angle as ok to od , or as the rays velocity in one medium to its velocity in the other; it follows that, if at all obliquities of the incident ray, these velocities bear the same proportion to one another, then at all obliquities the sine of incidence will bear the same proportion to the refracted sine. Now where the two mediums X and Z are given, that is, if X is air and Z glass, or if X is any medium whatever, and Z is any other medium whatever, the obliquity of the ray makes no alteration in the proportion of ok to od . For where the mediums are given, their densities and consequently the difference of their attractions is given. And, since the proportion, which the velocity of the ray in one medium, bears to its velocity in the other, depends upon the difference of their attractions; where this difference is given, the proportion of these velocities cannot be altered by changing the obliquity of the ray. So that if the velocity in one medium is to the velocity in the other as ok to od , and the sine of incidence to the sine of the refracted angle as ks to df in any obliquity of the ray, this proportion will continue the same, however the obliquity of the ray may be changed.

The reader will find it of use hereafter to remember what this proportion is in the following instances.

If a ray of light passes obliquely out of air into glass; the sine of incidence rc is to the sine of the refracted angle fd as 3 to 2 nearly. Now the angle fog , or the angle of refraction is the difference between roc or its equal kog , which is the angle of incidence, and kos , which is the refracted angle. Therefore, supposing the sines proportional to the angles, by proposition 26, the sine of fog the angle of refraction is as the difference between rc and df , or as $3-2$. That is, the sine of incidence rc is to the sine of refraction as rc to $rc-df$, or as 3 to 1.

If a ray of light passes obliquely out of glass into air, the sine of incidence fd is to the sine of the refracted angle rc as 2 to 3, by what has just now been laid down, and by proposition 39. Here tor is the angle of refraction, for it is the difference between fok , or its equal cot , which is the angle of incidence, and cor , which is the refracted angle. Therefore considering the sines and angles as proportional, by proposition 26, the sine of tor is as the difference between rc and db , or as $3-2$. That

is, the sine of incidence df is to the sine of refraction as df to $rc - df$, or as 2 to 1.

If a ray of light passes obliquely out of air into water, the sine of incidence is to the sine of the refracted angle as 4 to 3: and the sine of incidence to the sine of refraction as 4 to $4 - 3 = 1$.

If a ray of light passes obliquely out of water into air, the sine of incidence is to the sine of the refracted angle as 3 to 4: and the sine of incidence to the sine of refraction as 3 to $4 - 3 = 1$.

41. *Rays of light, which pass perpendicularly out of one medium into another, suffer no refraction.*

The ray no , Plat.VIII. fig. 5. which falls perpendicularly upon the surface ab , will not be refracted in passing out of the medium X into the medium Z, but will continue in the same direction and will go on in the right line ok perpendicular to ab on the other side.

By the last proposition, rc the sine of incidence, bears the same proportion to df the sine of the refracted angle in all obliquities of the incident ray. Now as ro is nearer to no , or as the incident ray is nearer to the perpendicular, the sine rc must be less: and consequently the sine df must be less in the same proportion. That is, the sine of the refracted angle decreases in the same proportion that the sine of incidence decreases. But when ro and no coincide, that is, when the incident ray describes no or is perpendicular to the refracting surface ab , then rc , which is the distance of r from the perpendicular, becomes nothing. And when rc is nothing, df , which always decreases in the same proportion with rc , will be nothing, or the refracted ray will have no distance from the perpendicular ok , but will describe this perpendicular. Therefore when the incident ray is perpendicular to the surface of the medium, into which it passes, the refracted ray, if it may be called so, will be perpendicular to the same surface; or the ray will go straight forward out of one medium into the other, without being bent or refracted in its passage.

If a ray of light no , that is passing out of air into water falls perpendicularly upon ab the surface of the water, it will pass straight forwards in the line od , and will suffer no refraction. For, though the water attracts the ray more than the air does and will consequently accelerate it, yet because the ray is moving in a line perpendicular to the waters surface, this attraction acts in the same direction that the ray is moving, and therefore cannot change the direction of the ray or bend it from its course, but will only accelerate its motion in the same right line no continued below the surface. In like manner, if the ray ko was passing out of water

ter into air and was perpendicular to ab the surface of the water out of which it passes, or, which amounts to the same thing, to the surface of the air into which it passes: this ray would be drawn back towards the water by the greater attraction on that side. But this attraction, as it acts in a line perpendicular to the surface of the water, acts in the same direction, in which the ray is moving, and therefore, though it retards the ray, will not bend it from its strait course: but the ray will go on in on , which is ko continued, and as it was perpendicular to the surface ab , whilst in the water, it will still be perpendicular to it, when it comes into the air.

C H A P. III.

Of the refraction of light, when it passes through a plane surface.

42. *When parallel rays pass obliquely out of one medium into another; they will continue parallel, after they are refracted.*

IF the rays AB and CD , Plat. VIII. fig. 8. are parallel one to another, and pass obliquely out of air into water through the plane surface RZ , these rays will not go on in the same directions so as to proceed strait forwards in BG and DH , they will each of them be refracted towards a perpendicular into BE and DF ; by proposition 37. But BE , DF , or the refracted rays will be parallel to one another as well as the incident ones.

Let KL be drawn perpendicular to the surface RZ at the point B . Then ABK is the angle of incidence for the ray AB ; and ABZ is the complement of incidence to a right angle. CDZ is likewise the complement of incidence for the ray CD . But AB and CD are parallel; and consequently the external angle CDZ made by the right line or line drawn in a plane surface RZ , is equal to the internal opposite angle on the same side ABZ . Euc. b. I. prop. 29. Therefore these rays, as their complements of incidence are equal, have their angles of incidence equal. And for this reason the refracted angles will be equal, or one ray will be just as much refracted as the other is; by proposition 40. But if the refracted angles are equal, the complements of those angles will likewise be equal. Now EBL is the refracted angle for the ray BE and RBE is its complement: in like manner RDF is the complement of the refracted angle for the ray DF . And since the external angle RBE , made by the line RZ crossing the two lines BE and DF is equal to RDF the internal opposite angle on the same side, these two lines BE and DF , or the refracted rays, are parallel. Euc. b. II. prop. 28.

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The reader will easily apply this method of reasoning in one case of parallel rays to all other cases, where they pass out of one medium into another through a plane surface, whether they pass out of a rarer medium into a denser, or out of a denser into a rarer. For in all cases the incident rays, if they are parallel, will be equally inclined to a plane surface, and consequently will be equally refracted, which will make them be equally inclined to the same plane surface after refraction, and for that reason they will still be parallel to one another.

43. *Rays, that diverge, will be made to diverge less, and rays, that converge, will be made to converge less, by passing out of a rarer medium into a denser, through a plane surface.*

If the rays RA and RB, Plat. VII. fig. 6. which diverge from the radiant R, pass out of air into water, or out of any rarer medium Z into any denser medium X through the plane surface Ss; these rays, when they get into the water, will not go straight forwards in the lines AC and BE, but will each of them be refracted towards a perpendicular, by proposition 37, and will describe AD and BG, that are nearer perpendicular to the surface Ss than AC and BE.

Now AD and BG diverge less than AC and BE, that is, than RA and RB; or the refracted rays diverge less than the incident ones. IK and RH and all other lines, which are or can be drawn perpendicular to the plane surface Ss are parallel to one another. Euc. b. XI. prop. 8. Therefore lines which are perpendicular to a plane surface do not diverge at all: for parallel lines make no angle with each other. But the refracted rays AD and BG are nearer being perpendicular to the surface Ss than the incident ones RA and RB; that is, the refracted rays are nearer being parallel to each other than the incident ones. But since rays, if they are parallel do not diverge at all, the nearer they are to being parallel, the less they must diverge. Therefore the refracted rays diverge less than the incident ones.

In like manner, if the rays TM, and WN, which converge towards each other and tend to the focus f , pass out of air into water, or out of any rarer medium Z into any denser one X, through a plane surface Ss, they will be made to converge less, that is, instead of going straight forwards in the lines Mf and Nf, they will be refracted into some such lines as MF, and NF, which make a less angle with one another than Mf and Nf do.

For the rays pass out of a rarer medium into a denser, and therefore are refracted towards the perpendiculars, by proposition 37. The surface,
where

where they are refracted is a plane one; and therefore all lines that can be drawn perpendicular to it either at N, as PO, or at M, or any where else are parallel to one another. But since the rays are refracted towards perpendiculars, or are nearer being perpendicular to the surface Ss than they were at their incidence, and since the perpendiculars are all of them parallel to one another, it follows, that the refracted rays are nearer parallel than the incident ones. Now parallel rays do not converge at all, and consequently those rays, which are nearest being parallel, will converge the least. But the refracted rays are nearer being parallel than the incident ones. Therefore the refracted rays converge less than the incident ones.

44. *Diverging rays will be made to diverge more, and converging rays will be made to converge more, by passing out of a denser medium into a rarer, through a plane surface.*

Let F, Plat.VII. fig. 6. be a radiant, and FM, FN, be rays diverging from it. Then, if these rays, at the points M and N, pass out of water into air, or out of glass into air, or out of any denser medium X into any rarer one Z, through a plane surface Ss; instead of going straight forwards in MQ and NV, they will be refracted into MT and NW, so as to diverge more, or so as to make a greater angle with one another, than the incident rays FM, and FN did.

The rays pass out of a denser medium into a rarer, and therefore are refracted from the perpendiculars. But the perpendiculars to a plane surface are parallel to one another. Therefore the refracted rays, being farther from the direction of perpendiculars, will be farther from being parallel, that is, will diverge more than the incident ones.

Or otherwise, we have seen, in proposition 43, that, if the rays TM, and WN, are the incident ones, and converge to f ; MF and NF which converge to F, will be the refracted ones. But, by prop. 39, all refraction is reciprocal. Therefore, if these rays, FM, and FN, are turned back again, so as to diverge from F, and become the incident rays, or if any other rays diverge from thence in the same manner; then MT, and NW, will be the refracted ones. Or, since, by prop. 43, converging rays are made to converge less, or to contain a less angle with one another, by passing out of a rarer medium into a denser through a plane surface; it follows, from prop. 39, that if these rays are turned back again, so as to become diverging ones, that pass out of a denser medium into a rarer, this contrary refraction will increase the angle as much as the other diminished it, or will make the rays diverge more.

If DA and GB, Plat.VII. fig. 6, are rays that converge to r , but must,
be-

before they come thither, pass through a plane surface at the points A and B out of water into air, or out of glass into air, or out of any denser medium X into any rarer one Z; then these rays will not go straight forwards in Ar and Br, but will be refracted into AR and BR so as to converge more towards each other, or so as to make a greater angle with one another after refraction than they did before.

The rays pass out of a denser medium into a rarer, and therefore are refracted from the perpendiculars, by prop. 38. But the perpendiculars are parallel to one another, because the surface is a plane one. Consequently the rays after refraction being farther from the direction of the perpendiculars than they were before, will be farther from being parallel to one another, that is, will make a greater angle with one another, or will converge more.

Or otherwise. If RA and RB had been the incident rays, then, by prop. 43, AD and BE would have been the refracted ones. Therefore if DA, and BE are the incident rays, AR and BR will be the refracted ones, by prop. 39. Or, since, by prop. 43, diverging rays are made to diverge less or to contain a less angle with one another by passing out of a rarer medium into a denser through a plane surface, it follows, from prop. 39, that if these rays are turned back again so as to become converging ones, that pass out of a denser medium into a rarer, or if any other rays converge in the same manner, this contrary refraction will encrease the angle just as much as the other diminished it, or will make the rays converge more.

45. *When diverging rays are refracted at a plane surface, the distance of the real radiant from that surface is to the distance of the imaginary radiant, as the sine of the refracted angle is to the sine of incidence.*

If R, Plat. VII. fig. 6, is the real radiant, or point from whence the incident rays RA and RB diverge; since these rays, when they are refracted, diverge less, if they pass out of a rarer medium into a denser, they will proceed after refraction in AD and BG as if they had come straight from r, which is the imaginary radiant, a point more remote than R. Because the less the divergency the more remote the radiant, by prop. 28.

Suppose, for instance, that R is the highest leaf of a tree, which stands upon the bank of a river, and that rays of light reflected from that leaf describe the lines RA, and RB, whilst they are in the air, but that at A and B those same rays pass into the water, and are there refracted towards a perpendicular into the lines AD and BG. The re-
fracted

fracted rays diverge less than the incident ones did; and the eye of a spectator will receive them, after this refraction, just as if they had come straight from r ; so that the leaf, from whence they came, will appear at r instead of R , or will appear farther from the surface of the water than it really is. And the distance of r the imaginary radiant will be to the distance of R the real radiant, as the sine of incidence to the sine of the refracted angle, or as 4 to 3; for this is the proportion, which the sine of incidence bears to the sine of the refracted angle. That is, these rays will fall upon the eye as if they had come from the top of a tree one part in four higher than it really is.

In order to determine the distance of the real and imaginary radiant, we must first observe that the divergency of the incident rays from the perpendicular RH is equal to the angle of incidence. The divergency of the ray RA from the perpendicular RH is the angle ARH ; the point of incidence is A , and IK is a perpendicular at the point A ; therefore RAI is the angle of incidence. But IK and RH are parallel, because they are both of them perpendicular to the plane surface Ss . Euc. b. XI. prop. 8. Therefore the alternate angles RAI and ARH are equal. Euc. b. I. prop. 29. Secondly the divergency of the refracted rays from the perpendicular rH is equal to the refracted angle. The divergency of the refracted ray AD from the perpendicular rH is DrH , and the refracted angle is DAK . But since IK and rH are parallel, the external angle DAK is equal to the internal and opposite one DrH . Euc. b. I, prop. 29. Now the distance of the radiant is inversely as the divergency of the rays, by prop. 28. And the divergency before refraction is to the divergency after it, as the angle of incidence to the refracted angle. Therefore the distance of R the real radiant before refraction, is to the distance of r the imaginary radiant after refraction, as these angles inverted, that is, as the refracted angle to the angle of incidence. Or considering the angles in the same proportion with their sines, by prop. 26, the distance of R from the surface, is to the distance of r from it, as the sine of the refracted angle to the sine of incidence.

This demonstration is general and may be applied to the case of diverging rays, that pass through a plane surface out of water into air, or out of any denser medium into any rarer. If F was any small object, such as a shilling, laid upon the bottom of a vessel, and the vessel was to be filled with water: the rays FM and FN , which are reflected from any point in the shilling and make it visible, must pass through the surface of the water Ss into the air, before they come to the eye; and in this passage they will be refracted from a perpendicular and be

made to diverge more, so as to proceed, when they are in the air in the lines MT and NW; by prop. 44. Now the more any rays diverge the nearer is the radiant from whence they come or appear to come, by prop. 28. Therefore these rays MT and NW will appear to come from some radiant f nearer to the surface of the water than F is; that is, the shilling will appear nearer the surface Ss than it really is; and that in the proportion of the sine of incidence to the sine of the refracted angle. Or, since the sine of incidence is to the sine of the refracted angle out of water into air as 3 to 4, the point f will be nearer to the surface of the water than F is, in the proportion of 3 to 4; or the shilling will appear one part in four nearer to the surface than it really is.

For here, as before, LFN, the divergency of an incident ray from the perpendicular, is equal to FNO, the angle of incidence of the same ray. And LfW, the divergency of this ray from the perpendicular, when it is refracted, is equal to PNW the refracted angle. But, since the divergency before refraction is to the divergency after refraction, as the angle of incidence to the refracted angle, or, by prop. 26, as the sine of incidence to the sine of the refracted angle, the distance of the real radiant F, before refraction, is to the distance of the imaginary radiant f , after refraction, as these sines inverted, or as the sine of the refracted angle to the sine of incidence.

46. *When converging rays are refracted at a plane surface, the distance of the imaginary focus from that surface is to the distance of the real focus, as the sine of the refracted angle to the sine of incidence.*

If DA and GB are rays tending to r , which, before they arrive there, are to pass out of glass into air, or out of any denser medium into any rarer; then r is only an imaginary focus: for though the rays tend thither before refraction, yet they will never meet there: because they will be refracted in their passage, and will be made to converge more, by prop. 44. And this greater convergency will bring them to a focus sooner, or will make them meet at some point, as R, which is the real focus, and is nearer to Ss the surface of the glass than r is, by prop. 29. Now r , the imaginary focus, is as much farther from the surface than R the real focus, as the sine of the refracted angle is greater than the sine of incidence. Or, since the sine of the refracted angle is to the sine of incidence, out of glass into air, as 3 to 2, the distance of r from the surface, is to the distance of R, as 3 to 2. Or the real focus is one part in three nearer than the imaginary one. I mention this instance, rather than

than any other, because we shall have occasion hereafter to apply the proposition particularly to this instance; for which reason the reader will do well to remember it.

The convergency of any incident ray DA to the perpendicular Hr , is equal to the angle of incidence of the same ray: for DAK is equal to DrH , Euc. b. I. prop. 29. And the convergency of this ray to the perpendicular, when it is refracted into AR , is equal to the refracted angle: for ARH is equal to IAR , Euc. b. I. prop. 29. But the distance of the focus is inversely as the convergency of the rays, by prop. 29, that is, the distance of the imaginary focus r before refraction, is to the distance of the real focus R after refraction, inversely as the angle of incidence to the refracted angle, or as these angles inverted, or as the refracted angle to the angle of incidence, or, by prop. 26, as the sine of the refracted angle to the sine of incidence.

The same demonstration is likewise applicable, when converging rays pass out of air into glass, or out of any rarer medium into any denser: the only difference is, that when they are to pass, as in the former case, out of a denser medium into a rarer, they are made to converge more, and the refracted angle is greater than the angle of incidence, and consequently the focus r of the incident rays, or imaginary focus, will be farther from the surface Ss than the focus of the refracted ones. But in this case, when WN , and TM converge to f , and before they meet pass out of air into glass, they will be made to converge less, and the refracted angle will be less than the angle of incidence, and consequently f , the focus of the incident rays, or imaginary focus, will be nearer to the surface Ss than F , the focus of the refracted rays, or real focus.

For in this case, as in the former, WfL , the convergency of the incident ray WN to the perpendicular Lf , is equal to WNP , the angle of incidence. And NfL , the convergency of the same ray to the perpendicular FL , when it is refracted, is equal to FNO , the refracted angle. But the distance of the focus is inversely as the convergency of the rays, by prop. 29. Therefore the distance of the focus before refraction is to the distance of it afterwards, inversely as the angle of incidence to the refracted angle; or, as these angles inverted, or as the refracted angle to the angle of incidence, or, by prop. 26, as the sine of the refracted angle to the sine of incidence.

47. *Rays of light after they have passed through a plane glass and have been twice refracted in their passage, keep the same direction in respect of one another, that they had at their incidence.*

When rays are to pass through a plane glass $aklb$, Plat.VIII. fig.9. they first pass out of air into glass through the plane surface ak and are there refracted towards a perpendicular, by prop. 37; and after this they are to pass again out of glass into air through the plane surface lb , where they will be as much refracted from a perpendicular as they were refracted towards it before, by prop. 38, 40. Therefore the second refraction will always undo what the other has done and will leave the direction of the rays, in respect of one another, the same after they have passed through the glass that it was at their incidence upon the surface ak . If they were parallel at their incidence, they will be parallel after the two refractions; or if they diverged or converged before, they will diverge or converge just as much afterwards.

One instance in a case of diverging rays may clear up this whole matter. Let c be a radiant and cd, ce diverging rays. When they enter the surface ak they will be made to diverge less, by prop. 43, and will proceed within the glass in the lines dg, en . But when these rays dg and en come to the second surface of the glass lb , they pass out of glass into air, and will be made to diverge more, by prop. 44. And since this second refraction is just equal to the first, by prop. 40, and is made the contrary way, it will restore the first divergency of rays, or gi and no , when they come out of the glass, will diverge just as much as cd and ce did, when they entered it.

CHAP. IV.

Of the refraction of parallel rays, when they pass through lenses, which are either convex or concave.

48. *If a beam of parallel rays, which are to pass out of a rarer medium into a denser through a convex surface, falls directly upon that surface, the rays after refraction will converge to their axis.*

LET ABKP, Plat.VIII. fig. 10. be any medium denser than the medium that surrounds it, and let ANB the convex surface of this denser medium be the segment of a sphere, whose center is C. Let XN be the axis of a beam of parallel rays, and DE be one of the collateral rays. Suppose, for instance, that ABKP was a mass of glass, with a convex spherical surface ANB, and that parallel rays of light DE and XN were passing out of the air into this glass. The beam is supposed to fall directly upon the glass; therefore the axis or middle ray XN is perpendicular

lar to the surface ANB , and consequently will enter the glass at N without being refracted, and will proceed strait forwards in the direction $NCGQ$; by proposition 41. But the ray DE or any other collateral ray, which falls either above the axis, as this does, on the part AN of the surface, or below it on the part NB , is oblique to the surface. For if through the point E , where the ray DE falls, a line IEC is drawn to C the center of the convexity, this line is perpendicular to the surface; and consequently the ray DE , which does not coincide with IE but makes with it the angle DEI , is oblique to the surface: and the angle DEI is the angle of incidence. This ray is to pass out of a rarer medium into a denser, and, will be refracted towards a perpendicular, by proposition 37. But within the glass, all lines perpendicular to its surface such as EC , MC , NC , converge to C the center of the convexity, which center is in the axis NG . And all the rays, which were parallel, and therefore neither converged nor diverged at their incidence, are refracted towards these converging perpendiculars, except the axis XN which passes strait forwards. From whence it follows that these refracted rays being nearer to the perpendiculars than the incident ones were, must converge to their axis. The ray DE in particular, when it gets into the glass, does not go on in the line EF , in which it was moving before, but in some line EG , which is nearer to the perpendicular EC . But the axis $XNGQ$ passes on without being refracted, and all the other rays which fall either above or below it, or on any part of the surface except N are made to converge to some point G , which is in their axis NG .

49. *If a beam of parallel rays, which are to pass out of a rarer medium into a denser through a concave surface, falls directly upon that surface, the rays after refraction will diverge from their axis.*

If anb the surface of a denser medium $kanb$ is the concave segment of a sphere, whose center is c , and parallel rays such as gn and de are to pass into this medium, out of a rarer that encompasses it; then, if gn is the axis of the beam and de one of the collateral rays; the ray gn will be perpendicular to the surface anb , because the beam falls directly upon this surface, by the supposition; but de or any other collateral ray will be oblique. In particular de will be oblique: for at the point e the line cei drawn from the center is perpendicular to the surface, but de does not coincide with ce but makes with it the angle dec . Therefore this ray is oblique and dec is the angle of incidence. Suppose the medium $kanb$ to be glass and the medium that encompasses it to be air. Then as the rays pass out of air into glass or out of a rarer medium into a denser, de and
all

all the other rays of the beam will be refracted towards perpendiculars, by proposition 38, except the axis gn ; for the axis, because it is perpendicular to the surface at its incidence, will pass on strait in the line nx without suffering any refraction, by proposition 41. But all the perpendiculars such as cx , cm , and ci diverge within the glass from c the center, which lies in the axis gnx . Therefore the rays, which were parallel to one another at their incidence, when they are refracted towards these diverging perpendiculars, must likewise diverge from some point in this axis. The ray de in particular, when it gets into the glass, does not go strait forwards in the line ef parallel to its axis nx , but is refracted towards the perpendicular ei and describes the line eb which diverges from the point g in the axis gx . That is the rays gn and de which were parallel to one another before refraction, will proceed after it in the diverging lines nx and eb , in the same manner that they would have done, if they had come at first from a radiant at g . What has here been shewn of this ray de is equally true of all the collateral rays, that fall on any part of the surface anb either above the axis or below it, or on any side of it. The axis alone will proceed strait forwards in the glass, and all the other rays will be refracted so as to diverge from some point g in that axis.

This proposition may be deduced from the foregoing one. And if the reader will attend to the manner of deducing it from thence, he will save himself much trouble in demonstrating all the propositions which follow relating to concave and plano-concave lenses.

In proposition 75 of mechanics, we observed that a quantity might change from affirmative to negative by passing through nothing; that is, if a quantity has been decreasing, till it becomes nothing, and then continues still to decrease, it will become less than nothing or will be negative. We observed farther that quantities are called negative only in reference to some other quantities to which they are opposite, but are in themselves or without this reference as much positive quantities as those which are called affirmative. Both these particulars we may now shew to be true in some other instances, which are more immediately applicable to our present purpose: and one of these instances will lead us to prove, what may perhaps seem a paradox, that a quantity may change from affirmative to negative by passing through infinity; that is, if a quantity has been encreasing till it becomes infinite, and then continues still to encrease and so becomes greater than infinite, it will then be negative.

With the radius or semidiameter cm , Plat. IX. fig. 1. strike emf the arc of a circle; this arc is convex on the side which is towards l . Open the compasses

passes to the distance dm , and with the radius dm strike gmb another arc of a circle, which is likewise convex towards l ; but then it is less convex than emf , for it approaches nearer to the right line amb . From hence it seems evident that the longer the radius is, so much less convex any given arc of the circle will be. The converse likewise of this seems evident, that the less convex the arc is so much longer must be the radius: for the more convex arc emf has mc for the radius; and the less convex arc gmb has a longer radius as md . From hence then we may reason in the following manner: as emf straitens or grows less convex towards l , it constantly approaches towards the right line amb . Thus, if emf was a wire bent into the arc of a circle, if the point m of the wire was fixed, and you lay hold of the two ends e and f , by pulling these ends towards l you would straiten the wire or make it less convex towards l : for when you had drawn it into the position gmb it would be straiter or less convex than it was before. As this line or wire is still farther unbent, or as the ends g and b move still farther towards l the convexity will keep decreasing: and when it was quite unbent or was come into the position amb , that is, when the two ends of it had moved one to a and the other to b , the line or wire would be quite strait, and therefore its convexity would be nothing. Suppose it to keep moving on in the same manner as before, that is, suppose the two ends of it a and b to move farther towards l : this motion had reduced the convexity to nothing before; therefore, if it continues, it will make the convexity less than nothing or negative.

But bending the line amb by drawing the ends of it a and b towards l so as to bring one of them to i and the other to k will make it concave on the side, which is towards l . Therefore the line imk , which is concave towards l is the negative of the line emf or of any other line that is convex the same way. Now the bending of imk considered by itself without any respect to emf is as much positive as the bending of emf or any other line can be. But if we compare them with one another; if as emf is convex towards l , we ask what is the convexity of imk the same way, we shall find that the convexity of imk towards l is less than nothing; and consequently the bending of imk compared with that of emf is likewise less than nothing or is negative. Therefore upon the whole, if the bending of a line one way is considered as affirmative, the bending of a line the contrary way must be considered as negative.

But a quantity may change from affirmative to negative, not only by passing through nothing so as to become less than nothing, but likewise by passing through infinity so as to become greater than infinite. For let us observe a little what happens to the radius mc , whilst the arc emf

is unbending itself or growing less convex. This radius lengthens as the convexity towards l or bending of the arc decreases. For mc is the radius, when the arc is in the position emf : but when it is grown flatter or is farther unbent or has a less convexity and is in the position gmb , then the radius is md . And since in the same manner, whilst the convexity of the arc keeps decreasing, the radius will constantly encrease; or since the radius is inversely as the convexity of the arc, it follows that when the convexity is nothing or when the arc is quite unbent and coincides with the right line amb , the radius will then be the reciprocal of nothing or will be infinitely long. And thus every right line amb may be considered as a finite or given part of a circle, whose radius and consequently whose circumference is infinite. Now, since a motion of e and f , or of the two ends of the arc emf , towards l , makes the convexity of the arc towards l decrease and the radius mc encrease, till the convexity becomes nothing and the radius becomes infinite: the same motion of these ends towards l , after they are passed a and b will make the convexity less than nothing, and the radius greater than infinite. But when the convexity towards l becomes less than nothing, or is negative, the arc is in some such position as imk or is concave towards l and the radius is ml . And if the radius, when it is on the side c , is affirmative, then the radius, when it is on the other side, must be negative. If mc is affirmative, ml being drawn the contrary way is negative. Therefore since the radius by becoming greater than infinite changes sides, it changes from affirmative to negative by passing through infinity.

What has here been said concerning the arc of a circle, the reader will easily apply to the segment of a sphere, or to the convex and concave surfaces ANB and anb , Plat. VIII. fig. 10. 11. For ANB the surface of the glass is convex towards the air, and its radius NC is in the glass. But whilst ANB flattens or whilst its convexity decreases the radius NC lengthens. Now, when this motion of ANB towards C , by which its convexity is diminished, has continued, till the surface is quite flat, then the convexity is nothing and NC is infinite. And for this reason a plane glass may be considered as the segment of a sphere of an infinite radius. But if the same motion was supposed to continue, or if ANB after it is become plane was to sink still farther in towards C ; the convexity towards the air would then become less than nothing or negative; that is, the surface towards the air would be changed into a concave one anb : and the radius would at the same time become greater than infinite or would be negative and change sides: for the center C is in the glass when the surface of the glass is convex and the radius is NC : but when the surface

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is concave the center c is in the air and the radius nc is drawn the contrary way.

Now as anb is the negative of ANB , these two glasses will have just contrary effects : and since the convex one makes rays that were parallel at their incidence diverge after they are refracted, the concave one will make their convergency less than nothing or negative, that is, it will make them diverge. And since the focus, when the convergency is affirmative is at G , or in the glass ; the focus, when the convergency is negative or when the refracted rays diverge will be at g on the contrary side or in the air. And if the former focus G , or the focus of converging rays is considered as affirmative, the latter focus g , that is the focus of diverging rays, or imaginary radiant from whence such rays appear to flow, must be negative. This imaginary radiant g or negative focus is by some writers called a virtual focus.

50. *When parallel rays pass out of a rarer medium into a denser through a spherical surface, the distance of the focus whether affirmative or negative, that is, whether real or virtual, will be to the semidiameter of the sphere of which that surface is a segment, as the sine of refraction to the sine of incidence.*

ANB , Plat. VIII. fig. 10. is a convex spherical surface, and the parallel rays DE and XN , by passing out of a rarer medium into a denser through this surface, are made to converge to a focus G , by proposition 48. This focus is affirmative. And NC the radius of the sphere, of which the surface ANB is a segment, will bear the same proportion to NG the distance of the focus, that the sine of refraction in this passage of the rays out of the rarer medium into the denser bears to the sine of incidence. This is one case of the proposition. And when we have shewn the proposition to be true where the spherical surface of the denser medium is convex and the focus affirmative, we will then shew it to be true, where that surface is concave and the focus negative.

Now the ray XN , which is the axis of the beam, and falls perpendicularly upon the surface, is not refracted at all, by proposition 41, but goes strait on in NG the same direction continued after it gets into the denser medium. But DE is oblique to the surface, and consequently, instead of going strait forwards in the direction EF , it will be refracted into EG so as to converge to the axis NG and to meet it at G , which is the focus. IC is perpendicular to the surface at the point E . Therefore DEI is the angle of incidence. But since DE and XG are parallel by the supposition, and IC crosses them, the external angle DEI is equal to the in-

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ternal opposite angle on the same side ECN, Euc. b. I. prop. 29. that is, ECN is equal to the angle of incidence. Now the angle ECG is the complement of ECN to two right ones, and upon this account they have both of them the same sine, as was observed in explaining proposition 32. But the sine of ECN is the sine of incidence: because ECN is equal to the angle of incidence. Therefore the sine of ECG is the sine of incidence. IC and DF, which cross one another at E make the vertical angles DEI and FEC equal. Euc. b. I. prop. 15. Therefore FEC is equal to the angle of incidence. The refracted ray is EG. Therefore GEC, being the angle which it makes with IC the perpendicular, is the refracted angle. And FEG is the difference between FEC and GEC, or between the angle of incidence and the refracted angle. Therefore FEG is the angle of refraction. Now DF and XG or the directions of the incident rays are parallel by what is supposed in the proposition. And consequently EG, which crosses them, makes the alternate angles FEG and EGC equal to one another. Euc. b. I. prop. 29. But FEG is the angle of refraction. Therefore the sine of EGC is the sine of refraction: because EGC is equal to the angle of refraction. From hence then it appears that, in the triangle EGC, the sine of the angle at G is the sine of refraction, and the sine of the angle at C is the sine of incidence. But the sides of plane triangles are to one another as the sines of the opposite angles. Therefore EC, opposite to G, is to EG, opposite to C, as the sine of refraction to the sine of incidence. But EC is equal to NC, for they are semidiameters of the same sphere, of which the spherical surface ANB is a segment. And EG is very nearly equal to NG, when E and N are not far asunder. Therefore NC, the semidiameter of the convexity is to NG the distance of the focus as EC to EG, that is, as the sine of refraction to the sine of incidence.

This demonstration is universal, whatever the two mediums are. But we will apply it particularly to the case of rays passing out of air into glass. In this passage of the rays the reader may remember that the sine of refraction is to the sine of incidence as 1 to 3. Therefore NC is to NG in this case as 1 to 3. Or since NC is one semidiameter of the sphere of which ANB is a segment; NG the distance of the focus, must be equal to three times NC and consequently is 3 semidiameters. Therefore if parallel rays pass out of air into glass, through a convex spherical surface, they are so refracted in their passage as to be made to converge to a focus, which will be at the distance of 3 semidiameters of the convexity behind the surface.

Perhaps this effect of such a piece of glass upon parallel rays may be made out in another manner, and so as to give an easier general proof

proof of the first case in the proposition. If the ray DE was not refracted at all it would go on, when it got into the glass, in the line EF. IED is the angle of incidence, and FEC is equal to it. GEC is the refracted angle and FEG is the angle of refraction. Now the reader may observe that FEG, or the angle of refraction, is contained between EF the direction of the incident ray, and EG the direction of the refracted one. Suppose then, that the angle of refraction was equal to the angle of incidence, what line would the refracted ray describe? FEC is the angle of incidence, EF is the direction of the incident ray; therefore the refracted ray cannot make an angle with EF equal to FEC, or cannot make the angle of refraction equal to the angle of incidence, unless it describes EC. Therefore if the angle of refraction was so great as to be equal to the angle of incidence, the refracted ray would be perpendicular to the convex surface, and the focus would be at C in the center of the convexity. But in fact the angle of refraction is less than the angle of incidence, that is, the rays are not so much refracted, and consequently will not converge so much as is here supposed. Therefore the focus will not be so near as the center: for the less the convergency the more remote the focus, by prop. 29. Now if the angle of refraction was equal to the angle of incidence the focus would be at C or at the distance of a semidiameter from the surface. But in the passage of a ray out of air into glass, the refracted angle is to the angle of incidence as 1 to 3 or is only $\frac{1}{3}$ of the angle of incidence. Therefore the focus will be 3 times as remote from the surface as it would have been if these angles were equal, or will be at the distance of 3 semidiameters from the surface.

This is universal, whatever proportion the angle of refraction bears to the angle of incidence. For since, if they were equal, the focus would be at the distance of a semidiameter, and since the less the angle of refraction is, the less the refracted rays converge, it follows, by prop. 29, that the less the angle of refraction is in proportion to the angle of incidence, so much more remote the focus will be in proportion to a semidiameter; or as the angle of refraction to the angle of incidence, so is a semidiameter of the convexity to the distance of the focus.

The case of this proposition, which has been already explained, relates to a convex spherical surface of the denser medium, when the refracted rays converge, and the focus is affirmative. The other case is, when *anb*, Plat. VIII. fig. 11. the spherical surface of the denser medium is concave, and the rays *gn* and *de*, which are parallel at their incidence,

are made by the refraction to proceed in the lines nx and eb , so as to diverge from an imaginary radiant, or negative focus g . To make what has been said about affirmative and negative quantities more familiar to the reader, before we come to apply it in those instances where it will be most useful to us; we will first deduce this case of the proposition from the foregoing one; and then to shew the certainty of this sort of reasoning we will give a distinct demonstration of it.

When ANB , Plat. VIII. fig. 10. is convex, as the sine of refraction is to the sine of incidence, so is NC , the semidiameter of the convexity to NG the distance of the affirmative focus G . Now, if we imagine the surface ANB to flatten, the radius NC will increase and NG , which is 3 times NC , if the denser medium is glass and the rarer air, will increase in the same proportion: that is, the planer the surface ANB is so much more remote will the focus be. If the convexity of the surface ANB was to keep decreasing in the same manner till it became nothing, or till the glass was quite plane; the radius would then be infinite, and the focus G would be at an infinite distance, and consequently the convergency of the rays would be nothing: that is, if the surface of the glass was plane the rays, after refraction, would be parallel, as they were at their incidence, and if they are considered as converging ones, the focus to which they converge must be an infinitely distant one. But, if ANB sinks still farther in, its convexity will then be less than nothing, that is, will be negative, or will be changed into concavity, as anb , fig. 11. The convergency of the rays likewise will be less than nothing, or will be negative, that is, the refracted rays will diverge. The radius NC will be greater than infinite, that is, will be nc , or will change sides, and consequently will be negative. And the focus G will be at a distance greater than infinite, and therefore will change sides and will be at g , or will be negative too. This then is the only difference between a convex and a concave surface, that the former is affirmative and the latter negative; and the only difference in their effect is, that the affirmative, or convex surface will make the refracted rays converge, and will produce an affirmative focus; and the negative, or concave one, will make them diverge and produce a negative focus. And consequently, since as the sine of refraction is to the sine of incidence, so is NC , the semidiameter of the convexity, to NG the distance of the affirmative, or real, focus; it follows that as the sine of refraction is to the sine of incidence, so is nc the semidiameter of the concavity, or negative convexity, to ng the distance of the negative focus.

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This case of the proposition may be proved distinctly in the same manner that the other case was. If a beam of parallel rays, of which gn is the axis, and de , one of the collateral rays, is to pass out of a rarer medium into a denser, and falls directly upon the concave surface of the denser medium anb ; then the axis gn , being perpendicular to the surface, will pass straight forwards in the direction gn continued, or in nx , by prop. 41. But the collateral ray de falls obliquely upon the surface, and consequently, instead of going straight forwards in ef , will be refracted towards the perpendicular ci , and will describe some such line as eb , so that, after refraction, it will go on as if it had come straight from g ; or g will be the imaginary radiant, that is, the virtual or negative focus, from whence this and any other collateral ray will diverge after refraction. And as the sine of refraction is to the sine of incidence, so is nc , the semidiameter of the concavity, to ng , the distance of this negative focus. dec is the angle of incidence; de and gn are parallel, by the supposition; therefore the alternate angles dec and ecn are equal. But ecg , which is the complement of ecn to two right angles, has the same sine with ecn ; and that is the sine of incidence; because ecn has just been proved to be equal to the angle of incidence. fei is equal to dec , the angle of incidence; because these two angles are vertical ones. Euc. b. I. prop. 15. bei is the refracted angle, or the angle, which the refracted ray makes with the perpendicular; and feb , which is the difference between fei and bei , is the angle of refraction. But ef or de continued to f , is parallel to gn . Therefore feb , the external angle, is equal to cge , the internal opposite angle on the same side; Euc. b. I. prop. 29. so that the sine of cge is equal to the sine of feb , or to the sine of refraction. From what has been said it appears, that in the triangle cge , the sine of the angle at g , is equal to the sine of refraction, and the sine of the angle at c , is equal to the sine of incidence. But the sides of this triangle, are to each other, as the sines of the opposite angles, by prop. 32. Therefore ec is to eg , as the sine of refraction to the sine of incidence: for the angle at g is opposite to the side ec , and the angle at c is opposite to the side eg . And, since ce and nc are equal, and eg and ng are very nearly equal, when e and n are near to one another; it follows that nc the semidiameter of the concavity, is to ng the distance of the negative focus from the surface of the denser medium, as the sine of refraction to the sine of incidence.

When rays pass out of air into glass, the sine of refraction is to the sine of incidence as 1 to 3. Therefore if the rays are parallel and the surface of the glass is concave, the semidiameter of the glass's concavity will

will be to the distance of the point, from whence the refracted rays will appear to diverge, as 1 to 3: that is, nc will be $\frac{1}{3}$ of ng , or since nc is 1 semidiameter, ng the distance of the virtual or negative focus will be equal to 3 semidiameters of the glass's concavity.

The reader will easily understand the manner of applying the other demonstration, of the first case of this proposition to the case now before us, without having it particularly pointed out to him.

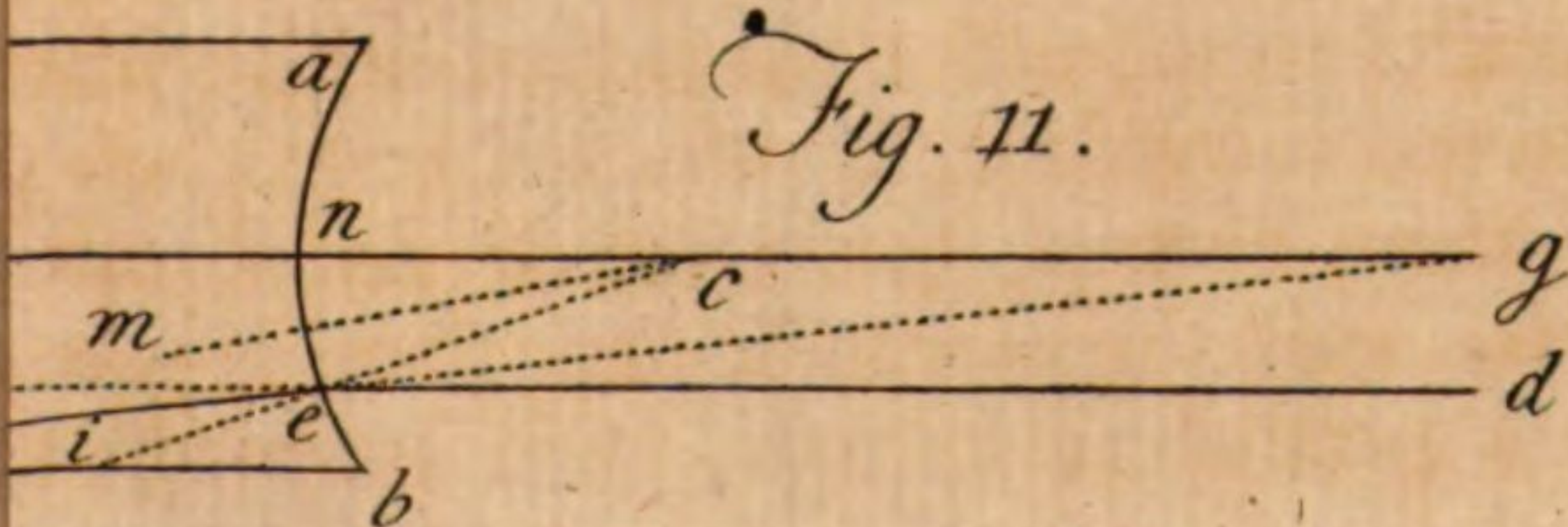
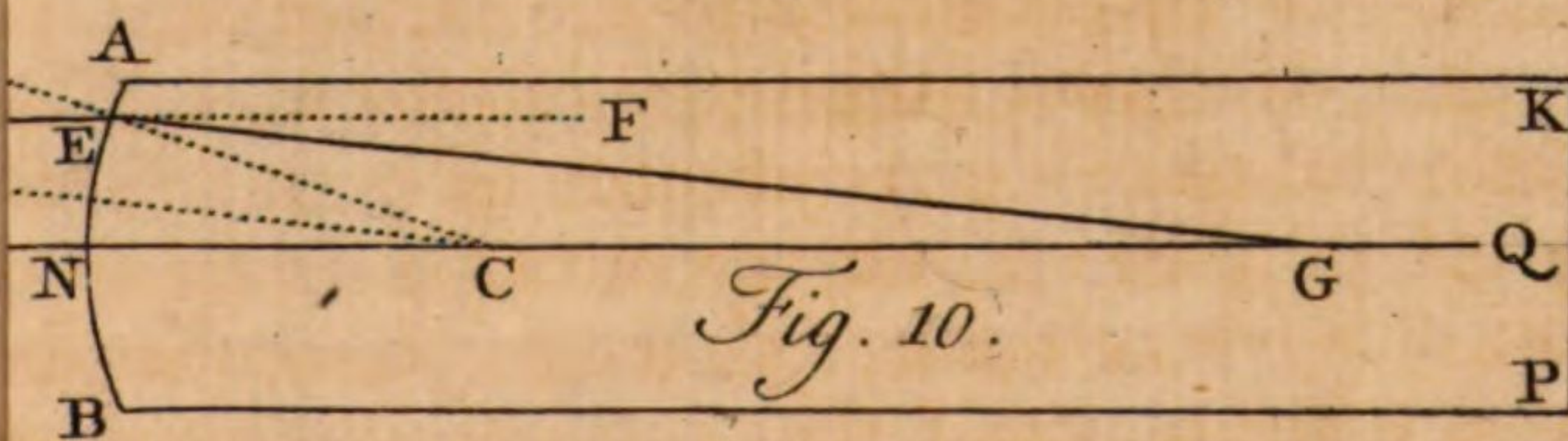
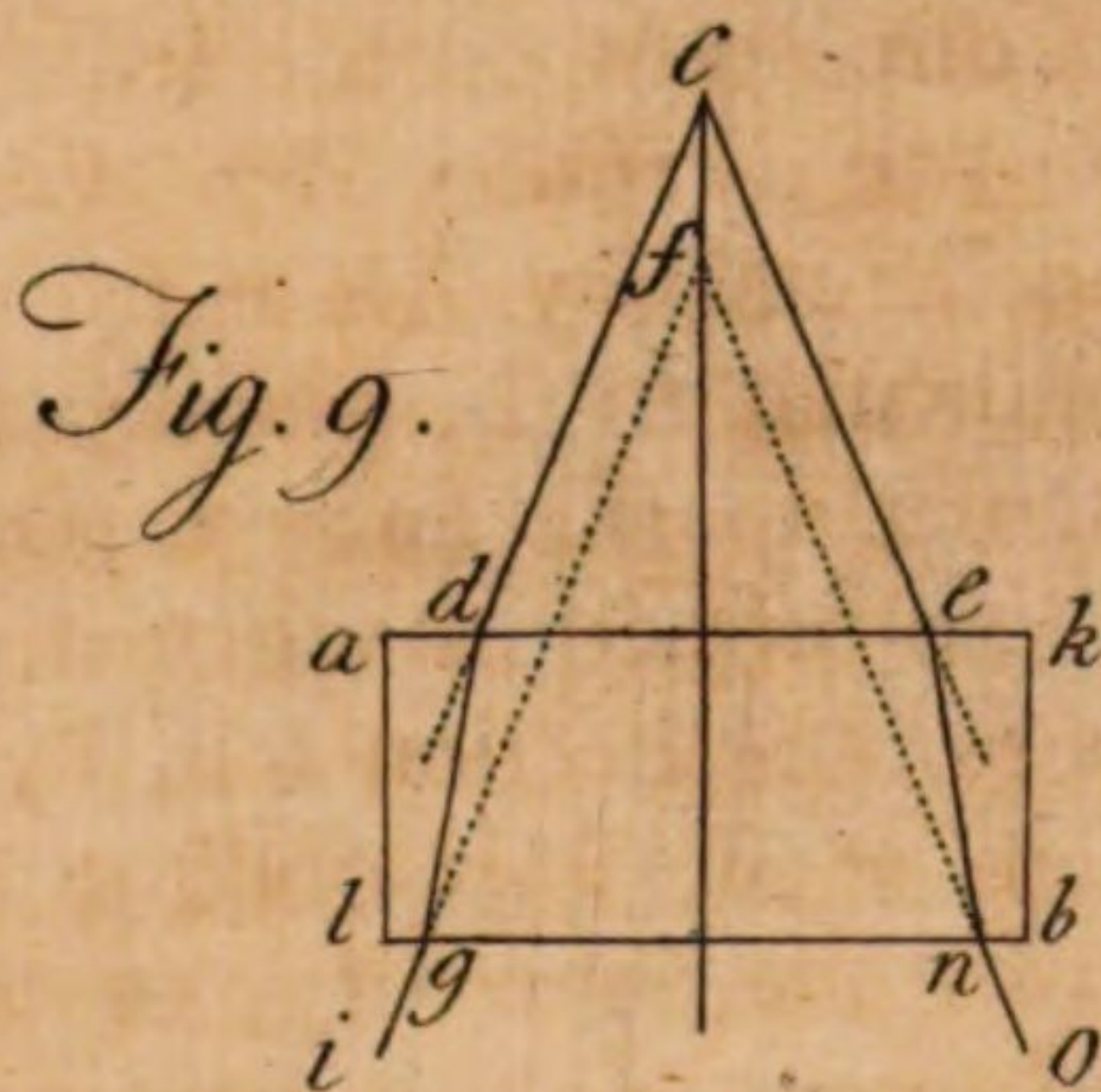
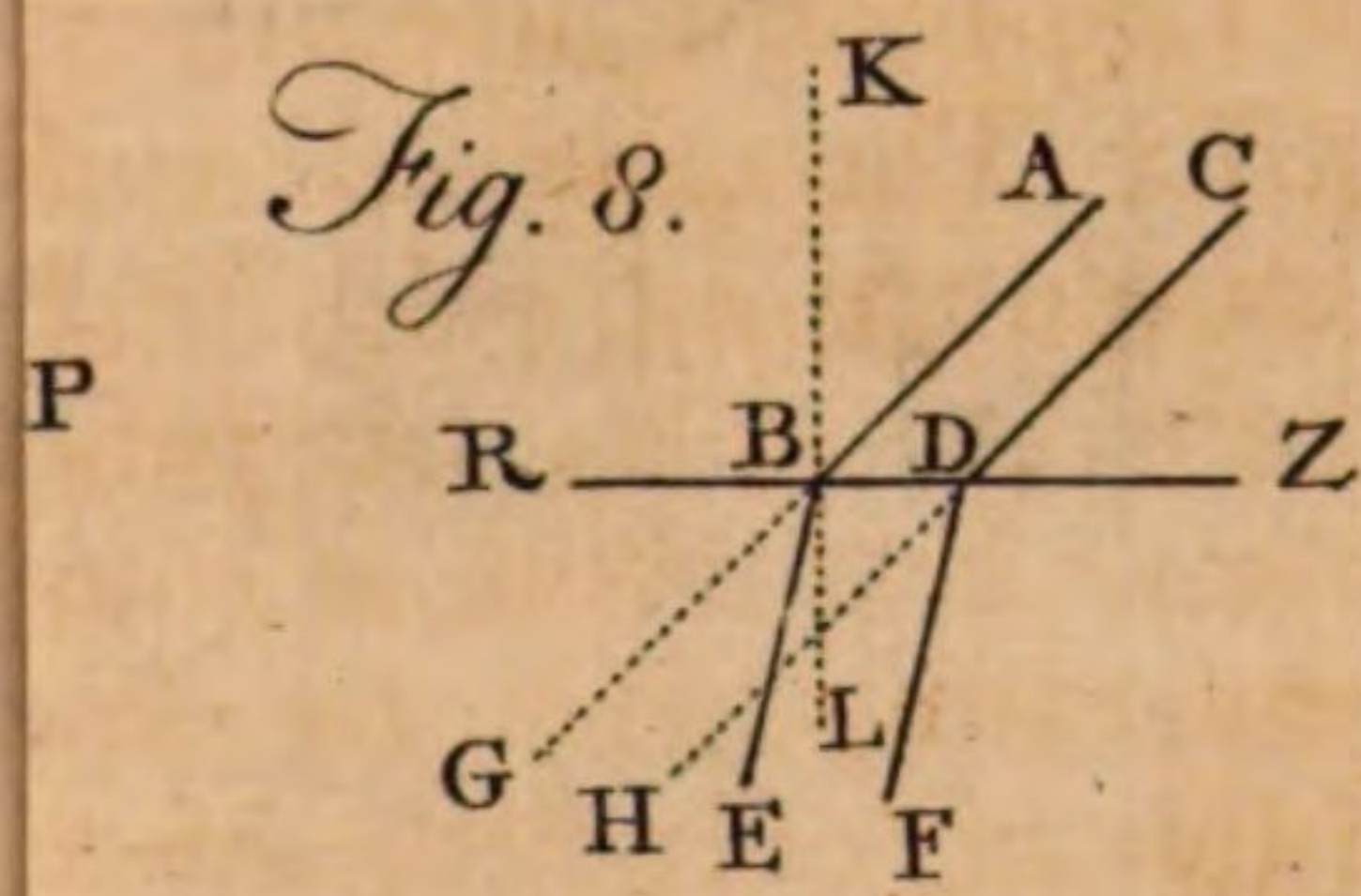
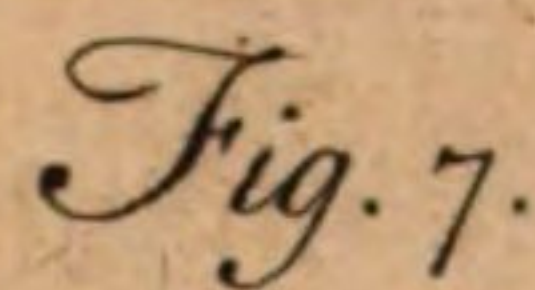
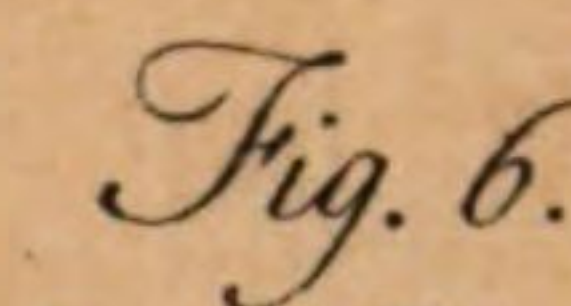
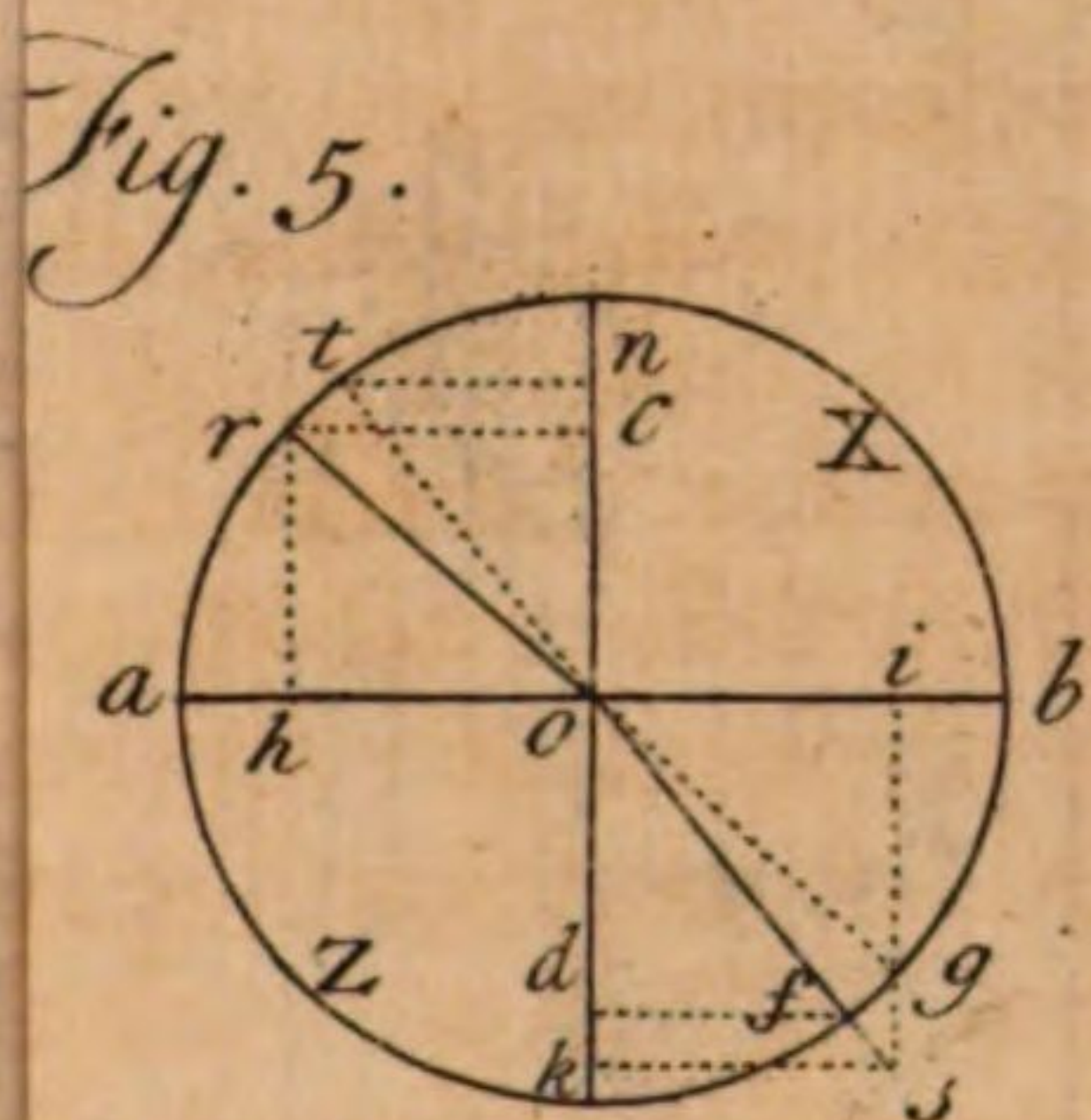
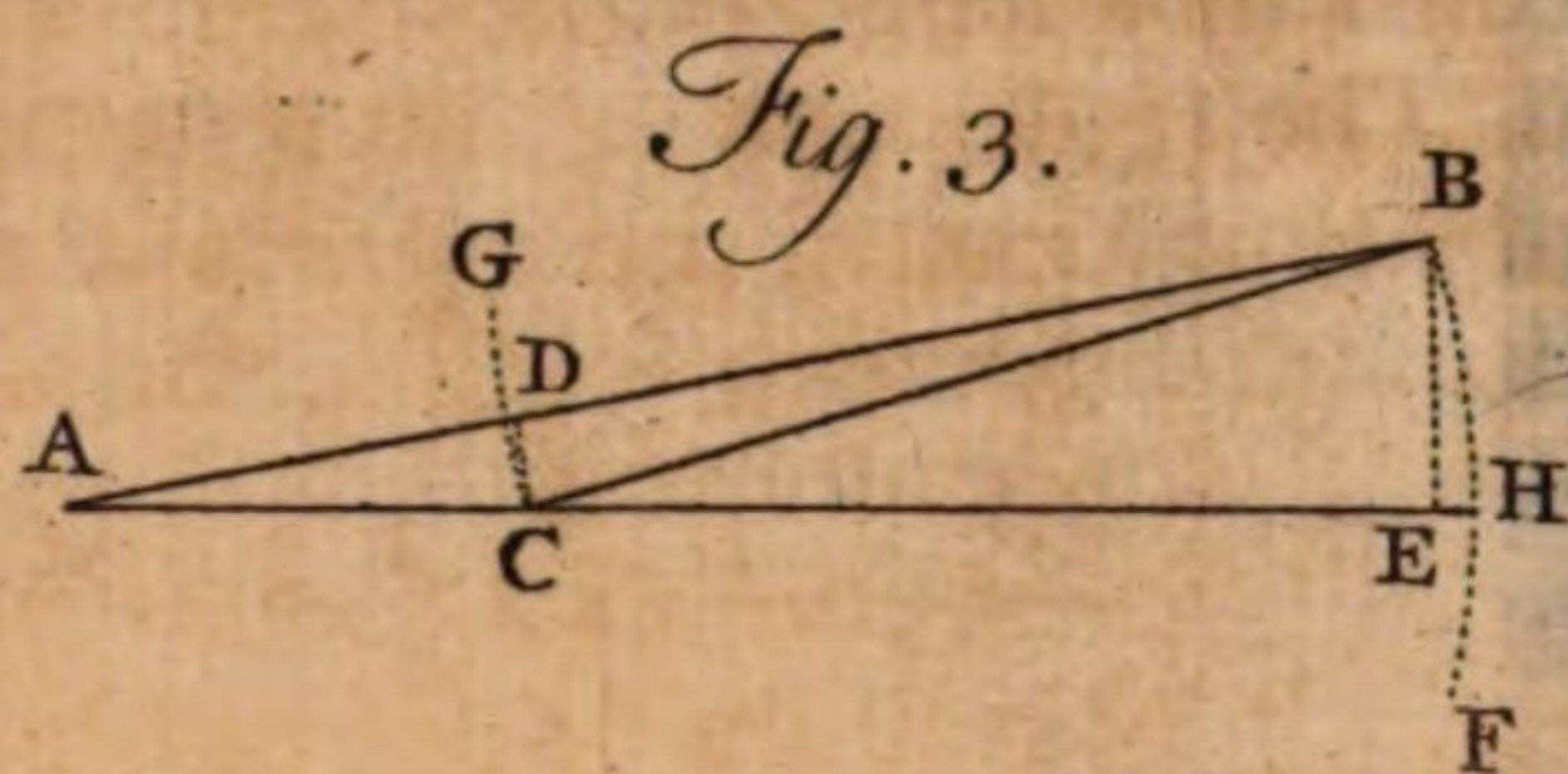
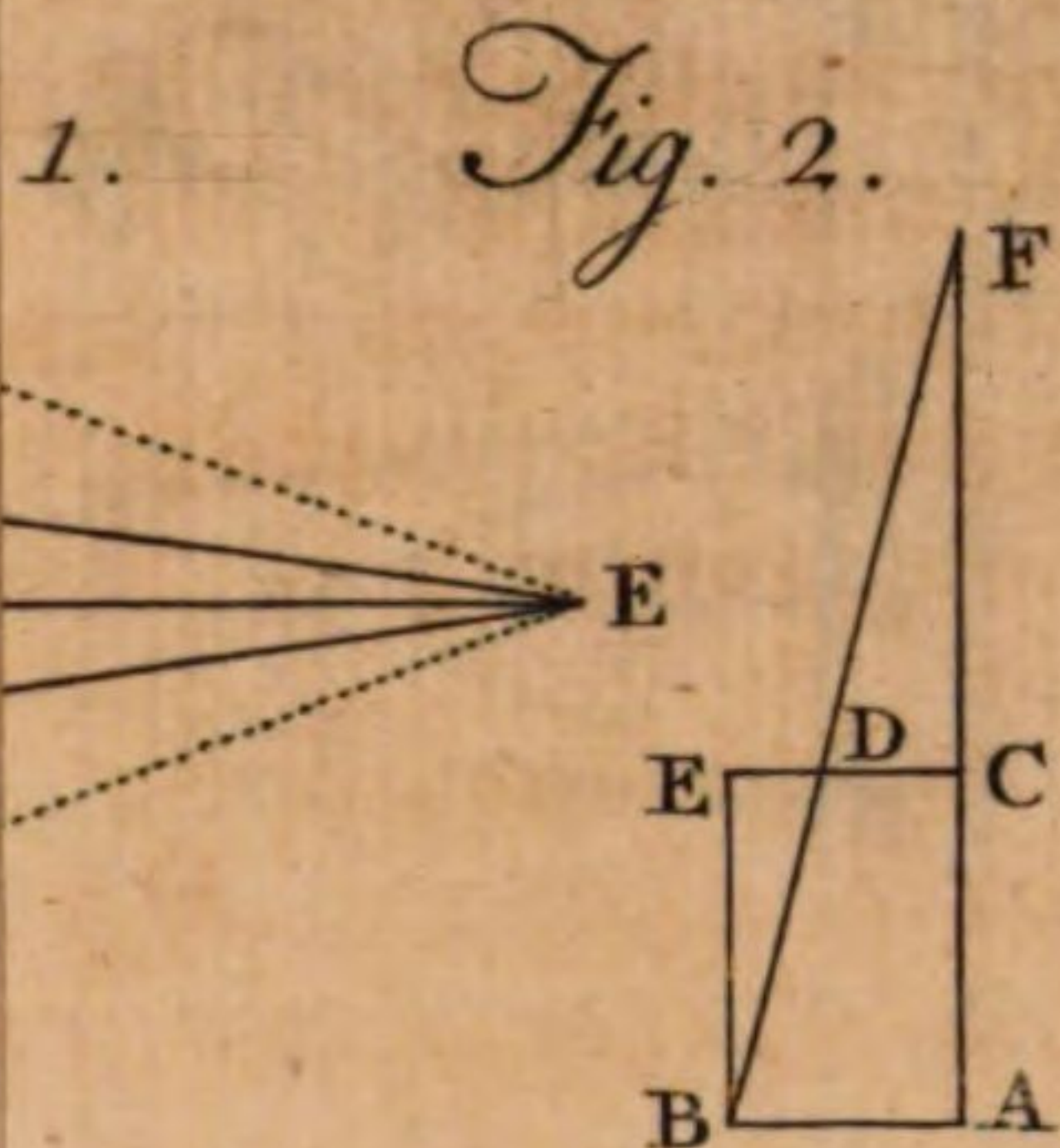
51. *Parallel rays that fall directly upon a plano-convex lens, are made to converge by passing through it; and the distance of the focus, to which they converge, when they are refracted, is equal to a diameter of the lens's convexity.*

This proposition consists of two different cases; for the rays may enter the glass two ways, either at its plane surface or at its spherical one.

The first case is when the rays enter at the plane side. If GL, Plat. IX. fig. 2. is a plano-convex lens, and AR, DS, are two parallel rays; DS, which is the axis of the beam, is perpendicular both to the surface GL and likewise to the surface GPL, because the beam is supposed to fall directly upon the lens. Therefore DS will go strait through the lens in the direction SPF without being at all refracted, by prop. 41. The collateral ray AR is parallel to DS, by the supposition, and DS is perpendicular to the plane surface GL. Therefore AR is likewise perpendicular to the same surface GL. Euc. b. XI. prop. 8. So that AR suffers no refraction when it enters the lens at R, but passes strait forwards in the direction RB, by prop. 41. But, when it is to pass out of the lens at B, it is oblique to the surface GPL: for since C is the center of the convexity, or of the sphere, of which the surface GPL is a segment, CO is perpendicular to the surface, and consequently RBE, which is the direction of the ray, is oblique to it. The ray therefore will be refracted from a perpendicular, by prop. 38; for it is passing obliquely out of glass into air, or out of a denser medium into a rarer. Now on this side of the lens all the perpendiculars, because they are drawn from C the center of the convexity, which is on the other side, such as CO and CF, diverge. Consequently if a ray, that before refraction, was parallel to the axis DF, or that neither diverged from the axis, nor converged towards it, is refracted from these perpendiculars, which all of them diverge, the refracted ray, because it is farther from the direction of the perpendiculars than it was at its incidence when it was parallel to its axis, will be made to converge

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PLATE VIII.





towards its axis. Thus the ray RB, which if it had gone straight forwards would have described BE and have made the angle EBO with the perpendicular BO, is refracted from this perpendicular, so as to make a greater angle OBF with it. Now if the ray had not been refracted at all, it would have continued parallel to its axis PF, as it was at its incidence; or if it had been refracted towards the perpendicular BO, which diverges from the axis, then the ray itself would have diverged from thence. But it is refracted from the perpendicular, and consequently acquires the contrary direction, or is made to converge towards the axis in the line BF.

PF, or the distance of F from the lens, that is, the distance of the focus, or point where this or any other collateral ray, when it is made to converge, will meet its axis is equal to twice CP, or to two semidiameters or one diameter of the sphere, of which the lens is a segment. The angle of incidence where the ray is refracted is EBO, and AE is parallel to DP, by the construction, therefore OBE or the angle of incidence, which is the external one, is equal to BCF the internal opposite one on the same side. Euc. b. I. prop. 29. OBF is the refracted angle, or angle which the refracted ray BF makes with the perpendicular; EBF, which is the difference between the angle of incidence OBE and the refracted angle OBF, is the angle of refraction. And EBF is equal to the alternate angle BFC. Euc. b. I. prop. 29. From what has been said, it appears, that in the triangle BCF the angle at F is equal to the angle of refraction, and the angle at C is equal to the angle of incidence. But in plane triangles, the sides are as the sines of the opposite angles, by prop. 32. Therefore BC is to BF as the sine of the angle at F to the sine of the angle at C, or as the sine of refraction to the sine of incidence. Now the sine of refraction is to the sine of incidence, when a ray passes out of glass into air, as 1 to 2. Therefore BC is to BF as 1 to 2, or BF is twice BC. BC and PC are equal, because they are semidiameters of the same sphere. And BF and PF are nearly equal, when B and P are near together, or when the collateral ray AB is near the axis CP. Therefore since BF is equal to twice BC, PF, the focal distance, is likewise equal to twice PC, or to two semidiameters or one diameter of the lens's convexity.

In the other case of this proposition the rays enter the lens at the spherical surface. If EH, Plat. IX. fig. 3. and XP are two parallel rays, of which XP is the axis and EH a collateral ray. This collateral ray will be refracted as it passes into the lens ANBD, or out of air into glass through a convex surface, and will be made to converge to its axis:

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XS, by proposition 48. The lens is here made very thick only that the reader may more distinctly see the direction of the rays, when they are within it; for in the demonstration this thickness must be neglected, or we must consider the lens as having little or no thickness at all. Now HM the collateral ray converges to its axis PS after this first refraction. And these converging rays are to pass out of the glass into the air through the surface ND, that is, out of a denser medium into a rarer through a plane surface. Therefore they will be made to converge more, by proposition 44. The first refraction, which this ray EH suffers when it enters the glass, would make it cross its axis or would produce a focus at the distance of 3 semidiameters of the glass's convexity; so that I would be the focus, if the ray was to go straight forwards in the direction HMI, which it acquires by this refraction. But it will be refracted a second time in passing out of the glass, and will be made to converge more, as has been already proved. So that I will be only an imaginary focus and, by proposition 29, the real focus will be made nearer than I, or will be at a less distance than PI, or than 3 semidiameters of the convexity. The question is, how much nearer? Now the ray HM and the axis PS converge within the glass and pass out of the glass into the air through a plane surface. Therefore, by proposition 46, the distance of the imaginary focus or PI will be to the distance of the real one as the sine of the refracted angle to the sine of incidence. But out of glass into air the sine of the refracted angle is to the sine of incidence as 3 to 2. And consequently the distance of the real focus where the rays will meet after this second refraction will be only $\frac{2}{3}$ of PI. And PI is 3 semidiameters of the lens's convexity. Therefore taking PF equal to $\frac{2}{3}$ of PI or to 2 semidiameters F will be the focus.

By this time the reader sees the difference of the two cases. If the rays fall first upon the plane side of the lens they are refracted only once and that is when they are going out of the lens at the convex side. But if they fall first upon the convex side they are refracted twice, once when they enter the lens, and again as they are going out of it at the plane side. And in both these cases the focal distance is the same, for it has been proved in both of them to be equal to 2 semidiameters or 1 diameter of the sphere of which the lens is a segment.

Perhaps it may appear strange that the two refractions, which the rays suffer in the latter case, should not make them converge more or bring them to a focus nearer than the single refraction, which they suffer in the former case. But this is the truth of the matter, as may thus be made to appear. When the rays fall first upon the plane side and are re-

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fracted only at the convex one, as they pass out of the lens; the angle of refraction is to the angle of incidence as 1 to 2, or whatever is the angle of incidence at the convex surface the angle of refraction will be half of it. Whereas, when the rays fall first upon the convex surface, whatever is the angle of incidence there, yet though the rays are twice refracted both the angles of refraction added together will be equal to no more than half of it. For the first refraction is made, when the ray is passing out of air into glass: consequently the angle of refraction there is only $\frac{1}{2}$ of the angle of incidence. Thus RHE or its equal OHC is the angle of incidence at the convex surface; and LHM, which is the angle of refraction, is only $\frac{1}{2}$ of it. The second refraction is made at the plane surface when the ray is passing out of glass into air; consequently the angle of refraction is $\frac{1}{2}$ the angle of incidence at that surface. But what is the angle of incidence at the plane surface where the angle of refraction is half of it? HM is the incident ray, MT is the perpendicular, and the angle which MI or MH continued makes with TM, that is, the angle TMI is the angle of incidence. But since LO or EH continued and MT are both of them perpendicular to the plane surface ND, they are parallel to one another. Euc. b. XI. prop. 8. Therefore TMI the external angle is equal to OHM the internal opposite one on the same side. Euc. b. I. prop. 29. That is, TMI the angle of incidence at the plane surface is equal to OHM the angle of refraction at the first. Now the angle OHM or the first angle of refraction is $\frac{1}{2}$ of the angle of incidence at the convex surface. And the angle of refraction at the second surface or IMF is $\frac{1}{2}$ TMI, that is $\frac{1}{2}$ OHM, or half $\frac{1}{2}$ of the angle of incidence at the convex surface. Therefore both the angles of refraction taken together are $\frac{1}{2}$ and half $\frac{1}{2}$ of the angle of incidence. But a third and half a third of the angle are just half the angle. So that both these refractions taken together amount to no more than the single refraction, which the rays suffer, when they enter the lens at its plane surface and are refracted only at their passage out of it through the convex one.

52. *Parallel rays that fall directly upon a plano-concave lens are made to diverge by passing through it; and the distance of the virtual or negative focus, from which they appear to diverge, when they are refracted, is equal to a diameter of the lens's concavity.*

This proposition as well as the last admits of two cases. For the rays may enter the lens either at the plane side or at the concave one.

If gl, Plat. IX. fig. 4. is a plano-concave lens, and parallel rays such as *ar* and *ds* fall directly upon the plane side of it so that *ds* the axis of the

beam may be perpendicular to the surface; then ar , because it is parallel to ds will likewise be perpendicular to the same plane surface. Euc. b. XI. prop. 8. And consequently both these rays and all other rays of the same parallel beam will pass into the glass without being refracted, by proposition 41. But at the second or concave surface of the lens only sc the axis of the beam, will pass out unrefracted, because that only is perpendicular to the surface. Any collateral ray such as ar , when it comes to the concave surface, cannot pass out of it in the right line be parallel to its axis sc ; for be is oblique to the surface of the glass, and consequently as the ray passes out of glass into air, it will be refracted from a perpendicular. c being the center of the concavity obc is a perpendicular at the point b where the ray passes out, as pc is another perpendicular at the point p . But these and all other perpendiculars converge to c , which is the center of the concavity and is in the axis pc . Therefore the ray, which instead of passing out parallel to its axis is refracted from a perpendicular, will be made to diverge from its axis. The incident rays are parallel, that is, they neither diverge nor converge. They are refracted from the perpendiculars, or when they are refracted they are farther from the direction of the perpendiculars than they were at their incidence. But the perpendiculars converge to the axis. Therefore the refracted rays will diverge from it. The ray ar in particular, which comes out of the glass at b , will not go straight forwards in the line be , which makes the angle ebc with the perpendicular bc , but in some line as bm that makes the angle mbc greater than ebc with this perpendicular. Thus the two rays ar and ds , which were parallel, when they entered the glass, will come out of it in the diverging lines bm and pc ; and consequently have the same direction in respect of one another as they would have had, if they had come from the radiant f . This point f from whence the refracted rays appear to diverge is called either the imaginary radiant, or the virtual focus, or the negative focus. And pf the distance of this negative focus from the surface of the glass is equal to a diameter of the sphere of which the concave surface gpl is a segment.

That pf is equal to twice pc might be shewn in the same manner as in the first case of the last proposition we shewed that PF , fig. 2, is equal to twice PC . But we may prove it likewise in another manner by shewing that the lens gpl , fig. 4. is the negative of GPL , fig. 2, and consequently that the effect it produces will be equal, only it will produce it the contrary way. Now if GPL was to flatten, or if the convex surface was to become planer, the radius PC and consequently the focal distance PF , which is twice the radius, would lengthen; that is, the less convex the

the glass is, so much more remote will be the focus. Therefore the less convex the glass is, so much less the refracted rays converge; for the convergency of the rays is inversely as the distance of the focus, by proposition 29. And when GPL is quite plane, or when the convexity is nothing, then the lens is only a plane glass, and the refracted rays will be parallel, or their convergency will be nothing, or the focal distance PF will be infinite, by propositions 47, 31. But, if the surface GPL moves still farther in the same direction or sinks in, after it is become plane, then its convexity becomes less than nothing or negative, and the focal distance is greater than infinite or is negative too; that is, it must be taken on the contrary side. This therefore is the only difference between the two lenses GPL and *gpl*, the convexity of one is affirmative, and the convexity of the other is negative; the focus of GPL or the affirmative focus is at F on the side of the refracted rays, and the focus of *gpl* or the negative focus is at *f* on the same side with the incident rays; and as the rays are made to converge by passing through the lens GPL, the lens *gpl* will produce the contrary effect, or will make their convergency negative or will make them diverge. But this convergency and divergency, this affirmative and negative effect will be respectively proportional to the causes that produce them. And as the convergency caused by the plano-convex lens, or by the lens when it is affirmative, will bring the refracted rays to a focus at the distance PF, or at the distance of two semidiameters of the convexity; so the divergency caused by the plano-concave lens, or by the lens when it is negative, will make the refracted rays appear to diverge from a negative focus at the distance *pf*, or at the distance of two semidiameters of the concavity.

The second case of parallel rays falling directly upon a plano-concave lens is, when they enter the lens at the concave surface as *xp*, Plat. IX. fig. 5. which is the axis of a parallel beam and *eb* which is one of the collateral rays. The axis passes into the lens on this side, and out of it again on the other, without being refracted, by proposition 41: because as the beam falls directly upon the lens the axis is perpendicular to both its surfaces. But *eb*, which is oblique to the surface, and all the other collateral rays, as they pass out of air into glass through a concave surface, are made to diverge from their axis, by proposition 49. Thus the ray *eb* will not go straight forwards in the line *bl*, but will be bent into the line *bm* so as to diverge from its axis *ps*, as if it had come at first from the point *i*. And these diverging rays when they pass out of the lens into the air will be made to diverge more, by proposition 45. Thus the ray *bm* does not go straight forwards in the line *mk*, but is bent still farther and

is made to describe the line mg , which diverges from its axis as if it had come at first from the point f , which is therefore the imaginary radiant, virtual focus, or negative focus of these rays after they have passed through the lens.

The distance of the point i from the lens, which is the point from whence the ray ps and bm appear to diverge after the first refraction, is equal to 3 semidiameters of the lens's concavity, by proposition 50. But the second refraction makes the rays diverge more, and makes the imaginary radiant f nearer than i , in the proportion of the sine of incidence to the sine of the refracted angle, that is, in the proportion of 2 to 3: therefore pf is $\frac{2}{3}$ of pi , or $\frac{2}{3}$ of 3 semidiameters, or is 2 semidiameters of the concavity.

The letters of reference are the same in fig. 5. and in fig. 3. only the former are small ones and the latter capitals. And with this help the reader, if he thinks it worth his while, may easily shew that the second case of this proposition is the negative of the second case in the foregoing one; and may apply to the case now before us all that was said to prove that the two refractions of a ray, when it falls first upon the convex side of a plano-convex lens, are only equal to the single refraction, which it suffers, when it falls first upon the plane side.

53. *If parallel rays fall directly upon a lens, which is equally convex on both sides, they will converge, after they have passed through it, to a focus at the distance of a semidiameter of either convexity.*

The plano-convex lens EFG, Plat. IX. fig. 6. would make parallel rays converge and bring them to a focus at the distance of a diameter of its convexity, by proposition 51. And another plano-convex lens HKI, which has the same convexity with the former, or is a segment of an equal sphere, will produce an equal effect: and consequently, if the rays, after they have passed through the first lens EFG, fall upon the second HKI, and pass through that, they will be made to converge as much more as they did, and for that reason their focus will be as near again, by proposition 29. That is, since one of these lens's bring the rays to a focus at the distance of a diameter of the sphere of which it is a segment, both of them together will produce a double convergency in the rays, and therefore will bring them to a focus at half the distance, or at the distance of a semidiameter of the sphere of which either lens is a segment. Now, if the two plane sides EG and HI of these two lenses were close together, they would make the double convex lens ACBD. Therefore a convex lens of equal convexities on both sides may be considered as two.

plano-convex lenses of equal convexities: and as two such lenses would make rays, that were parallel at their incidence, converge to a focus at the distance of a semidiameter of the convexity of either; the convex lens, which is made up of both of them, will produce the same effect.

The reader may imagine there is some difference between two plano-convex lenses and one double convex one. The difference seems to be this; that after the rays are refracted at the surface EFG, they will suffer a second refraction in passing out of the lens through the plane surface EG, since the first lens makes them converge they will fall obliquely upon the surface HI and be refracted there a third time, and then again a fourth time, when they pass out of the surface HKI. Whereas in going through the lens ACB, they are refracted only twice, once when they enter it and once when they go out of it. But then this apparent difference makes no real one. For when the rays, that were made to converge within the lens EFG, come out of the glass into the air through the plane surface EG their convergency is indeed encreased; by proposition 44. But when the rays thus converging pass out of the air into the other lens HKI through the plane surface HI, their convergency is just as much diminished; by proposition 43, 40. Therefore of the four refractions, which the rays suffer, when they pass through two plano-convex lenses that are not close together, the second and third refractions mutually destroy one another; so that the whole effect is produced by the first when the rays enter the surface EFG, and by the fourth when they pass out of the surface HKI, just as it would have been if these two lenses had been close together and had made up one double convex one of equal convexities on both sides such as ACBD.

Burning-glasses are double convex lenses. The sun is so far off, that we may consider every point or every radiant upon the surface of it, as a radiant at an infinite distance, and consequently may suppose the rays emitted from each point of it to be parallel to one another, without any sensible error, by prop. 30. Therefore all the rays of the sun that fall upon a burning-glass, if the convexities of it are equal on both sides, will, by passing through the glass, be made to converge, and will be united in a focus behind the lens, at the distance of a semidiameter of either of its convexities. When all these rays are thus brought together, the heat they produce will be very great, and may be sufficient to set fire to a piece of paper, or of touchwood, or even to bodies less apt to burn than these are, if they are placed in the focus of the lens. This effect of the rays of the sun when they are thus collected, is the reason why that point where they are collected is called the focus: and the name,

name, after it had for this reason been given to this point, has been made use of as a general one to stand for any point where converging rays meet, or to which they tend.

A plano-convex lens will collect the rays of the sun, and produce a sufficient heat at the focus to set fire to bodies that are placed in it. But the distance of the focus, when such a lens is made use of, is equal to the diameter of the sphere, of which the lens is a segment.

A convex lens, whose convexities are unequal, will serve for the same purpose: and the distance of the focus in this case shall be determined in prop. 55.

The ancients made use of refracting burning-glasses, as appears from a passage in a play of Aristophanes, which is called *The clouds*. For Strepsiades, act. II. scene 1. tells Socrates, that he had found out an excellent contrivance to defeat his creditors, if they should bring an action against him. His contrivance was, that he would get from a jewellers (for thus the scholiast teaches us to translate the word *φαρμακοπώλαις*) a certain transparent stone, which they called *ύαλον*, that was used for kindling fire; and then standing at a distance he would hold it to the sun, and melt down the wax on which the action was written. The scholiast calls this instrument a round stone; but certainly it was not round like a sphere; for he says it was *τροχοειδές παχὺ* round like a wheel and thick. But as from this description it was not a sphere, so neither was it a piece of plane glass round at the edges, as appears from the use of it; because a circular piece of plane glass would not produce the effect, for which this *ύαλος* was used. For however ridiculous the author of the play intended to make this contrivance of Strepsiades, he has given us a description of the common use that this *ύαλος* was put to, as well as of the particular use which Strepsiades was to make of it. It was sold at the jewellers and was used for kindling fire. But a circular glass could not have been applied to this purpose, unless both or one of its sides were convex. I have called it a glass, though *ύαλος* was a natural substance and not an artificial one, as the scholiast informs us. For *ύαλος*, though it signified glass at the time when the scholiast lived, yet the ancients meant by it a transparent stone, which was properly called *κρύον*, and was like that artificial glass which had been found out when he wrote. The substance, of which these lenses were made, seems not to have been crystal but what is now called talc. Though this may be looked upon only as a softer sort of crystal. That it was a natural and a soft substance appears from Herodotus, book III. 24. For in describing the manner of burying the dead amongst the Æthiopians, he

he says, that the dead body was first dried, and then covered with mortar, and when it had been painted as like the deceased as could be, it was put into a coffin made of *ύελος*, great plenty of which, and such too as is easy to work, they dig up in that country. Coffins made of this were, as he tells us, transparent, so that the corpse might be seen through them. Talc is a soft transparent stone that will easily split into thin plates : and therefore a piece of it, thick enough for a lens, will not bear a good polish, but when it is made as smooth as it can be, it will be full of cracks and scratches. Which accounts for the manner in which the scholiast of Aristophanes says these lenses made of *ύαλος* were used. The surface of them was first oyled and the lens itself was warmed. The oyl would fill up the cracks and scratches and being of nearly the same density with the talc itself, would by that means make it more transparent : as paper, when it is dry, is not near so transparent, as it is, when it has been wetted with oyl. It is difficult indeed to say for what reason they warmed the lens, unless it was to make the oyl more fluid that it might run into the cracks more easily. But perhaps the ancients might imagine, that the rays of the sun would burn better through a warm lens than through a cold one. And though our modern philosophers, who are acquainted with the theory of light and refraction, know very well that the heat or coldness of the substance through which the rays are to pass will make no difference : yet many of those, who make use of burning-glasses, would even now think it a paradox and would scarce be persuaded to believe that the rays of the sun may be made to kindle fire by passing through cold water.

54. *If parallel rays fall directly upon a lens, which is equally concave on both sides, they will diverge, after they have passed through it, from an imaginary radiant, or negative focus, at the distance of a semi-diameter of either concavity.*

One plano-concave lens *bki* would make rays, that were parallel at their incidence and fell directly upon it, diverge from a negative focus at the distance of a diameter of the sphere, of which the lens is a segment, by prop. 52. Therefore two such lenses *bki* and *efg*, if the rays were to pass through both of them, would make the rays diverge as much more, and consequently, by prop. 28, the negative focus, or imaginary radiant, will be as near again, or at the distance of a semi-diameter of either concavity. This therefore will be the distance of the negative focus, when the rays pass through a concave lens, having equal concavities on each side, such as *acbd* : for this lens *acbd* may be considered

sidered as the two plano-concave ones *bki* and *efg* with their plane sides close together.

The reader sees by this time why a concave, or a plano-concave lens cannot be used as a burning-glass. The rays of the sun are made to burn by being made to converge so as to be brought together in a focus, that many of them may fall within a narrow compass upon some point in the thing that is to be burned. But a concave or plano-concave lens makes the rays of the sun diverge, or separates them farther from one another, instead of bringing them closer together.

55. *When parallel rays fall directly upon any lens that is convex; as the sum of the semidiameters of both convexities to the semidiameter of either; so is double the semidiameter of the other to the distance of the focus, after the rays are refracted.*

We have already seen what is the distance of the focus, when only one side is convex, or when both sides are equally convex. But some lenses have a greater convexity on one side than on the other. The chrystalline humour of the eye is of this sort. And the rule here laid down is applicable to all cases. We will apply it in the instance of a lens having equal convexities on both sides, and to a lens that has one side plane: and then from shewing the truth of it in these two instances we may infer the truth of it in all other instances.

First in a lens of equal convexities the distance of the focus has been shewn to be a semidiameter of either convexity, in prop. 53. And the focus found by this rule will be at the same distance. Call the semidiameter of one of the convexities s , and then, because the convexities are equal on both sides, the semidiameter of the other convexity must likewise be equal to s . Therefore the sum of the semidiameters is $s+s$ or $2s$. In this case the proportion will be as follows. As the sum of the semidiameters of both convexities, or as $s+s=2s$, is to the semidiameter of either convexity, or to s : so is double the semidiameter of the other convexity, or so is $2s$, to the focal distance. But in this analogy, as $s+s$ to s , so is $2s$ to a fourth proportional, it is easy to determine what this fourth proportional must be. For since the first term $s+s$ is equal to the third, or to $2s$, the second term s must likewise be equal to the fourth. Therefore the focal distance is s , or one semidiameter of the convexity on either side.

Secondly, if the lens is a plano-convex one, we cannot apply this rule for finding out the focus, unless we consider the plane surface as the segment of a sphere: and, from what has been shewn under prop. 49.

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it appears, that we may consider it as the segment of a sphere, whose femidiameter is infinite. Call this infinite femidiameter S , and call the femidiameter of the sphere, of which the convex side is a segment, s . Then the sum of the femidiameters of both convexities, or rather of both sides, is $S+s$. Therefore by the rule, as $S+s$ the sum of the femidiameters of both convexities, is to S the femidiameter of one of them, so is $2s$, or double the femidiameter of the other, to the focal distance of this plano-convex lens. But when the proportion stands thus, as $S+s$ is to S , so is $2s$ to a fourth proportional, what will that fourth proportional be? I say the fourth term will be equal to the third, or will be $2s$. The first term is $S+s$, and the second is S . But these two terms are equal to one another. For S is an infinite quantity, and s is a finite one. Therefore since this finite quantity s , bears no proportion to the infinite one S ; if you add the finite quantity s to the infinite one S , this infinite quantity is not encreased by the addition of a quantity so small in respect of the whole as to be quite insensible. Therefore $S+s$ is equal to S : and consequently, since the first and second terms $S+s$ and S are equal to one another, it could not be as $S+s$ to S , so $2s$ to the focal distance, unless this fourth term, or focal distance, was likewise equal to $2s$. But $2s$ is two femidiameters of the lens's convexity. Therefore the distance of the focus of a plano-convex lens is rightly determined by this rule: for it is found to be two femidiameters, or one diameter of the sphere, of which the convex side of the lens is a segment; and this was proved to be the distance of it in prop. 51.

But thirdly, this rule for finding the focus either of a convex lens of equal convexities, or of a plano-convex lens may be applied to any convex lens whatever. For the two cases already considered, may be looked upon as the two extremes. When the convexity on one side is equal to the convexity on the other side, this is one extreme. And when the convexity on one of the sides has decreased till it becomes nothing, or is quite plane, this is the other extreme. The intermediate cases are any of those, when one side is less convex than the other: and as the rule here laid down is true in the two extremes, it will be equally true in all the intermediate cases. When the convexity of the lens is equal on both sides, as ACBD, Plat. IX. fig. 6. it appears from what has been said already concerning this rule, that $s+s$, the first term, is double s , the second term, and $2s$, the third term, is likewise double the focal distance. If the side ADB of this lens was to sink in, or to flatten, till at last it became quite plane, so that the lens was changed into a plano-convex one, such as EFG, then, from what has

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been said already, it appears that $S+s$, the first term, is equal to S , the second, and $2s$, the third term, is likewise equal to the focal distance. Since therefore in one extreme, the first term is double the second, and the third is double the focal distance, and when the lens has changed to the other extreme, the first term is equal to the second, and the third is equal to the focal distance, it follows, that whilst the lens was changing from one extreme to the other, or in all the intermediate cases, the first and second terms are constantly approaching to a ratio of equality, and so likewise are the third and fourth. And since the first term becomes equal to the second, and the third equal to the fourth, in the same state of the lens, it follows farther, that their respective approaches to a ratio of equality have been in the same proportion. Therefore in all the intermediate states of the lens, that is, whatever was the convexity of ADB , the first term has born the same proportion to the second, that the third has to the fourth, or the sum of the semidiameters of both convexities, has always been to the semidiameter of either, as double the semidiameter of the other, to the focal distance.

56. *When parallel rays fall directly upon any concave lens, as the sum of the semidiameters of both concavities, is to the semidiameter of either, so is double the semidiameter of the other to the distance of the negative focus.*

This proposition may be easily proved in the same manner as the last is, the reader will be able to apply that demonstration without having it repeated. Or indeed, this proposition is the same as the other: for a concave lens is only a lens whose convexity is negative, by prop. 49. So that the only difference is, that if, when the lens is convex, the semidiameters are considered as affirmative, when the lens is concave they will be negative; the refracted rays, which converge in one case, will have a negative convergency, that is, will diverge in the other; and the focus, which is on the side of the refracted rays, when the convexity is affirmative, will be on the contrary side when the convexity is negative; and therefore the focus in the concave lens will be virtual, or negative. But these changes make no alteration in the proportion, by which the focal distance is determined: for the proportion will be the same whether the terms of it are considered as affirmative or negative.

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57. *The collateral rays of a direct parallel beam, when they are made to converge by a convex or plano-convex lens, do not all of them meet in a focus at exactly the same distance behind the lens.*

The collateral ray AR Plat. IX. fig. 2. crosses its axis after refraction, or is brought to a focus at F, which is at the distance of a diameter of the lens's convexity, by prop. 51. But any other collateral ray, which falls upon the lens nearer to the axis DS than AR is, will have its focus after refraction, a little farther from the lens than F; and any other collateral ray, which falls upon the lens farther from the axis than AR, will have its focus a little nearer than F. The ray DS is perpendicular to both surfaces of the lens, and consequently is not refracted at all, by prop. 41. Now the nearer any ray falls to DS, the nearer perpendicular, or the less oblique, it is to the two surfaces of the lens, and consequently it will be refracted the less. But the less any ray is refracted, the less it will converge; and therefore, by prop. 29, the focus will be so much the more remote from the lens. The farther any ray falls from DS, the farther it is from being perpendicular, or the more oblique it is, for which reason it will be refracted the more, and will converge the more, and therefore the distance of the focus will be less. Thus we see that all the collateral rays will not have their focus at exactly the same point. The focus of the ray DR is at F; but those which fall nearer to the axis will have their focus more remote, and those which fall more remote from the axis will have their focus nearer. But in such collateral rays as are none of them very far from the axis, this difference will be inconsiderable.

If the reader is desirous of knowing why the demonstration of proposition 51, is not universally applicable to all other collateral rays, as well as to AR, he may remember that in that demonstration there was a small error. For we supposed BF and PF to be equal, though they are not quite, but only nearly, so: they cannot be quite equal, unless B and P, or R and S, that is AR and DS, were close together. And the farther B is from P, or R from S, or AR the collateral ray from DS the axis, so much the longer will BF be in proportion to PF. Now the length of BF is always a given quantity: for it is always twice CB, as appears from the demonstration of proposition 51. Therefore the greater proportion this given line BF bears to PF, or the longer BF is in respect of PF, so much the shorter PF must be. Euc. b.V. prop. 10. From whence it follows, that the farther B is from P, or R from S, or the collateral ray AR from the axis DS, so much the shorter is PF,

or so much less is the focal distance. But in all such collateral rays as do not fall very far from their axis, or are not very oblique to the surface of the lens, the difference between BF and PF being inconsiderable in any of them, the distance of the focus is so nearly the same, that they may all be considered as meeting at the same point.

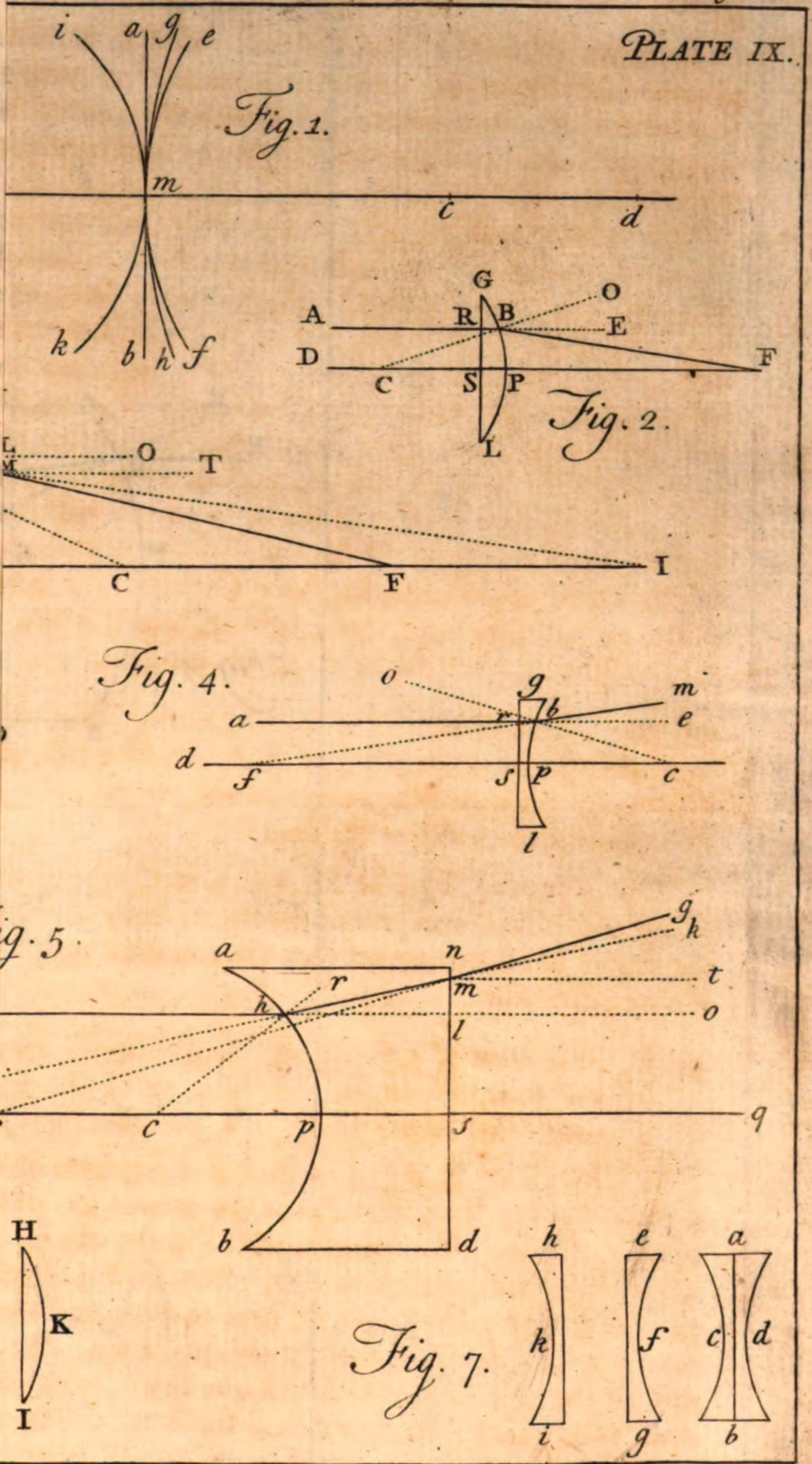
The reader will be able to apply this reasoning, without having it repeated, in the case of such a beam of direct and parallel rays as falls first upon the convex side of a plano-convex lens, or in the case of such an one as falls upon either side of a double convex lens. For in these cases as well as in that, which we have been explaining, the axis only of the beam is perpendicular to the lens, and passes through the lens unrefracted; and those rays, which fall nearest to the axis, are nearest perpendicular, or least oblique, and consequently will be refracted least, and will converge least, and for that reason will have their focus most remote from the lens. Whereas those rays, which fall upon the lens at any considerable distance from the axis, are farther from perpendicular, or more oblique, are more refracted, converge more, and consequently have their focus nearer to the lens.

58. *The collateral rays of a direct parallel beam, when they are made to diverge by a concave, or plano-concave lens, do not all of them diverge from an imaginary radiant, or negative focus, at exactly the same distance before the lens.*

This is proved, by Plat. IX. fig. 4, in the same manner that the foregoing proposition was proved by fig. 2, only as the lens is here concave, the reader must remember that the refracted rays diverge, and the focus is a negative one.

59. *When a beam of parallel rays falls obliquely upon a lens, the distance of the focus, if the lens is convex, or of the negative focus, if it is concave, will be nearly the same as when the rays fall directly.*

If LPG, Plat. X. fig. 1. which is a segment of the sphere PMFG, is a plano-convex lens, and PO is the axis of it: parallel rays fall directly upon it, if their axis coincides with PO the axis of the lens, or if the collateral rays are parallel to PO. But DN and BE, though they are parallel to one another are oblique to PO, and therefore are collateral rays not of a direct but of an oblique beam. These rays BE and DN, and all the rays of any such oblique beam, when they are refracted, will meet their axis or be united in a focus, at the same distance behind the lens, that the rays of a direct beam would, that is, by proposition 51, at



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at the distance of a diameter of the lens's convexity or of the sphere, of which the lens is a segment.

Suppose the pole or vertex of the lens P was placed at R so that PO should coincide with RC : then AR , BE , and DN , would be a direct beam; AR or RC , which is AR continued, coinciding with PO would be the axis of the beam, and RE and DN would be collateral rays. In this position of the lens the focus will be at F , or at the distance of a diameter of the lens's convexity, by proposition 51, and at this point F the rays BE and DN will meet. But, if the position of the lens is changed and the pole or vertex of it instead of being at R is at P , then indeed the ray AR will fall beside the lens; but the other rays BE and DN will still fall upon it; and as the refractive power of the lens is not altered by changing its place, BE and DN will still converge as they did before, and will meet at F . The only alteration is that the rays BE and DN are more oblique than before, and consequently will be more refracted and will have their focus nearer to the lens; and F is nearer to the lens, when the vertex of it is at P , than when it is at R . But if the obliquity of these rays is not very great, that is, if the lens is not far removed the distance of F from P will be very nearly the same as its distance from R ; that is, the focus of the oblique beam will be nearly at the same distance as the focus of the direct one.

The same method of reasoning will demonstrate this proposition in any case of lenses either convex or concave, for by altering the position of the lens an oblique beam may be changed into a direct one; and when the focal distance of the direct beam is known this determines the focal distance of the oblique one. For the changing the lens's position does not change its refractive power, though it changes the obliquity of the rays. And consequently, since if the position is not much changed the obliquity is nearly the same, the focal distance of a beam that is not very oblique will be nearly at the same distance as the focus of a direct one.

But we will apply this method to one other case: and then from the former case and this together we may deduce the truth of the proposition in all cases whatever. Let LPN , Plat. X. fig. 2. be a plano-convex lens. In the former case we supposed the rays to fall first upon the convex side of the lens. But now suppose AE and BG to be two parallel rays of an oblique beam, which fall first upon the plane side of the lens. These rays, by the supposition, are oblique to PO the axis of the lens. But if P the vertex was at the point R then DCR would coincide with PO ; and the ray which describes DCR would be perpendicular to both sides of the lens; and BG and AE , because they are parallel to DR , would

would likewise be parallel to PO , so that the beam would be a direct one, and DRS would be its axis. This axis would pass through the lens unrefracted, by proposition 41, and BG , AE , which in these circumstances are the collateral rays of a direct beam, would by passing through the lens converge to a focus at the distance of a diameter of the sphere $PTHR$, of which the lens is a segment, by proposition 51. And though by putting the vertex of the lens at P the ray DR does not pass through it, yet the rays AE , and BG , will meet at the same point. For by altering the position of the lens nothing is altered, except the obliquity of the rays; and if this is not made very great, their convergency, when they are refracted, and consequently the distance of the focus, will be very nearly the same as when the vertex is at R , and the beam falls directly upon the lens.

Now since the oblique focus is at the same distance as the direct one, when the convexity of the surface of a plano-convex lens is affirmative; it follows, by what was said under proposition 49, and 52, that if the convexity was negative or the lens a plano-concave one, the oblique focus would then only become negative or would be before the lens instead of being behind it, but still at the same distance as the negative direct focus.

But a double convex lens is only two plano-convex ones, and a double concave lens is only two plano-concave ones with their plane sides close together. And since the oblique focus whether affirmative or negative is at the same distance as the direct one, when the lens's are used separately, it will likewise be at the same distance, when they are used together.

The reader remembers that the axis of an oblique beam is oblique to the surface of the lens, and consequently is refracted. But then the direction of the axis, is not altered by this refraction; for the refraction is such as makes the refracted ray parallel to the incident one. We may explain this matter in two instances of an oblique axis, that will make it intelligible in all other instances. Let ef , Plat. X. fig. 3. be an oblique ray passing into the convex lens at the point f , and coming out of it again at the point g . Now if da , which is a tangent to the lens at the point of ingress f , is parallel to kc , which is a tangent to it at the point of egress g , then the points of contact f and g are parallel to one another, that is, the points of ingress and egress are parallel like the two sides of a plane glass; and for this reason the refraction which the ray suffers at f and g will not alter its direction more than it would have been altered by passing through a plane glass. Therefore gb the refracted ray will be parallel to

ef

ef the incident one, by proposition 47. In like manner *ef* falls upon the concave surface of the lens *acbd* at the point *f*, and comes out of the lens at the point *g*, then if *ab* a tangent to the concave surface at *f* the point of ingress is parallel to the plane surface *cd*, this ray passes through the lens as it would through a plane glass, because the point of ingress *f* is parallel to the side *cd* as the two sides of a plane glass are parallel to one another. Therefore *gb* the refracted ray will be parallel to *cf* the incident one, by proposition 47. Such a ray as this there will be in all oblique beams upon whatever lens they fall, and this ray is the axis of the beam. It was proper just to mention this matter; but as it will be of no great use in our future enquiries, we need not give ourselves the trouble to determine upon what point of the lens this ray will fall.

C H A P. V.

Of the refraction of diverging rays by a convex lens, and of converging rays by a concave lens.

60. *The principal focal distance of a convex or plano-convex lens is the distance at which rays, that were parallel at their incidence, are united after refraction. And the principal focal distance of a concave or plano-concave lens is the distance, from which rays, that were parallel at their incidence, appear to diverge after refraction.*

IF *PV*, Plat. X. fig. 4. is a diameter of the sphere of which the plano-convex lens *AVB* is a segment, then if parallel rays were to fall upon this lens, they would be so refracted by passing through it as to meet in a focus at a distance equal to *PV*, by proposition 51. And agreeably to this definition the distance *PV* taken on either side of the lens is called the principal focal distance. In like manner, if *pv*, Plat. X. fig. 5. is a diameter of the concavity of the lens *avb*, then if parallel rays were to fall upon this lens, they would be so refracted by passing through it as to diverge from a virtual or negative focus at the distance *pv*, by proposition 52. And agreeably to the definition, the distance *pv* taken on either side of the lens is called the principal focal distance. The principal focal distance of a lens, that has equal convexities on each side, is a semidiameter of either convexity: and the principal focal distance of the lens, that has equal concavities on each side, is a semidiameter of either concavity. By proposition 53, 54.

The only difference in the following propositions between a convex and concave lens is this. When in any case of a convex lens we have
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taken the principal focal distance on the side of the incident rays, in the similar case of a concave lens we must take the principal focal distance negative, that is, we must take it on the opposite side, or on the side of the refracted rays.

We shall demonstrate the following propositions upon plano-convex and plano-concave lenses only, but the reader will easily see that the demonstrations are universal and may be applied to double-convex and double-concave ones.

We have already seen, in the last chapter, what effect a convex or plano-convex lens will have upon parallel rays, or rays which come from a radiant at an infinite distance, as those may be said to do, which come from the sun. But suppose that the radiant was nearer and that the rays, instead of being parallel, diverge when they fall upon the lens; as rays would that come from a candle or any other luminous body at no very great distance: it still remains for us to enquire what effect the lens would have upon such rays as these. Now this enquiry may be divided into three different cases, according to the different position of the radiant in respect of the lens's principal focus. For first the radiant, the luminous point, or the candle, which is indeed rather many radiants together than one alone, may be in the lens's principal focus, that is at the distance PV , Plat. X. fig. 4. Or secondly it may be more remote than the principal focus, and be at the distance RV , which is greater than PV . Or thirdly, it may be nearer than the principal focus, and be Plat. X. fig. 6. at the distance RV , which is less than PV . Now parallel rays neither diverge nor converge: but rays in this state are in the middle between divergency and convergency. For if the divergency of rays diminishes, they become nearer parallel; and when they are quite parallel their divergency is nothing: consequently, if they converge, their divergency is less than nothing or negative. And since convergency is negative divergency, and a concave or plano-concave lens is a negative convex or plano-convex one; it follows that there will be just as many negative cases as there are affirmative ones: that is, there are three cases of converging rays falling upon a concave or plano-concave lens. First, the converging rays may at their incidence tend to the lens's principal focus; or the imaginary focus may be at the principal focal distance, thus Plat. X. fig. 3. when converging rays fall upon the lens ab the focus to which they are directed may be p , so that pv the distance of this imaginary focus may be equal to a diameter of the lens's concavity or to the principal focal distance. Secondly, the incident rays nb and fv may tend to r an imaginary focus at the distance vr , which is greater than vp , or
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than the principal focal distance. Or thirdly, the incident rays *ma* and *xv*, Plat. X. fig. 7. may tend to the imaginary focus *r* at the distance *vr*, which is less than *vp* or than the principal focal distance.

61. *If rays diverge from the principal focus of a convex or plano-convex lens; these rays, when they have passed through the lens, will be parallel.*

If *PF*, Plat. IX. fig. 2. is the principal focal distance of the lens *GL*, then rays, as *AR* and *DS*, which are parallel at their incidence will, when they are refracted, be united at *F*. Therefore, by proposition 41, since all refraction is reciprocal, if these rays were to be turned back again in the lines *FB* and *FP* so as to diverge from *F*, by passing through the lens they will be refracted back into their former directions *RA* and *SD* so as to be parallel. And the effect of the lens upon any other rays diverging from *F*, or from a radiant at the principal focal distance, would be the same, all such rays will become parallel, when the lens has refracted them.

So that if a candle was placed at *F*, or behind any other convex lens at the principal focal distance, the rays, which diverge from any one point of the candle, when they fall upon the lens, will be parallel to one another, after they have passed through it.

62. *If rays converge to the principal focus of a concave or plano-concave lens; these rays, when they have passed through the lens, will be parallel.*

If the rays *ar* and *ds*, Plat. IX. fig. 4. are parallel, when they fall upon the plano-concave lens *gl*, they will after they are refracted, go on in the lines *bm* and *pc*, so as to diverge from *f* the principal focus. Therefore, since all refraction is reciprocal, by proposition 41, if these rays were to be turned back again in the lines *mb*, *cp*, so as to converge to *f* the principal focus of the lens, they would be refracted into their former directions *ra* and *sd* and would become parallel. And whatever rays converge in this manner to the point *f*, or to the principal focus of this or any other lens, will suffer just such a refraction as will produce the same effect in them or will make them parallel to one another.

The chief use we shall make of this proposition will be best understood, when we come to apply it in explaining one sort of telescopes.

63. *If the radiant from whence rays diverge is farther from a convex or plano-convex lens than its principal focus, the rays, after they are refracted, will converge.*

If PV , Plat. X. fig. 4. is the principal focal distance of the lens AB , and any radiant point at R is at a greater distance than this from the lens; then rays, such as RV and RA , which diverge from thence will be made to converge by passing through the lens. If the radiant had been at P instead of R , that is, if the radiant had been in the principal focus, the refracted rays would have been parallel, or their divergency would have been nothing, by proposition 61. But when the radiant is more remote than P , the incident rays diverge so much the less, by proposition 28. And consequently the divergency of the refracted rays will likewise be so much the less. Now when the radiant is in the principal focus the divergency of the incident rays is nothing. Therefore if the radiant instead of being at P is at R or more remote than the principal focus, the divergency of the refracted rays will be less than nothing or negative, that is, the refracted rays will converge as VF and AF do, and will meet in a focus at F .

That, when the incident rays diverge, the focus after refraction will be more remote than when they are parallel, is evident. For when the radiant is at an infinite distance, or when the incident rays are parallel, the divergency before refraction is the least possible, and consequently the negative divergency or convergency after refraction will be the greatest possible; and for this reason the focus will be as near to the lens as it can be, by proposition 29. But, when the radiant is at any distance less than infinite, the nearer it is so much greater is the divergency of the incident rays, by proposition 28. Therefore so much less will be their negative divergency, or their convergency, after refraction; and consequently the focus will be so much the more remote, by proposition 29. From hence it appears that the rays of the sun will be collected into a focus nearer to the lens than the rays of a candle or any other luminous body placed at R .

64, *When the radiant is more remote from a convex or plano-convex lens than the principal focus, as the difference between the distance of the radiant and principal focal distance is to the principal focal distance, so is the distance of the radiant before refraction, to the distance of the focus after it.*

The distance of the radiant R , Plat. X. fig. 4. from the lens AR is RV ; the principal focal distance is PV ; the difference between the distance of the radiant and principal focal distance, or between RV and PV , is RP . Therefore the proposition affirms that as RP is to PV , so is RV to a fourth proportional, which will be the distance of the focus from the lens,

lens, after the rays are refracted: and, taking VF this fourth proportional, F will be the point where the refracted rays will concur. The truth of this will appear from the following considerations.

If the radiant was in the principal focus or at the distance PV, the refracted rays would be parallel; that is, their divergency would be nothing, by proposition 61. Therefore the refraction of the lens will just destroy as much divergency as rays have that proceed from a radiant in the principal focus or at the distance PV. But the radiant at R is more remote than the principal focus; its distance from the lens is RV, which is greater than PV. Now the more remote the radiant is, the less the incident rays diverge; for the divergency is inversely as the distance of the radiant, by proposition 28. Therefore as much greater as RV is than PV, so much less rays diverge, which proceed from a radiant at R, than rays, which proceed from a radiant at P: or, the divergency of rays falling upon the lens AB, when the radiant is at R, is to the divergency of rays falling upon the same lens, when the radiant is at P, as the distances RV and PV inverted, that is, as PV to RV. But the refraction of the lens would just destroy all the divergency of the rays, if the radiant was at P. Therefore the divergency, which the rays have that come from R, is to the divergency, which the refraction of the lens will destroy, as PV is to RV. Now PV is less than RV, by the supposition, and consequently, the rays that come from R, have not so much divergency as the lens will destroy. Therefore after refraction their divergency will be less than nothing, or will be negative. And this negative divergency will be proportional to the difference between the divergency, which the incident rays have, and the divergency, which the lens will destroy; or to the difference between PV and RV, that is, it will be proportional to RP. From hence then it follows, that the negative divergency of the rays after refraction is to their divergency before refraction as RP to PV. But as the divergency after refraction to the divergency before it, so is the distance of the radiant before refraction to its distance after refraction, for, by proposition 28, the distance of the radiant is always as the divergency inversely. That is, as RP to PV, so is RV the distance of the radiant before refraction to a fourth proportional, which will be the distance of the radiant after refraction. But then we must remember, that the divergency after refraction is negative, that is, that the rays converge; and consequently this fourth proportional will be negative, or will be on the contrary side of the lens to P, that is, it will be a negative radiant. But as we have frequently observed before that if a focus is considered as affirmative, a negative focus means a ra-

diant: so here, when we consider a radiant as affirmative, a negative radiant means a focus. Therefore this fourth proportional, or VF , is the distance of the focus. Upon the whole therefore, as RP is to PV , so is RV the distance of the radiant before refraction to VF the distance of the focus after refraction.

Let PV , the principal focal distance of the lens AB , be 4 inches. Then the rays of the sun, or any other parallel rays, that were to pass through the lens, would be collected into a focus at the distance of 4 inches from the lens. But suppose there was a candle at R , and that RV , the distance of it from the lens, is 6 inches. The radiant is then very near to the lens, so that the rays, which come from it and fall upon the lens, are not parallel but diverging. Therefore when they have passed through the lens, the refraction, which they suffer, will not make them converge so much, as it would have done, if they had been parallel before, and consequently the focus will be more remote, by prop. 29. Now the focus of the sun's rays would be at the distance of 4 inches; and how much the focus of the rays coming from the candle will be farther off is found by this proposition. For the distance of the principal focus, or $PV=4$ inches, the distance of the radiant or $RV=6$ inches. The difference of these distances $RV-PV=RP=6-4=2$ inches. Therefore, as RP to PV , or as 2 to 4, so is RV to VF , or 6 to 12, which is the focal distance. That is, the rays diverging from the candle at R will by the lens be collected into a focus at F , which is 12 inches from the lens.

65. *If rays, when they fall upon a concave or plano-concave lens, converge to an imaginary focus, which is at a greater distance from the lens than its principal focus; these rays by passing through the lens will be made to diverge.*

If the rays fv and na converge to the point r , then r is their focus. But if, before they arrive at that focus, they are to pass through the plano-concave lens ab , they will be refracted, and so being turned out of the way will never meet at r ; for which reason the focus in these circumstances is called imaginary: the rays are directed towards it, but never meet at it. If this imaginary focus was at p , the principal focus of the lens, or at the distance pv , then the rays, when they have passed through the lens, would be parallel, by prop. 62. But if it is at the distance rv , which is greater than pv , or than the principal focal distance; then, since the imaginary focus is more remote, the convergency of the incident rays is less, by prop. 29. Now if the imaginary focus

focus had been at p , the refraction of the lens would have made the rays parallel after they had passed through it, or would have made their convergency nothing: therefore if the imaginary focus is more remote than p , that is, if the incident rays converge less, the same refraction will make their convergency, when they have passed through the lens, less than nothing, or negative; that is, the refracted rays, instead of converging, as the incident ones did, will have the contrary direction, or will diverge, and will proceed in the lines vr and am , as if they had come straight from the point f , which point is the imaginary radiant, or negative focus, after the rays are refracted.

66. *When the imaginary focus is more remote from a concave lens than its principal focus, as the difference between the distance of the imaginary focus and the principal focal distance to the principal focal distance, so is the distance of the imaginary focus, before refraction, to the distance of the imaginary radiant after it.*

The distance of the imaginary focus r , Plat. X. fig. 5, from the lens ab is rv , the principal focal distance is pv , the difference between rv and pv , or $rv - pv$, is rp . Therefore as rp is to pv , so is rv to a fourth proportional, which will be the distance of the imaginary radiant after refraction. Thus if vf is this fourth proportional, the rays fv and na , which converge to r before refraction, will become vr and am , so as to diverge from f after refraction.

This proposition might be proved distinctly in the same manner as proposition 64. But instead of repeating that demonstration, we may better shew, that every thing in this proposition is the same with every thing in that, only taken on the contrary side; that is, this proposition is only proposition 64. in negative terms, or fig. 5. is the negative in all respects of fig. 4. For first, in fig. 4. R is a radiant, from which the incident rays diverge, and fig. 5. r is a focus on the contrary side of the lens, to which the incident rays converge. Secondly, AB is a plano-convex lens, and ab is a plano-concave one. Thirdly, the principal focal distance PV is taken on the same side with the incident rays, and the principal focal distance pv is taken on the side of the refracted rays. Fourthly, the refracted rays AF and VF converge, and the refracted rays am and vr , or, which is the same thing, fa , and fv , diverge. Therefore fifthly, the focus F is on the side of the refracted rays, and the negative focus, or imaginary radiant f , is on the side of the incident ones. Since therefore, whether diverging rays fall upon a plano-convex lens, or converging ones upon a plano-concave lens, every thing is the same

same, only taken on the contrary sides, provided the radiant in one case, and the imaginary focus in the other, are more remote than the principal focus; the proportions will be the same in both cases; that is, since RP is to PV , as RV to FV ; rp is likewise to pv , as rv to fv .

67. *If the radiant, from whence rays diverge, is nearer to a convex, or plano-convex lens, than its principal focus; the rays, after they have passed through the lens, will continue to diverge, but less than they did before.*

If AB , Plat. X. fig. 6, is the lens, PV its principal focal distance and R the place of the radiant, so that RV less than PV is its distance from the lens; any incident rays as RA and RV , which diverge from R before refraction, will diverge less after they have been refracted by passing through the lens, and will go on in the lines VX and AM , as if they had come at first in the lines FA and FV straight forwards from an imaginary radiant F .

Here let us look back to fig. 4. When R is more remote from the lens than P , the refracted rays converge. But if the radiant R was to approach the lens, then, by prop. 28, the divergency of the incident rays would encrease, and consequently their negative divergency, that is, their convergency would be so much less after they were refracted. Till at last, when the radiant R by approaching the lens was got to P , or was at the principal focal distance, the divergency of the incident rays would be so great that the refraction of the lens would only just destroy it all and no more; and the negative divergency, or the convergency of the rays after refraction would be nothing, for the refracted rays would be parallel, by prop. 61. Therefore, if the radiant was still to approach the lens and to get nearer to it than the principal focus, as at R , fig. 6. the divergency of the incident rays would by this means be farther encreased, by prop. 28, and would be so great, that the refraction of the lens would not destroy it all. But as this refraction would in any case destroy just as much divergency as the rays would have if the radiant was at P ; it follows, that when the radiant is at R , nearer than P , a part of their divergency will then be destroyed, and consequently the rays, though they diverge after refraction, will however diverge less than they did at their incidence.

We may observe farther, that when the radiant is at R , fig. 4, more remote than the principal focus and the refracted rays converge, the focus is at F , on the same side of the lens with the refracted rays. When the radiant is at P , the refracted rays are parallel, and therefore,
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by prop. 31, the distance of the focus is infinite. Since therefore as the radiant approaches to the lens AB, the focus departs from it, till the distance of the focus becomes infinite, when the radiant is at P; if the radiant was still to approach nearer than P, as suppose it to be at R, fig. 6, then the distance of the focus still encreasing will be greater than infinite, that is, by what was said under proposition 49, the focus will then be negative, or will change sides, and instead of being on the side of the refracted rays will be on the side of the incident ones at F. But such a negative focus is an imaginary radiant. Therefore the rays after refraction, will appear to diverge from the imaginary radiant F.

68. *When the radiant is nearer to a convex, or plano-convex lens, than its principal focus, as the difference between the distances of the radiant and principal focus to the distance of the principal focus, so is the distance of the real radiant before refraction to the distance of the imaginary one afterwards.*

RV, Plat. X. fig. 6. is the distance of the point from whence the rays come, or of the real radiant R, from the lens AB; PV is the distance of the principal focus; $PV - RV = RP$ is the difference between them. Therefore the proposition affirms, that as RP is to PV, so is RV to a fourth proportional, which will be the distance of the point from whence the refracted rays will appear to come, or of the imaginary radiant. And if FV is this fourth proportional, then the rays, which proceed from the point R before refraction, will go on in the lines AM and VX after refraction, as if they came from the point F.

The demonstration of this proposition is the same with that of proposition 64. The letters of reference are the same in both; so that if the reader wants a distinct proof of this proposition, he need only read that demonstration and apply it to fig. 6. instead of fig. 4. The only difference is, that in this proposition F is a radiant; whereas in proposition 64, having first considered F as a radiant, we afterwards shewed that it would be a negative radiant, or that, as it was on the contrary side of the lens to the radiant R, it would be a focus.

Or otherwise; in proposition 64. the radiant is beyond the principal focus P, and in the proposition now before us, it is nearer to the lens than P: so that PR in these two propositions is on the contrary sides of the principal focus; and PR by changing sides, makes VF change sides, as has been seen already, in proposition 67. And as this is the only difference between the two cases, the proportions will be the same in both, PR to PV, as RV to FV.

69. *If*

69. *If the imaginary focus, to which rays converge, is nearer to a concave or plano-concave lens than its principal focus; the rays, after they have passed through the lens, will continue to converge, but less than they did before.*

Let ma , Plat. X. fig. 7, and xv be rays converging to r , so that if there was no lens in the way they would meet at r : then r is the focus of these rays. But if they are to pass through the plano-concave lens ab before they come to r , then the refraction, which they suffer, will turn them out of the way, and will prevent them from ever coming to r at all. Therefore r is then only an imaginary focus; the rays, though they are directed thither, never meet there. Now if rv , the distance of this imaginary focus from the lens is less than pv , which is the principal focal distance, the rays, after they are refracted, will continue to converge in lines af , vf , but less than they did at their incidence: so that af and vf , the refracted rays, will converge less than mar , xvr , which are the directions of the incident rays. For if the rays, when they fell upon the lens, had converged to p , instead of r , that is, if the imaginary focus had been at the principal focal distance, the refracted rays would have been parallel, or the refraction of the lens would have just destroyed all the convergency which the rays had at their incidence. But rv , the distance of the real focus, is less, by the supposition, than pv the principal focal distance. Therefore the rays, as they converge to a nearer focus than p , converge more than if they had converged to p , by prop. 29. and consequently they have a greater convergency than the refraction of the lens can destroy; so that they will continue to converge after refraction: but their convergency will be diminished, for it will be proportional to the difference between what the incident rays had and what the refraction of the lens destroys.

Now since the refracted rays af and vf converge less than the incident ones ma and xv , the focus f , where the refracted rays meet, or the real focus, will be at a greater distance from the lens than the focus r , to which the incident rays tended, or than the imaginary focus, by proposition 29.

70. *When the imaginary focus is nearer to a concave, or plano-concave lens, than its principal focus, as the difference between the distances of the imaginary focus and principal focus to the distance of the principal focus, so is the distance of the imaginary focus to the distance of the real focus..*

The distance of the imaginary focus is rv , Plat. X. fig. 7. the distance of the principal focus is pv ; the difference between them is pr

$pv - rv = rp$. So that, if rv and pv are known, vf , the distance of the real focus from the lens, may be thus found: as rp is to pv , so is rv to a fourth proportional, which is the distance vf . This proposition might be deduced from prop. 66. in the same manner that prop. 68. is deduced from prop. 64. But a readier way of proving it is to shew that it is in all respects the same with prop. 68, which has been proved already, except only, that every thing is here taken on the contrary side, that is, this proposition is only proposition 68. in negative terms; or fig. 7. is the negative of fig. 6. For first, R in fig. 6, is a radiant from whence the incident rays diverge, and r fig. 7, is a focus on the contrary side of the lens, to which the incident rays converge. Secondly AB is a plano-convex lens, and ab is a plano-concave one. Thirdly, the principal focal distance PV is taken on the same side of the lens with the incident rays, and the principal focal distance pv is taken on the same side with the refracted ones. Fourthly, the refracted rays AM and VX diverge, and the refracted rays af and vf converge. Therefore fifthly, the imaginary radiant F is on the side of the incident rays, as the real focus f is on the side of the refracted ones. Since therefore, whether diverging rays fall upon a plano-convex lens, or converging ones upon a plano-concave lens, every thing is the same, only taken on the contrary sides, provided the radiant in one case, and the imaginary focus in the other, is nearer to the lens than its principal focus, the proportions will be the same in both cases; that is, since RP is to PV , as RV to FV ; rp is likewise to pv , as rv to fv .

Before we leave this subject, we may deduce three consequences from some of the foregoing propositions, which will be very useful to us in some of our future enquiries. These consequences are the subject of the propositions, which follow in this chapter.

71. *When a radiant is farther from a convex or plano-convex lens, than its principal focus, as the radiant approaches to the lens on one side, the focus departs from it on the other side, and on the contrary, as the radiant departs the focus approaches.*

Whilst the radiant R , Plat. X. fig. 4, is at a greater distance from the lens AB than PV , which is the principal focal distance of the lens, as R approaches to the lens on one side, F , which is the focus, will depart farther from it on the other side.

Provided R is at a greater distance from the lens than P its principal focus, the refracted rays will converge, by prop. 63. The nearer R is the more the incident rays diverge, by prop. 28. Now the conver-

gency, which the refracted rays have, is proportional to the difference between the divergency, which the incident rays have, and that which the lens will destroy, by what was said in proving prop. 64. And the divergency, which the lens destroys is a given quantity, by prop. 63, for it is always just as much as rays would have that came from the lens's principal focus. From hence then it follows, that the more the incident rays diverge the difference between the divergency which they have, and that which the lens will destroy, is so much the less. Therefore the more the incident rays diverge, or the nearer the radiant is the less the refracted rays will converge; and consequently the more remote will be the focus, by prop. 29. So that the nearer the radiant is to the lens, the more remote the focus will be, or as the radiant approaches, the focus will depart. In like manner, the more remote the radiant is, the less the incident rays will diverge, and consequently the refracted rays will converge the more, for which reason the focus will be the nearer, by prop. 29, or as the radiant departs, the focus will approach.

Or otherwise; as RP to PV , so is RV to FV . by prop. 64. But as the radiant R approaches the lens RP decreases, and consequently RP will bear a less proportion to PV . But if the first term bears a less proportion to the second, the third, or RV will bear a less proportion to the fourth or to FV . Now when RP decreases it is evident that RV decreases; that is, when the first term decreases the third will likewise decrease at the same time. And if these two quantities RP and RV , as they decrease together were likewise to decrease in the same proportion, then because PV the third term is a given quantity FV the fourth term would be a given quantity too, and therefore however the radiant was nearer to the lens, or farther from it, RP might always be to PV as RV to FV , without any alteration being made in FV , or in the focal distance. But RP and RV , though they decrease together, cannot decrease in the same proportion. For RP is a less quantity than RV , and it is plain, that when R approaches to the lens, RP and RV will decrease equally, that is, just as much as RP is diminished, just so much RV will be diminished: therefore these two quantities are equally diminished indeed, but not in the same proportion: for if from two quantities, one a less and the other a greater, you take equal quantities, you will diminish the less more in proportion than you do the greater. So that RP decreases faster than RV . Consequently the proportion of RP to PV would decrease faster than that of RV to FV , if as PV always continues of the same length, so FV was likewise to continue al-

always of the same length. But the proportion of RP to PV is never less than that of RV to FV, by prop. 64. Therefore as R approaches to the lens, to preserve the proportion between RV and PV the same with that between RP and RV, not only RV must decrease but FV must likewise encrease; or as the radiant approaches the focus will depart. That RP decreases faster than RV, will appear farther by considering what would be the case, if R the radiant was at P the principal focus. In this approach of the radiant to the lens, R and P coincide; therefore RP decreases till it becomes nothing. But RV will not be decreased to nothing: for when R and P coincide RV is then equal to PV. But if, when these two quantities RP and RV are both of them decreasing, if RP becomes nothing before RV does, then RP must decrease faster in proportion than RV.

It may here not be amiss to take notice how far the focus will have departed from the lens, when the radiant by approaching towards it has gotten to the principal focal distance. When R and P coincide $RP=0$. Therefore RP bears no proportion to PV: and consequently RV will bear no proportion to FV. But RV is then equal to PV. Therefore when the radiant is at the principal focal distance, the distance of the focus to which the rays converge, or FV, is so great that PV bears no proportion to it; and consequently FV must be infinite. That is when the radiant is in the principal focus of the lens, the refracted rays converge to a point at an infinite distance, and consequently are parallel, by proposition 31. And so we found they ought to be, by proposition 61.

If R departs from the lens, then RP encreases and so likewise does RV. But RP and RV encrease equally, therefore they cannot encrease in the same proportion: because RP is less than RV, and if equal quantities are added to RP and RV, this equal addition to both will make a greater encrease in proportion in RP the lesser of these two quantities. Therefore RP encreases faster in proportion than RV. Consequently, since RP is always to PV, as RV to FV, if the first quantity encreases faster than the third, to keep the proportion between RV and FV, the same with that between RP and PV, besides the encrease of RV it will be necessary that FV should decrease; or as the radiant R departs from the lens so as to make RV encrease, the focus F will approach to it, so as to make FV decrease. That RP encreases faster than RV will be evident from the following consideration. RP is less than RV, when they begin to encrease; but when R is at an infinite distance from V or from the lens it will likewise be at an infinite distance from P: for if RV is infinite RP must be infinite too; because if from RV an infinite quantity,

tity you take the finite quantity PV , the remainder will still be infinite; because the part taken away is too small in respect of the whole to bear any proportion to it, or to make any alteration in it by being taken away. But when RP and RV are both of them infinite they are equal to each other, and consequently RP must have encreased faster in proportion than RV ; for, as they were unequal at first setting out, they could not have been equal at last, unless the smaller of the two quantities or RP had encreased faster in proportion than the larger or than RV .

Here again we may observe when the distance of R , or when RV , is infinite, what will be the distance of F the focus. As RP to PV , so is RV to FV . Now when R is at an infinite distance, RP and RV are both of them infinite, and consequently are equal to one another, so that the first term is then equal to the third: therefore the second term or PV will be equal to the fourth or FV . Now PV is the principal focal distance of the lens, and FV is the distance of the point to which the refracted rays converge. Therefore when the radiant is at an infinite distance, that is, by proposition 30, when the incident rays are parallel, the refracted rays will converge to the principal focal distance, as we have already seen they will do in the definition of the principal focal distance, proposition 60.

72. *When a radiant is nearer to a convex or plano-convex lens than its principal focus, as the real radiant approaches to the lens the imaginary radiant will approach to it, and as the real radiant departs the imaginary one will depart.*

If the radiant R , Plat. X. fig. 6. is nearer to the lens AB than P its principal focus; then, by proposition 67. the refracted rays diverge, but less than the incident ones did; so that, if RA and RV are the incident rays, the refracted rays will be AM and VX . Now since RA and RV diverge more than AM and VX , the real radiant R , from whence the incident rays diverge, will be nearer to the lens than the imaginary one F , from whence the refracted rays appear to diverge, by proposition 28. And as RP is to PV , so is RV to FV , by proposition 68. If in this case the real radiant R moves towards the lens the imaginary one F will likewise move towards it; and if the real radiant moves farther from the lens, provided it does not move to a greater distance than P or the principal focus, the imaginary one will likewise move from it.

The nearer the radiant R is to the lens, so much more the rays, which fall upon the lens, diverge from one another, by proposition 28. But whatever their divergency is, whether it be great or small, the refraction
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of the lens will always destroy just the same part of it; for, by proposition 61, it destroys just as much divergency as rays would have that come from P the principal focus. From hence then it follows that the greater the divergency is before refraction so much the greater it will be afterwards; because the greater the whole is, the greater likewise will be the remainder when a given part is taken from it. Therefore the more the incident rays diverge, or the nearer the real radiant is, the more the refracted rays will diverge, and consequently the imaginary focus will be the nearer. So that when the distance of RV decreases, or when the real radiant R approaches to the lens, the distance FV will decrease, or the imaginary radiant will likewise approach it. If R departs from the lens the incident rays diverge the less, by proposition 28, and consequently the refracted rays will diverge the less, by what has been already shewn; therefore the imaginary radiant will be the more remote from the lens; or will depart from it, when the real radiant departs.

Or otherwise; RP is to PV as RV to FV, by proposition 68. Now as R approaches to V, RP will bear a greater proportion to PV; because PV being the principal focal distance of the lens, is therefore, in all the motions of R, an invariable quantity. But when the proportion of RP to PV encreases, the proportion of RV to FV must likewise encrease. Because at all times whatever proportion RP bears to PV, RV bears the same proportion to FV. Consequently when R approaches the lens, the proportion of RV to FV must encrease. But when R approaches the lens, RV or its distance from the lens decreases: from whence it follows that the proportion of RV to FV cannot encrease, unless when RV decreases FV was to decrease. Therefore, since this proportion does encrease, whilst RV decreases FV will decrease likewise: or when the real radiant R approaches the lens, the imaginary one F will likewise approach it. In like manner, when R moves from the glass PR decreases; therefore in this motion of the radiant RP will bear a less proportion to PV; and consequently RV, though it encreases, will bear a less proportion to FV. But this would be impossible unless FV was to encrease. Therefore when RV encreases or when the real radiant moves from the lens, FV will likewise encrease or the imaginary radiant will move from the lens at the same time.

73. *If the real radiant is close to a convex or plano-convex lens, the imaginary radiant will likewise be close to it.*

When R, or the point from whence the incident rays diverge, Plat. X. fig. 6 touches V the vertex of the lens, then F, or the point from whence the refracted rays appear to diverge will likewise touch the lens at V.

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The radiant R, when it is at V will be between the lens and the principal focus. Therefore, by proposition 68, as RP to PV, so is RV to FV. But when R is close to V, the distance of it from P is equal to the distance of V from P, or RP is equal to PV. Therefore RV will likewise be equal to FV. Now when R touches the lens at V the distance of R from V or RV is nothing, and consequently, the distance of F from V or FV will be nothing, that is, F will likewise touch the lens at V.

CHAP. VI.

Of the refraction of converging rays by a convex lens and of diverging rays by a concave lens.

WE have in the last chapter considered what effect a convex or plano-convex lens will have upon such rays as diverge at their incidence; and what effect a concave or plano-concave lens will have upon such as converge. It still remains for us to consider what effect these lenses will produce in the opposite cases, that is, what will be the effect of a convex or plano-convex lens, when the incident rays converge, and what the effect of a concave or plano-concave one, when they diverge.

74. *If rays converge, when they fall upon a convex or plano-convex lens, they will converge more, after they have passed through it and have been refracted.*

If the rays MA and XV, Plat. X. fig. 6. converge to the point F, but before they arrive there, are to pass through a plano-convex lens AB, these rays will not meet at F after they are refracted, but will converge more than they did at their incidence and will meet at R. For, by propositions 67, 68, if the rays RA and RV were to diverge from R, they would be refracted into the lines AM and VX by passing through the lens, and would diverge less than they did at their incidence, just as if they had come straight forwards from the point F. Therefore if these rays AM and VX are turned back again in the lines MA and XV so as to converge to F, they would be refracted into their first directions AR and VR, so as to be made to converge more and to meet at the point R, by proposition 41. In like manner, any other rays, which diverged before refraction, and diverge likewise after it, will diverge less, by proposition 67. And all converging rays may be considered as such rays turned back again; and, since all such rays, when turned back again, will be refracted

ted into the lines, which they described at their incidence, by proposition 41, therefore all converging rays, when they have passed through the lens, will be made to converge more.

75. *When converging rays pass through a convex or plano-convex lens; as the sum of the distances of the principal focus and imaginary focus is to the distance of the imaginary focus, so is the distance of the principal focus to the distance of the real focus.*

If MA and XV are the incident rays, they converge to F; and, because the refraction, which they suffer in passing through the lens, will prevent them from meeting there, F is only an imaginary focus; the refracted rays will meet at R, and consequently R is the real focus. Now if FV the distance of the imaginary focus, and PV the principal focal distance are known, RV the distance of the real focus may be thus found, as $PV + FV$ is to FV, or as the distance of the principal focus added to the distance of the imaginary focus is to the distance of the imaginary focus, so is PV to RV, or so is the distance of the principal focus to the distance of the real focus. That R will be the real focus, if F is the imaginary one, appears from propositions 68, 41. For, if the incident rays RA and RV had diverged from R, the refracted rays AM and VX would diverge from F, by proposition 68. And if the rays AM and VX are turned back again in the lines MA and XV so as to converge to F, they will be refracted into the lines of their former incidence AR and VR, because all refraction is reciprocal, by proposition 41. Now, by proposition 68, RP is to PV, as is RV to FV. Therefore PV one consequent is to FV the other consequent, as RP one antecedent is to RV the other antecedent. Euc. b. V. prop. 16. And since PV is to FV, as RP to RV, $PV + FV$ is to FV, as $RP + RV$ is to RV. Euc. b. V. prop. 17. But $RP + RV$ is equal to PV. Therefore $PV + FV$ is to FV, as PV is to RV.

76. *If rays diverge, when they fall upon a concave or plano-concave lens, they will diverge more, after they have passed through it and have been refracted.*

If there is any radiant at f , Plat. X. fig. 7. from which the rays fa and fv diverge; when they have been refracted by passing through the plano-concave lens ab , they will go on in the lines am and vx so as to diverge more, and will have the same direction in respect of each other, as if they had come straight forwards from a radiant at r . If the rays ma and xv , which converge to the focus r before refraction, were to pass through the lens, they would converge less after refraction and would
meet

meet at the focus f , by propositions 69, 70. And if these rays were to be returned back from f , in the lines fa and fv so as to diverge from f , they would be refracted back into their former directions am and vx , by proposition 41. In like manner, all rays, which converge at their incidence, and converge likewise after they are refracted, will converge less after refraction than they did before it: and all rays, which diverge at their incidence, may be considered as such rays turned back again, and such rays, when turned back again, will be refracted into their first directions, and so be made to diverge more. Therefore all rays, which diverge at their incidence, will likewise be made to diverge more, when they have passed through the lens ab .

Since the refracted rays am and vx diverge more than the incident ones fa and fv , the imaginary radiant r will be nearer to the lens than the real radiant f , for the distance of the radiant is inversely as the divergency of rays, by proposition 28.

77. *When diverging rays pass through a concave or plano-concave lens; as the sum of the distances of the principal focus and real radiant is to the distance of the real radiant, so is the distance of the principal focus to the distance of the imaginary radiant.*

If f , Plat. X. fig. 7. is the radiant from whence the rays come, or the real radiant, then its distance from the lens is fv : the principal focal distance of the lens is pv . When these two are known the distance of r the imaginary radiant, or point from whence the refracted rays will appear to come, that is, the distance rv may be thus found. As $pv + fv$ is to fv , or as the principal focal distance added to the distance of the real radiant, is to the distance of the real radiant, so is pv to rv , or so is the principal focal distance to the distance of the imaginary radiant. That r will be the imaginary radiant, if f is the real one, follows from propositions 70, 41. For if the incident rays ma and xv had converged to r the refracted rays af and vf would have converged to f . Therefore, if the refracted rays af and vf were to be turned back again, and so were to diverge from f in the lines fa and fv , they would be refracted back into the lines of their first direction am and vx , by proposition 41. Now by proposition 70, rp is to pv as rv to fv . Therefore pv is to fv as rp to rv . Euc. b.V. prop. 16. And $pv + fv$ is to fv as $rp + rv$ is to rv . Euc. b.V. prop. 17. But $rp + rv$ is equal to pv . Therefore $pv + fv$ is to fv as pv is to rv .

This proposition might have been proved otherwise, for it might be deduced from proposition 75: because in effect it is the same as proposition 75, only every thing is taken on the contrary side; that is, this
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proposition is only proposition 75 in negative terms, or fig. 7. is the negative in all respects of fig. 6. For first, F , in fig. 6, is a focus to which the incident rays converge, and f , fig. 7. is a radiant on the contrary side of the lens, from which the incident rays diverge. Secondly AB is a plano-convex lens, and ab is a plano-concave one. Thirdly, the principal focal distance PV is taken on the side of the refracted rays, and the principal focal distance pv is taken on the side of the incident rays. Fourthly, the refracted rays AR and VR converge, and the refracted rays am and vx diverge. Therefore fifthly, the real focus R is on the side of the refracted rays, and the imaginary radiant r is on the side of the incident ones. Since therefore whether converging rays fall upon a plano-convex lens, or diverging rays upon a plano-concave lens, every thing is the same only taken on the contrary sides, the proportions will be the same in both cases; and, because $PV + FV$ is to FV as PV is to RV , $pv + fv$ is to fv , as pv to rv . Or, since the proposition is true, when its terms are affirmative, it will likewise be true, when they are negative.

There are two consequences from what has been said in relation to concave and plano-concave lenses, which will be of use to us hereafter; and therefore we will take notice of them distinctly in the two following propositions.

78. *When diverging rays fall upon a concave or plano-concave lens, if the real radiant moves either towards the lens or from it the imaginary radiant will move the same way.*

If f , which is the real radiant, *Plat. X. fig. 7.* approaches to the lens ab , r , which is the imaginary radiant will likewise approach to it; or if f departs from the lens, r will depart from it. For the nearer f is to the lens so much more the incident rays diverge, by proposition 28. The more the incident rays diverge the more the refracted rays will diverge. And the more the refracted rays diverge, the less will be the distance of r , or of the radiant from which they appear to come, by proposition 28. Therefore as f approaches the lens, r will move the same way. In like manner the farther f is from the lens the less the incident rays will diverge, and consequently the refracted rays will likewise diverge the less, therefore r , the imaginary radiant from which they appear to diverge will be the more remote, by proposition 28. Therefore as f departs from the lens, r will move the same way.

Or otherwise $pv + fv$ is to fv , as pv is to rv , by proposition 77. Now when f approaches to the lens fv decreases, and consequently $pv + fv$ must decrease at the same time. But since pv or the principal focal di-

stance is always the same, both $pv + fv$ and fv will decrease only just as fv decreases. That is in the approach of f to the lens $pv + fv$ and fv will decrease equally. But if from two unequal quantities such as $pv + fv$ and fv , equal parts are taken, the lesser quantity or fv will decrease faster in proportion to the whole than the greater quantity or $pv + fv$. Now if in the approach of f to the lens fv decreases faster in proportion than $pv + fv$, it follows that fv will bear a less proportion to $pv + fv$, or that the proportion of fv to $pv + fv$ will decrease. But since it is always as $pv + fv$ to fv so pv to rv , when the proportion of the second term to the first decreases, the proportion of the fourth term to the third, or of rv to pv must decrease likewise. Therefore in the approach of f to the lens, rv will bear a less proportion to pv . But pv is an invariable quantity, and consequently rv cannot bear a less proportion to it unless rv decreases. Therefore in the approach of f to the lens rv or the distance of r from the lens will decrease, that is r will likewise approach the lens. In like manner, when f departs from the lens, or when fv encreases, $pv + fv$ will encrease at the same time. But then because pv is a given quantity, $pv + fv$ and fv will encrease equally: and since fv is a less quantity than $pv + fv$, where an equal addition is made to both of them, this equal addition will make fv or the lesser quantity encrease more in proportion to the whole, than $pv + fv$ or the greater quantity. But if fv encreases faster than $pv + fv$, such an encrease will make fv bear a greater proportion to $pv + fv$; that is, in the departure of f from the lens the proportion of fv to $pv + fv$ encreases. But since as $pv + fv$ to fv , so is pv to rv , if the proportion of the second term to the first, or of fv to $pv + fv$ encreases, the proportion of the fourth to the third or of rv to pv must encrease likewise. Now pv or the third term is the principal focal distance of the lens, and consequently is an invariable quantity: so that the proportion which rv bears to it cannot encrease unless rv encreases. Therefore in the departure of f or the real radiant from the lens, rv or the distance of the imaginary radiant from the lens will encrease, that is r will likewise depart from the lens at the same time.

79. *If the real radiant is close to a concave or plano-concave lens, the imaginary radiant will likewise be close to it.*

If f , Plat. X. fig. 7, the real radiant, or the point, from which the incident rays come, is close to the plano-concave lens so as to touch it at v , then r the imaginary radiant, or point, from which the refracted rays appear to come, will likewise touch it at the same point. Or, when fv the distance of the real radiant from the lens is nothing, then rv the distance
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of the imaginary radiant will be nothing. For as $pv + fv$ to fv so is pv to rv , by proposition 77. Now when f touches the lens, fv is nothing: therefore $pv + fv$ is to fv , as $pv + 0$ is to 0 , and consequently pv will be to rv as pv to 0 , or when the second term in this proportion or fv is nothing, the fourth term or rv will likewise be nothing, that is when the real radiant f touches the lens, the imaginary radiant r will touch it too.

C H A P. VII.

Of the principal focus of meniscuses, and entire spheres of either glass or water.

80. *In a meniscus, as the difference between the semidiameters of the two surfaces, to the semidiameter of either, so is double the semidiameter of the other to the principal focal distance.*

A Meniscus has one of its sides convex and the other concave. So that if the plane sides of the plano-convex lens AB, Plat. X. fig. 8. and of the plano-concave one CD were close together they would make the meniscus EF or GH; for EF and GH differ in shape from one another only upon account of the different thickness of the glass. From hence it is evident that the properties of a meniscus are the same either with those of a plane glass, or with those of a convex lens, or with those of a concave one. For first; if the convexity on one side is exactly equal to the concavity on the other, then these two sides will produce in the refracted rays, contrary effects and in an equal degree; so that the direction of the rays, when they have passed through the meniscus, will be the same that it was, before they entered it. And this, as was shewn in proposition 47, is the property of a plane glass. Or, when the convexity on one side of the meniscus is equal to the concavity on the other, the two surfaces of the meniscus are parallel to each other, as the two surfaces of a plane glass are, so that the meniscus is nothing but a plane glass that is bent. Of this sort are watch-glasses and the side of a glass decanter, which in refracting the rays of light produce no more alteration in the direction of them, than any other plane glass would have done. Secondly, if the convexity is greater than the concavity, then likewise the two surfaces of the meniscus produce contrary effects, but not in an equal degree; for the convex surface refracts the rays more than the concave one; and therefore the meniscus will have all the properties of a convex lens. And thirdly, if the concavity is greater than the convexity, then the refraction will be greater at the concave surface than at the convex one, and

consequently the meniscus will have the properties of a concave lens. From hence it follows that when the convexity is greater than the concavity, if the incident rays were parallel, the refracted ones will converge, by proposition 53, and the focus will be a real or affirmative one. But if the concavity is greater than the convexity, then the refracted rays will diverge, by proposition 54, and the focus will be a virtual or negative one. The distance of this focus whether affirmative or negative is the principal focal distance of the meniscus; and the rule for finding it is this, which is laid down in the proposition; as the difference between the semidiameters of the convexity and concavity is to the semidiameter of either, so is double the semidiameter of the other to the principal focal distance.

This proposition does not want to be demonstrated, because it is the same in effect with proposition 55. Here, in the meniscus, the first term in the proportion is the difference between the semidiameters of the two surfaces; and in the convex lens it was the sum of them. But the reader will easily understand that this makes no material alteration in the proposition, if he recollects that both surfaces of a convex lens are of the same sort, and that the semidiameters of these surfaces are both of them affirmative; whereas the two surfaces of a meniscus are not of the same sort, for one of them is convex and the other concave, and consequently, if the semidiameter of the convexity is considered as affirmative, the semidiameter of the concavity must be considered as negative. Thus in a convex lens if the semidiameter of one convexity is called r , and that of the other s , then both r and s are affirmative and one of them added to the other makes $r+s$, which is the sum of these two semidiameters. But in a meniscus, if the semidiameter of the convexity is called r , and the semidiameter of the concavity is called s , then r is affirmative and s is negative, that is, s must have the negative sign before it, and the notation of it or manner of expressing it is by this mark $-s$. In this case therefore, when r and $-s$ are added together they make $r-s$, which is the difference between r and s . So that the first term, both in proposition 55, and in this, is the same in effect; for it is the two semidiameters added together: but the two semidiameters added together is the sum of them, when they are both affirmative, and is the difference between them, when only one of them is affirmative, and the other negative. Upon the whole therefore this proposition may be considered as one case of proposition 55, or a meniscus may be considered as a sort of convex lens having the convexity affirmative at one surface and negative at the other.

Now when the incident rays are parallel, the refracted rays will converge and the focus will be affirmative, if the convexity of the meniscus is

is greater than its concavity; but they will diverge and the focus will be negative, if the concavity is greater than the convexity. If the semidiameter of the convexity is called r , as before, and the semidiameter of the concavity is called $-s$; then the more convex one side of the meniscus is, so much less is the sphere of which that side is a segment; for any given part in a sphere's surface is so much the more convex as the sphere is less: and consequently the more convex one side is, so much the shorter will be the semidiameter of the sphere of which that side is a segment, or the quantity r will be so much the less. And for the same reason the less convex this side of the meniscus is, the greater will be the semidiameter of the convexity or the quantity r . In like manner the more concave one side of the meniscus is, the less is the semidiameter of the concavity or $-s$, and the less the concavity the greater is $-s$. Now when r is greater than $-s$, $r-s$ will be affirmative. Thus, for instance, if r or the semidiameter of the convexity is 10 inches and $-s$, or the semidiameter of the concavity is 6 inches. $r-s=10-6=4$. In this case $r-s$ or 4 is an affirmative quantity: for in all subtraction if a less quantity is taken from a greater the remainder is affirmative. But when the semidiameter of the convexity is greater than that of the concavity, or when $r-s$ is affirmative, the convex surface is a segment of a greater sphere than the concave one, and consequently the convexity on one side of the meniscus will be less than the concavity on the other. And when the convexity is less than the concavity, the meniscus will have the effect of a concave lens, as appears from what has been already proved. Therefore when $r-s$ is affirmative, the rays, if they were parallel at their incidence, will diverge after refraction and the focus will be negative. But when r is less than $-s$, $r-s$ is a negative quantity: for if an equal quantity is subtracted from an equal one the remainder will be nothing, therefore if a greater quantity is taken from a less the remainder will be less than nothing. Thus if r is 10 inches, and $-s$ is 12 inches, $r-s=10-12=-2$. Now if the semidiameter of the convexity or r is less than that of the concavity or than $-s$, the convex surface of the meniscus is a segment of a less sphere than the concave one; and consequently the convexity on one side is greater than the concavity on the other; so that the meniscus will have the effect of a convex lens. Therefore when r is less than $-s$, or when $r-s$ is negative, the rays, if they were parallel at their incidence will converge after refraction, and the focus will be affirmative.

The reader, if he understands the first rules of algebra, will easily find that this follows from the rule laid down in the proposition for finding the

the principal focus of a meniscus. For, calling the principal focus x , as $r-s$ is to r , so is $-2s$ to x , by the proposition; or; since the second term may be either semidiameter, either r or $-s$ and then the third must be double the other; as $r-s$ is to $-s$, so is $2r$ to x . The quantity x is to be found by the rule of three, by multiplying the first term into the second, and then dividing the product by the first, for the quotient of this division is x . Now here we must observe that in a given meniscus the product of the first term multiplied into the second is the same in either notation for either $rx-2s$ or $-sx+2r$ produces $-2rs$: and farther we must observe that the product of the second term multiplied into the third is always negative; because one term is always affirmative and the other negative. But whether x is negative or affirmative depends upon the nature of the first term $r-s$, by which this negative product $-2rs$ is to be divided in order to find out the quantity x . Now if $r-s$ is affirmative, the quotient x arising from $-2rs$ divided by $r-s$ will be negative, because all quotients are so, where a negative quantity is divided by an affirmative one. But if $r-s$ is negative, the quotient x will be affirmative, because all quotients are so, where a negative quantity is divided by a negative one. From hence therefore it appears, as before, that if r is greater than $-s$, or the semidiameter of the convexity is greater than that of the concavity, that is, if $r-s$ is affirmative, the rays, that were parallel when they fell upon the meniscus, will diverge after refraction, and the principal focus will be a negative or virtual one. But if r is less than $-s$, or the semidiameter of the convexity is less than that of the concavity, that is, if $r-s$ is negative, the rays, that were parallel when they fell upon the meniscus, will converge after refraction, and the focus will be an affirmative or real one.

When we have thus determined the principal focal distance of a meniscus, or shewn what effect such a glass will have upon parallel rays, it would be tiresome as well as needless to shew what effect it would have upon diverging or converging ones. For since a meniscus, unless the surfaces of it are parallel to one another, has the same effect, either that a convex lens or a concave one would have, all the cases of diverging or converging rays, that are refracted by it, will be the same with those already explained in the instances of convex and concave lenses.

81. *When rays, that were parallel, have passed out of the air and gone through a sphere of glass, they will converge to a focus, the distance of which from the sphere will be equal to half its semidiameter.*

If AGN, Plat. X. fig. 9. is a glass sphere, and QA, BC are two parallel rays, or two rays which come from the sun; the ray BC continued
passes

passes through E the center of the sphere; and consequently BC is perpendicular to the surface, upon which account it suffers no refraction as it enters the sphere at C; and for the same reason likewise it suffers no refraction when it goes out of it at D: so that the ray BC passes strait forwards through the sphere in the direction BT; by proposition 41. But the ray QA which falls obliquely upon the surface of the sphere at A, as it passes out of air into glass, will be refracted, and will be so made to converge to its axis, that if it was to proceed in the direction AGT, which it acquires by this refraction, it would meet its axis, or the focus would be, at T, by proposition 48. And upon the supposition here made, that the ray is only once refracted CT, or the distance of the focus from the surface where the rays enter, is equal to three semidiameters of the sphere, by proposition 50.

Now if CT is three semidiameters of the sphere, CD or the diameter contains two of them, and consequently DT, or the focus produced by the first refraction, is at the distance of a semidiameter from the second surface of the sphere. But this focus T is only an imaginary one, for the rays, though they converge to T after their first refraction, will never meet there; because when AG or any other oblique ray comes out of the glass into the air it will be refracted a second time, and this second refraction will make it converge more: so that the ray AG, instead of going strait forwards in the direction GT, will be refracted at G and will describe the line GF so as to meet its axis at F.

The rule for finding DF or the distance of the real focus after the rays are thus refracted a second time is this; divide DT the distance of the imaginary focus from the sphere into two equal parts, and DF or half DT will be the distance of the real focus.

Continue the incident ray QA to H. Since QAH is the direction of the incident ray, and AG is its direction after it is once refracted at entering the sphere, the angle HAG contained between the incident and refracted ray is the angle of refraction. Now when the refracted ray AG goes out of the sphere at G, it passes out of glass into air, as it passed out of air into glass, when it entered it at A; and because the two mediums are the same in both cases, the ray as it passes out of the sphere will be just as much refracted from a perpendicular as it was refracted towards one upon its entrance. If this is not self-evident it may be shewn to be true from the following considerations. Suppose the ray AG, instead of passing out of the sphere at G, to be turned back again in the direction GA and to pass out at A. Upon this supposition it will be just as much refracted in passing out at A as it was at its entrance, by proposition 41. But upon
this

this supposition the ray passes out of the glass sphere into the air, and so it would have done, if it had not been turned back again, but had passed out at G. Therefore the ray would be equally refracted, whether it was turned back again and passes out at A, or is not turned back again and passes out at G: for in both cases it passes out of glass into air and through the surface of the same sphere. But if it was turned back again, the refraction, when it passes out at A, would be equal to the refraction, when it entered the sphere at A. Therefore the refraction, when it passes out at G, is likewise equal to the refraction which it suffered upon entering the sphere at A. That is; the ray AG when it passes out of the sphere does not go straight forwards in the direction GT, but is refracted into some line GF so as to make the angle of refraction TGF equal to the angle of refraction HAG. But since QH and BT, the directions of the rays at their incidence upon the sphere, are parallel, by the supposition, and AT crosses these two parallel lines ATF, or GTF which is the same angle, is equal to the alternate angle HAG. Euc. b. I. prop. 29. Therefore since TGF and GTF are each of them equal to HAG they are equal to one another; that is, the angles at the base of the triangle GFT are equal; and consequently the sides FG and FT, which subtend these equal angles, are likewise equal. Euc. b. I. prop. 6. But when G and D are very near each other DF is nearly equal to GF. Therefore DF is likewise equal to FT, or DF is half DT, or the focus F divides DT into two equal parts. Now if the sphere is glass DT, as has been proved, is a semidiameter of the sphere, and consequently DF, or the focal distance is half a semidiameter.

82. *When rays that were parallel at first, have passed out of the air, and have gone through a sphere of water, they will converge to a focus, the distance of which from the sphere will be equal to its semidiameter.*

If AGN, Plat. X. fig. 9. is a hollow sphere of glass, and this sphere is filled with water, then since AC or GD or any other part in the side of the glass is a meniscus whose convexity on one side is equal to its concavity on the other, the rays of light as they pass through any part of the glass will not have their direction at all altered, by proposition 80. And consequently the water contained within the glass may be considered as a sphere of water encompassed with air. Now parallel rays as QA, BC, passing out of air into water through the convex surface AC, since they pass out of a rarer medium into a denser, will be made to converge, by proposition 48: and the distance CT or the distance of the focus to which the rays converge by this first refraction is thus determined, by proposition

tion 50, as the sine of refraction is to the sine of incidence, so is the semidiameter of the sphere to DT the focal distance. But the sine of refraction is to the sine of incidence in the passage of a ray out of air into water, as 1 to 4. Therefore the semidiameter is to the focal distance, after the rays have been once refracted, as 1 to 4; that is, CT the distance of the focus is equal to 4 times CE or to 4 semidiameters or 2 diameters of the sphere. Now CD is one diameter, and consequently DT will be another diameter. But when the ray comes out of the water at G, it will be refracted a second time and will be made to converge more; so that it will cross its axis DT nearer to the sphere than T, at some point F, which divides DT into two equal parts, which may be shewn by what was said in proposition 81. Therefore DF will be half DT; and, since DT is a diameter, DF, or the focal distance after both refractions, which is half DT, must be a semidiameter of the sphere.

The body of some glass decanters is pretty nearly spherical, and the water contained in it is a sphere of water. Therefore from what has been said it appears that parallel rays, or the rays of the sun, by passing through such a decanter full of water will be collected into a focus behind it at the distance of a diameter of the sphere, in the same manner that a convex lens made of glass would collect them: and any such bodies as easily take fire will be burned by those rays, if the bodies are placed at that focus.

A lens of water may likewise be contrived which would collect the rays of the sun and kindle fire in the same manner that a burning glass does. For if each surface of the lens AB, Plat. IX. fig. 6. was a watch glass with its convexity outwards and the edges, where the two glasses are joyned, were cemented together so as to make a hollow lens; and if this hollow lens was to be filled with water, the contents would be a lens of water. The two glasses ACB, ADB, are meniscuses of such a sort as would make no alteration in the direction of the rays that pass through them, as was proved in proposition 80. But the water, that is between the lenses, being denser than the air would make the rays of the sun converge for the same reasons that we have already seen a lens of glass would produce this effect. The only difference would be, that since the density of water is less than that of glass, rays by passing through a lens of water will be less refracted, and will therefore converge less, and consequently, by proposition 29, will have their focus more remote than if they had passed through a lens of glass. It would not be worth our while to enquire how much more remote the focus would be; all that we wanted to prove was that the rays of the sun may be made to burn by passing through cold water.

C H A P. VIII.

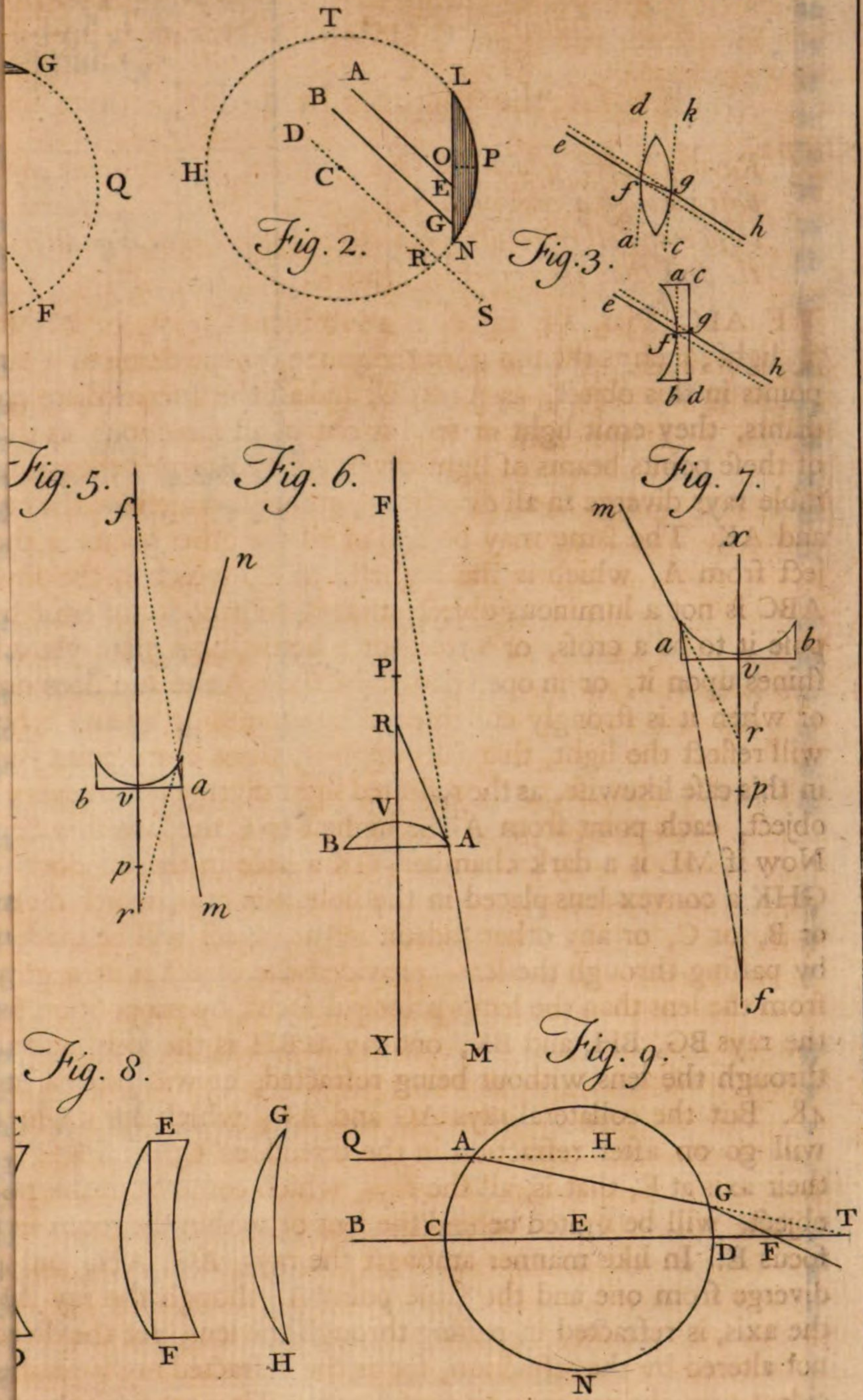
Of the pictures in a dark room.

83. *Rays of light, which flow from several points in any object, and then pass through a convex or plano-convex lens, will afterwards converge to so many other points, provided the object is at a greater distance from the lens than its principal focus.*

IF ABC, Plat. XI. fig. 1. is a luminous object, or object that emits light; such as the sun is, or the moon, or the flame of a candle; all the points in this object, as A, B, C, and all the intermediate points, are radiants, they emit light or send it out in all directions, so that from each of these points beams of light diverge. Thus from the point A innumerable rays diverge in all directions; amongst which are the rays AG, AH, and AK. The same may be said of all the other points in the whole object from A, which is the highest, to C, which is the lowest. Or, if ABC is not a luminous object, that is, if it does not emit light, as suppose it to be a cross, or a tree, or a house, or a man, yet when the sun shines upon it, or in open day-light though the sun does not shine out, or when it is strongly enlightened by candles or by any other means, it will reflect the light, that falls upon it, from every point; and therefore in this case likewise, as the reflected light diverges from every point in the object, each point from A the highest to C the lowest will be a radiant. Now if ML is a dark chamber, GK a hole in the window-shutter, and GHK a convex lens placed in the hole, the rays, which diverge from A, or B, or C, or any other radiant in the object will be made to converge by passing through the lens, provided the object is at a greater distance from the lens than the lens's principal focus, by proposition 64. Amongst the rays BG, BH, and BK, one ray as BH is the axis, and this will pass through the lens without being refracted, as was proved in proposition 48. But the collateral rays AG and AK, which are made to converge, will go on after refraction in the directions GE and KE so as to cross their axis at E, that is, all the rays, which come from the point B in the object, will be united behind the lens or within the room in the point or focus E. In like manner amongst the rays, AG, AH, and AK, which diverge from one and the same point A, though the ray AH, which is the axis, is refracted in passing through the lens, yet the direction of it is not altered by the refraction, for as the refracted ray is parallel to the incident one, the ray may be considered as if it went strait through the lens,

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PLATE X.



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as was shewn in proposition 59. But any other rays proceeding from the same point, such as AG, and AK, will be made to converge to their axis and will be united in a focus or point at F. And thus likewise all the rays, which proceed from C, will be united in a focus or point at D. The same may be said of the rays which diverge from each point in the whole object: the rays from each point will by the refraction of the lens be made to converge to a point on the other side of it: and therefore upon the whole the rays of light, which flow from several points in the object and then pass through the lens, will afterwards converge to so many other points.

It is necessary the object should be more remote from the lens than the lens's principal focus. For if it was in the principal focus, then the rays diverging from the several radiants A, B, C, &c. would not converge to so many other points after refraction, for they would not converge at all, but would be parallel, by proposition 61. Or if the object was nearer than the principal focus, then likewise the refracted rays would not converge at all, but would diverge, by proposition 67.

The reader will see, without having it explained to him, that a plano-convex lens would produce the same effect. And as a double convex one is most commonly used for the purposes to be mentioned in the following propositions, it will be sufficient here to have taken notice once for all, that what is hereafter proved to be true of one sort of lens, will be equally true of the other sort, allowing for the different distances of the principal focus of a plano-convex lens and a double convex one.

84. *Wherever the rays, which come from the several points of any object, meet again in so many points, after they have been made to converge by the refraction of a convex lens, there they will make a picture of the object upon any white body on which they fall.*

If the rays which diverge from the several points A, B, C, &c. in the object AC, Plat. XI. fig. 1. pass through the convex lens GHK fixed in the hole of the window shutter, and, when they come into the dark room ML, converge to just as many points F, E, D, &c. they will make a picture of the object upon a piece of white paper held at the place DF.

There will be just as many focuses upon the paper as there are radiant points in the object, from which the rays come; and these focuses will be disposed in the same manner in respect of one another that the radiants are, by proposition 83. Those focuses will be the most bright, in which the most rays of light are united, and those will be the least bright or the deepest shaded, in which the fewest rays are united. And the most rays must be united in such focuses, as correspond to those ra-

dians, from which the most light come; and the fewest rays must be united in such focuses as correspond to those radiants, from which the least light come. Therefore the light and shade upon the paper or at the focuses will be every where answerable to the light and shade upon the surface of the object or at the radiants. And lastly, since the colours of the rays are not changed, each focus upon the paper will have the same colour with the corresponding radiant. Now when the rays from these focuses are reflected by the paper and enter the eye of a spectator, who looks at the paper, he will there see the picture or likeness of the object. For the figure made up of these focuses will be like the figure of the object; because the focuses are disposed in the same manner in respect of one another, that the radiants in the object are. The light and shade upon the paper is every where answerable to the light and shade upon the surface of the object. And the colouring of each particular part through the whole figure upon the paper is the same with the colouring of the correspondent part in the object.

It is necessary that the room should be darkened so that no light may come into it, but what comes through the hole GK. For if besides the rays which flow from this object any other strong light, such as open day-light, was to fall upon the paper; this would dilate and weaken the shadings and colourings of the picture so as to make it disappear: the few rays that produce the picture become imperceptible, when they are mingled with the much more numerous rays of open day-light. Indeed though the room is not so absolutely darkened as to exclude all light but what enters at the hole, though there should be some other rays come into it and mingle themselves with the picture; yet if the room is almost dark, that is, if there are but few of these rays, the picture will not disappear, but will only be made fainter. The picture is the most vivid when the room is quite dark; it becomes fainter by letting in a small quantity of light; and will be made wholly to disappear by letting in too much.

If in a room that is quite darkened and has no hole in the window-shutter to let any light at all into it, you place a lighted candle, then by holding a convex lens at some distance from the candle, and a piece of white paper at a proper distance beyond the lens, you may collect the rays that flow from the several points in the candle into as many focuses upon the paper; and these focuses will make the picture of the candle upon the paper. This picture may easily be seen notwithstanding it is produced only by rays from the candle, and the same candle enlightens every part of the room. For the rays, which have passed through the lens and are

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collected by it, will throw a stronger light upon the paper than the direct rays, or those which have not been thus collected, will produce any where else: therefore though the paper is at the same time enlightened by the direct rays, which come from the candle, as well as any other part of the room, yet these will be much fainter than the picture, and, though they are mingled with it, will not hinder it from appearing.

In like manner, when we use a common burning-glass, which is a convex lens, it collects the suns rays into a focus, which, when it is received upon a white paper, is a bright round spot. We call it a focus, though properly speaking it consists of innumerable focuses; for the rays which come from the several points in the sun are made by the glass to converge to as many points or focuses upon the paper: so that the bright spot, which is produced at the focus of a burning glass by collecting the suns rays, is nothing else but the picture of the sun. But this picture or bright spot is easily seen, though it is produced in open day-light and not in a dark room: because the light of the rays, when they are thus collected, is so much greater than the light of the direct rays of the sun, or than common day-light, that common day-light, though it is mingled with the picture, will make little or no alteration in it.

The reader has by this time seen the reason, why, in proposition 25, I gave the name of a pencil of rays not to the incident beam, AG, AH, AK, alone, but to this incident beam and the refracted one, GF, HF, KF, taken together: for it is not the incident beam alone, which produces the effect of a painters pencil; it is the incident beam and refracted one together which paint the picture of the point A in the object at the point F upon the paper.

For the better understanding many things which follow, it will be necessary here to consider several properties of such pictures as are produced in the manner just now described.

85. The picture in a dark room is inverted in respect of the object.

In whatever position the object AC, Plat. XI. fig. 1. is placed, the picture DF will be in the contrary position. The focus, in which the rays, that come from A, such as AG, AH, and AK, are united, is in the axis AH; for this axis goes strait through the lens GHK, and the rest of the rays are made to converge towards it. So likewise, as appears from what was said concerning the axis of an oblique beam the focus, where the rays that come from C, such as CG, CH, and CK, are united, is in the axis CH for the same reason. Now the highest point A is painted upon that part of the paper where the rays, that come from A, are united; and
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the lowest point C is painted upon that part of the paper where the rays, that come from C , are united. Therefore the picture of the highest point is somewhere in the axis AH continued beyond the lens into the room; and the picture of the lowest point is somewhere in the axis CH continued in the same manner. But the axis AH is the middle ray of the cone of rays GAK , by proposition 22. And this cone has the lens GHK for its base, by proposition 21: and consequently the axis AH passes through the middle of the lens. For the same reason the axis CH must pass through the middle of the lens. And if both the axis AH and the axis CH pass through the middle of the lens, they must cross one another in passing: so that though AH is the higher and CH the lower of the two on the side of the lens next the object; yet on the other side of the lens, when the axes are within the room, they will have crossed each other; and HF , which is AH continued, will be the lower of the two, and HD , which is CH continued, will be the higher. But we have proved already that the point A will be painted somewhere in the axis AH continued beyond the lens, or somewhere in the line HF , and that the point C will be painted somewhere in the axis CH continued beyond the lens, or somewhere in the line HD . Therefore A or the highest point of the object will be painted lower upon the paper than C , and C or the lowest point of the object will be painted higher upon the paper than A ; that is, the picture FED will be in a contrary position to the object, or, in respect of the object, will be inverted.

86. *The picture in a dark room is not distinct, unless the paper, upon which it is painted, is placed where the several beams of rays are each of them united in their respective focuses.*

The picture is then only distinct when all the rays that come from one and the same point in the object are collected into one and the same point upon the paper. For, if the rays, which come from A , for instance, are not united in a focus at F , but are scattered so as to be spread over a spot upon the paper instead of falling all together upon a single point; and if the rays likewise, which come from the point next below A in the object, are scattered in the same manner; then the rays which come from different points in the object, will not be separated from one another upon the paper, but as the rays from the point A are scattered at F and the rays that come from the point next below A are likewise scattered just above F , these two beams, though they come from different points in the object, will be mingled upon the paper, and consequently in the picture these two points will not appear so distinct from

from one another as they do in the object. This will be the case of all the other parts of the picture, unless the paper is placed where the several beams are united in their respective focuses.

87. *As the object approaches to the lens, the distinct picture departs from it; and as the object departs, the distinct picture approaches.*

The picture, by proposition 86, is distinct where the several beams of rays, that come from different points or radiants in the object, are united in their respective focuses, where the rays, for instance, which come from one and the same radiant A are all of them united in one and the same focus F, and those, which come from B, are united in one and the same focus E. But as the object approaches to the lens the several radiants A, B, C, &c. approach it, and consequently the several focuses F, E, D, &c. depart from it, by proposition 71. Therefore as the object approaches to the lens the distinct picture, which is always in the same place with these focuses, departs from it. In like manner as the object, or the several radiants A, B, C, &c. departs from the lens, the several focuses F, E, D, &c. and consequently the distinct picture will approach to it, by proposition 71.

From hence then it follows, that if the picture is distinct upon the paper, when the distance of the object is HB, and the distance of the paper HE; then supposing the object to be brought nearer to the lens, the picture upon the paper will become confused; for the distance of the distinct picture is encreased by bringing the object nearer: and in order to make the picture appear distinct again upon the paper, it is necessary that the distance HE should be encreased as much, or that the paper should be removed farther from the lens. But if the object is removed farther from the lens, then the distinct picture gets nearer to it, and if the paper still continues at the same distance HE the representation upon the paper will be confused: therefore in order to restore the distinctness of this picture or representation, the distance HE must be diminished, or the paper must be brought nearer to the lens.

If rays that come from several objects enter the lens at the same time and pass through it into the dark room, then the distinct picture at the focus, will consist of the representations of all those objects. But to make all the parts of the picture distinct, the several objects, that are represented in it, must be at equal distances from the lens: for if some of them are farther off than others, then by holding the paper so near to the lens as to have the remoter objects represented distinctly upon it; these parts of the picture, where the nearer objects are represented, will appear confused;

fused; or on the contrary by holding the paper so far from the lens as to have the nearer objects represented distinctly, those parts, where the remoter ones are represented, will appear confused. Though if the distances of the several objects are not very different, the confusion in either case will be but inconsiderable.

When the object AC is removed to an infinite distance from the lens, or when the several radiants A, B, C, &c. are so remote that the rays, which come from them, may, by proposition 30, be considered as parallel; then these rays will be collected into their several focuses, in the principal focus of the lens, by proposition 60: and therefore the distinct picture of the object will likewise be at the principal focus of the lens, by the proposition now before us. This is the case of the sun: it is so remote an object, that the rays, which come from the same point, are parallel to one another, and consequently the distinct picture, which is the burning spot, when the solar rays are collected by a burning glass, is at the principal focus of the glass.

Here we may observe that though, whilst the object keeps departing from the lens, the picture keeps approaching to it; yet the picture can never be nearer to the lens than its principal focus; since this is the place of the picture, when the object has departed to an infinite distance.

88. *When the object is parallel to the picture, the diameter of the object is to the diameter of the distinct picture, as the distance of the object from the lens is to the distance of the picture from the lens.*

If the paper is held parallel to AC, so that the object and distinct picture may be parallel to one another; then AC, which is the height of the object, or a diameter drawn from the top of it to the bottom, bears the same proportion to FD, which is the height of the picture, that CH or the distance of the object from the lens bears to DH or to the distance of the picture from the lens; that is, AC is just as much longer or shorter than EF, as CH is longer or shorter than DH.

Now A is the highest radiant in the object, and C is the lowest radiant, and the height of the object is the distance between these two or AC. The radiant A is painted somewhere in the right line AH continued behind the lens, and the radiant C is painted somewhere in the right line CH continued in the same manner, by propositions 83, 84. So that the height of the object is AC or the distance between these two lines at the object, and the height of the picture will be FD or the distance between the same lines, when they are continued to the place where that picture is represented.

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Since therefore CD and AF are right lines, and AC is parallel to FD; it follows that the angle ACD is equal to the alternate one CDF; and that the angle CAF is likewise equal to the alternate one AFD. Euc. b. I. prop. 29. Therefore in the triangles AHC and FHD the angles at A and F are equal, and so are the angles at C and D. And CHA and DHF are equal, because they are vertical. Euc. b. I. prop. 15. So that all the angles in the triangles AHC are respectively equal to all the angles in the triangle FHD, or these two triangles are equiangular and consequently the sides about the equal angles are proportional. Euc. b. VI. prop. 4. AC is to CH, as FD is to DH, or the height of the object is to the height of the picture, as the distance of the object from the lens is to the distance of the picture from the lens.

In the same manner we might prove that the breadth of the object is to the breadth of the picture, or that any diameter or line drawn across the former is to any corresponding diameter or line drawn across the latter, as the distance of the object from the lens on one side is to the distance of the picture from it on the other.

From hence it follows that the diameter of the picture may either be shorter than that of the object, or longer than that of the object, or equal to it. If DF is nearer to the lens than AC, that is, if the distance of the picture is less than the distance of the object, the diameter of the picture will be less than the diameter of the object, or DF will be shorter than AC, in the same proportion. But if the picture was farther from the lens than the object, any diameter of the picture would be greater than the corresponding diameter of the object: the height, for instance, of the picture would be greater than the height of the object. Or if both the picture and object were at an equal distance from the lens, the picture would in all respects be equal to the object.

89. *The picture, when it is confused, is something bigger, than when it is distinct.*

When the paper is held either too near to the lens, or too far from it the picture will be confused. by proposition 86. And in either case it will be a little longer and a little broader than if it was made distinct by holding the paper at a proper distance.

When the paper is held at the focus, or when the picture is distinct, all the rays, that come from the same point in the object, are collected into one single point upon the paper. Thus the rays, for instance, that come from A, Plat. XI. fig. 1. or from the highest point in the object, are collected into the point F, or into the lowest point in the picture, and

all the rays, that come from C, or from the lowest point in the object, are collected into the point D, or into the highest point of the picture, by proposition 84, 86. It is plane therefore that the length of the picture is the distance between F and D, or the distance between that place upon the paper where the rays fall which come from the highest point of the object, and that place where those fall which come from the lowest point.

Now when the picture is distinct F and D, as has just been observed, are only single points. But if the paper was to be brought nearer to the lens, so as to receive the rays of each pencil before they are united in their several focuses, the rays that come from A would not be collected into a single point at F, but would be spread over a little circular space upon the paper. In like manner the rays that come from C would not be collected into a single point at D but would be spread over a little circular space. And the same would happen to the rays of every other pencil. This will make the picture appear confused, by proposition 86. It will likewise make the picture longer. For the length of the picture is the distance between that place upon the paper where the rays fall which come from the highest point of the object, and that place where those fall which come from the lowest point. This, when the picture is distinct, is the distance between the two points F and D. But when the picture is confused the rays at F are spread over a small circle, and therefore will be extended in all directions a little way round the point F, and consequently will reach below the point F: and for the same reason the rays at the other end of the picture will reach above the point D. But the picture is extended upon the paper as far as the rays at F reach one way and the rays at D the other way. Therefore the picture, when it is confused, is something longer than when it is distinct. In the same manner we might prove that it is something broader too. And consequently the confused picture is every way something bigger than the distinct one.

There is another way of making the picture confused besides bringing the paper too near to the lens, and that is by removing it to too great a distance. For when the rays of any pencil have met at their focus, as at F, for instance, if there is nothing there to stop them, they will proceed strait forwards, and, after they are passed the focus, will begin to diverge from thence as from a radiant, by proposition 15. So that when the paper is more remote than this focus, these rays, instead of being collected into a single point at F will be in a diverging state, and consequently will be spread over a small circle at F. The same likewise will happen to the rays at D. And this, for the same reasons as before, will make the picture a little bigger every way, than it would be if it was distinct.

90. *The object and the distinct picture are similar surfaces.*

It will here be necessary to explaine what is meant by the surface of the object. When an object is considered as reflecting light from that side of it which is towards the lens, whatever is the shape of the object on that side, we may always look upon it as a plane surface differently shaded. Thus if AC, Plat. XI. fig. 1. is the body of a man; though the side of the man, which is towards the lens, is not a plane surface, yet the light is reflected from it, just as if it was the figure of a man drawn upon the plane surface of a piece of canvas and differently shaded in different parts of it. Upon this account we consider the side of the man, or other object, which is next the lens, as such a plane surface differently shaded. And in this view we affirm that the surface of the object will be similar to the surface of the distinct picture. These two surfaces are similar, if the height of the object bears the same proportion to the height of the picture, that the breadth of the object in any part of it bears to the breadth of the picture in the corresponding part. The proportion that the height of the object bears to the height of the picture, is always the same with the proportion, that the breadth of the object in every part of it bears to the breadth of the picture in the corresponding part, for each of these proportions is the same with that which the distance of the object from the lens bears to the distance of the picture from the lens, by proposition 88. Therefore the object and picture are always similar to one another.

Indeed it is in this similarity that a principal part of the likeness between the object and picture consists. For though they are shaded and coloured in the same manner; yet one of them would not be like the other unless their outlines were similar, that is, unless the surface included within the outlines of the object was similar to the surface included within the outlines of the picture.

91. *When the object is given, the diameter of the distinct picture is inversely as the distance of the object from the lens.*

The diameter of the picture encreases as its distance from the lens encreases, and decreases as its distance decreases, by proposition 88. For when it is at the same distance with the object, it is equal to the object; when it is more remote than the object its diameter is greater than that of the object in proportion as its distance is greater; and when it is nearer than the object, its diameter is less in proportion as its distance is less. Now the distance of the picture encreases as the distance of the object decreases, and decreases as the distance of the object encreases, by pro-

position 47. Since therefore the diameter of the picture encreases as its distance encreases, and since its distance encreases as the distance of the object decreases; it follows that the diameter of the picture encreases as the distance of the object decreases, or that the diameter of the picture is greater as the distance of the object is less, or that the diameter of the picture is inversely as the distance of the object.

92. *The area of the picture, when the object is given, is inversely as the square of the objects distance from the lens.*

Whatever is the distance of the object, and whatever, in consequence of that, is the size of the picture, it has been proved, in proposition 90, that the surface of the picture is always similar to the surface of the object. And, because the surface of the object, by the supposition, is always the same, the surface of the picture, which in all sizes of it is similar to the surface of the object, must in all sizes of it be similar to itself. But the areas of similar surfaces are as the squares of their homologous diameters, that is, as the squares of their heights, or as the squares of their breadths. Euc. b. VI. prop. 20. corol. 1. Therefore the area of the picture is always as the square of its diameter. But the diameter of the picture is inversely as the distance of the object, by proposition 91. Therefore the square of the diameter of the picture, or the area of it, is inversely as the square of the distance of the object.

We may apply this proposition and the foregoing one at the same time to an instance or two by the help of numbers. When the object AC, Plat. XI. fig. 1. is at the distance BH from the lens, the height of the picture is DF. But if the distance of the object was twice BH, then the height of the picture would be only $\frac{1}{2}$ DF. And when the height of it is thus reduced to $\frac{1}{2}$, the breadth will at the same time and for the same reason be likewise reduced to $\frac{1}{2}$. For, by proposition 91, either the height or the breadth of the picture is inversely as the distance of the object: so that by doubling the distance of the object both the height and the breadth of the picture are halved. But if by doubling the distance of the object, the picture is made only $\frac{1}{2}$ as long and $\frac{1}{2}$ as broad as it was, the surface of it will be reduced to $\frac{1}{4}$. That is, if we call the distance BH 1, then twice BH will be 2: the square of 2 is 4; and by encreasing the distance of the object from BH to twice BH, the area of the picture is diminished to $\frac{1}{4}$ of what it was. But $\frac{1}{4}$ is the reciprocal of 4 or of the square of the distance. Therefore the area of the picture is as the reciprocal of the square of the distance, or inversely as the square of the distance. In like manner if the distance of the object was three times HB, the height of
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the picture would be $\frac{1}{3}$ of DF, and the breadth of it would likewise be reduced to $\frac{1}{3}$ of what it was before. Therefore the area of the picture would be reduced to $\frac{1}{9}$: for if at the distance 3HB the surface of it is but $\frac{1}{3}$ in length and $\frac{1}{3}$ in breadth what it is when the distance was HB, the area of that surface will be but $\frac{1}{9}$ of what it was. And thus in all other instances, the area of the picture, or the surface of that part of the paper, where the rays of light fall which form the picture, is always as the reciprocal of the square of the objects distance: so that if we call that surface 1 at any certain distance of the object, at twice the distance of the object it will be only $\frac{1}{4}$; at three times the distance of the object it will be $\frac{1}{9}$; at four times the distance it will be $\frac{1}{16}$; &c.

93. *When the distance of the object is given, the diameter of the distinct picture is directly as the diameter of the object.*

If the distance of the object is given, that is, if the distance of the object does not vary, the picture will always be at the same distance from the lens. If the object is always at the distance BH from the lens, Plat. XI. fig. 1. the image will always be at the distance EH, by proposition 64, 84. In this case the diameter of the picture DF will be greater or less as the diameter of the object AC is greater or less. For AC the diameter of the object always bears the same proportion to DF the diameter of the picture that BH the distance of the object bears to EH the distance of the picture, by proposition 88. But when BH and EH are given, that is, when the object and picture continue at the same distance from the lens, the proportion which BH bears to EH is likewise given; for the same lines always bear the same proportion to each other. Therefore when BH and EH are given, the proportion, which AC bears to DF, must be given too. Now where the proportion of AC to DF is invariable, it follows that if AC encreases DF must encrease in the same proportion, or if AC decreases DF must decrease in the same proportion; that is, where the distance of the object, and consequently the distance of the picture is given, as the height or any other diameter of the object is greater or less the height or any other corresponding diameter of the picture will be greater or less in the same proportion. But quantities which vary in the same proportion are directly as each other. Therefore where the distance of the object from the lens is given, the diameter of the picture is directly as the diameter of the object. So that if AC was twice as long as it is in the figure before us, the length of DF would be doubled, if AC was thrice as long, then DF would be tripled, or in whatever other proportion AC encreases, DF will encrease at the same time and in the same

pro-

proportion. Or if AC was half as long as it is in the figure, DF would be but half as long, if the length of AC was only one third of what it is, the length of DF would likewise be only one third, or in whatever other proportion AC decreases, DF will decrease at the same time and in the same proportion.

94. *Whatever is the distance of the object from the lens, the diameter of the distinct picture will always be the same, provided the diameter of the object is proportional to its distance.*

If the distance of the object AC, Plat. XI. fig. 1. was twice HB, this would make the picture DF half as long as it is, when the distance of the object is only once HB, by proposition 91. But if the length of AC was to be doubled, this would make the picture twice as long, by proposition 93. Now if, when the distance of the object is doubled, the length of it was to be doubled at the same time, the length of the picture would be halved upon one account, and at the same time would be doubled upon the other account, that is, the length of the picture would be diminished and encreased equally, and consequently upon both accounts taken together it would continue the same. In like manner, if the distance of AC the object was only half BH, this would make the picture DF twice as long as it is, by proposition 91. And if the length of the object was half AC this would make the picture half as long as it is. If therefore, when the distance of the object is halved the length of it is halved at the same time, that is, if the length of the object decreases in the same proportion with its distance, this will make the picture twice as long upon one account and half as long upon the other account, and consequently upon both accounts taken together its length will be encreased and diminished equally, and therefore will continue the same. This method of reasoning may easily be applyed to any other instance where the diameter of the object is proportional to its distance from the lens, that is, where the diameter of the object encreases or decreases in the same proportion with its distance.

Or otherwise. Call the height of an object X, and call the distance of it from the lens Z. Then the length of the picture is as X directly, by proposition 93, and as Z inversely, by proposition 91. Therefore universally the length of the picture will be as the fraction $\frac{X}{Z}$; for such a fraction, like all other fractions, is as its numerator directly and as its denominator inversely, that is, it is directly as X and inversely as Z. But if X is always directly as Z then $\frac{X}{Z}$ or the length of the picture will be a given

ven quantity, that is $\frac{X}{Z}$ will always be the same, whatever be the values of X and Z , or let the distance and length of the object be what they will. For the fraction $\frac{X}{Z}$ is as X directly, or encreases in the same proportion that X encreases, and it is inversely as Z , or decreases in the same proportion that Z encreases. But X is directly as Z , that is, in whatever proportion X encreases Z will encrease in the same proportion. Therefore in whatever proportion the fraction $\frac{X}{Z}$ encreases when its numerator X encreases, it will be diminished at the same time in the same proportion by the encrease of its denominator Z . In like manner, if $\frac{X}{Z}$ is diminished because X its numerator decreases; it will at the same time be proportionally encreased by the decrease of its denominator Z . Therefore in all the variations of X and Z , the value of the fraction $\frac{X}{Z}$ continues the same; or in all the changes of the objects height and distance; the length of the picture will continue the same; provided X is directly as Z , or provided the height of the object is directly as its distance.

Suppose there were two objects at different distances from the lens, and suppose their heights to be in the same proportion to one another that their distances are, for instance, call the height of one object X and the height of the other x , call the distance of the former object Z and the distance of the latter z ; and let X be to x as Z to z : then the distinct picture of each object will be of the same length. The length of one picture will be as $\frac{X}{Z}$ and the length of the other will be as $\frac{x}{z}$, but since X is to Z as x to z , the first term divided by the third is equal to the second divided by the fourth, that is, $\frac{X}{Z} = \frac{x}{z}$; that is, since $\frac{X}{Z}$ is a given quantity whatever are the values of X and Z , when they are both diminished in the same proportion and reduced to x and z , the fraction $\frac{x}{z}$ will be equal to $\frac{X}{Z}$. Therefore each picture will be of the same length.

95. *When the diameter and distance of the object are given, the diameter of the picture will be directly as its distance from the lens.*

If the diameter and distance of the object are given, it is plane, from propositions 91, 93, that the diameter of the picture cannot be varied, whilst the same lens is made use of. But though the object continues the same and is placed at the same distance, yet by changing the lens GHK ,
Plat.

Plat. XI. fig. 1, for one that is less convex or that has a longer principal focus, the rays will be made to converge less by passing through it, and the focuses of the several pencils, or the picture made by those pencils will be at a greater distance from the lens. In like manner, if a more convex lens is made use of, the refracted rays will converge the more, and the picture will be at a less distance. This may be proved more certainly by applying proposition 64. When the distance of the picture is thus varied only by changing the convexity of the lens, without any alteration being made either in the diameter or in the distance of the object, the diameter of the picture will be greater as its distance is greater, or less as its distance is less. For since the distance of the object is always to the distance of the picture, as the diameter of the object is to the diameter of the picture; and since the first and third terms, or the distance and diameter of the object, are supposed invariable, in whatever proportion the second term, or distance of the picture, is encreased or diminished, the fourth term, or diameter of the picture, must encrease or decrease in the same proportion; that is, the diameter of the picture will be directly as its distance from the lens.

96. *When the diameter and distance of the object are given, the area of the picture is as the square of its distance from the lens.*

For the object and picture are always similar surfaces, by proposition 90; and consequently the picture, whatever is the size of it, will always be similar to itself, because it is always similar to the object. But the areas of similar surfaces are as the squares of their corresponding diameters. Euc. b. VI. prop. 20. corol. 1. Therefore the area of the picture is as the square of its diameter. Now the diameter of the picture, when the diameter and distance of the object are given, is as its distance from the lens, by proposition 95. Therefore the square of its diameter, or the area of it, is as the square of its distance from the lens.

97. *Though the distance of the object from the lens should be varied, yet the picture may be preserved distinct upon the paper without varying the distance of the paper from the lens.*

This indeed would be impossible, as appears from proposition 87, if the same lens was to be made use of in all the different distances of the object. But we have observed, in proposition 95, that a less convex lens will make the picture more remote from the lens, and a more convex one will make it nearer. Suppose therefore that the picture of any object is represented distinctly upon the paper in the dark room, and then let

let the object be removed to a greater distance from the lens; if in this greater distance of the object the same lens is made use of, the picture will be confused, unless the paper is brought nearer, by proposition 87. But if in this case the lens is changed for one of a less convexity, this will throw the picture farther back. And if it is thrown as much back by diminishing the convexity of the lens, as it is brought forwards by the removal of the object; then by both causes concurring to produce equal and contrary effects it will be kept in the same place, and will continue distinct, though the paper is not stirred. In like manner if the object comes nearer to the lens, then, by proposition 87, the paper must be removed backwards from the lens in order to preserve the picture distinct, supposing the same lens to be made use of. But a more convex lens would bring the picture forwards. Therefore, when the object approaches the lens, if a more convex lens is made use of, the picture may by this means be brought forwards just as much as it is thrown backwards by the approach of the object. And thus again, by both these causes concurring to produce equal and contrary effects, the distinctness of the picture may be preserved without stirring the paper.

98. *Though the distance of the object from the lens should be varied, yet the picture may be preserved distinct upon the paper, without either moving the paper or changing the lens.*

When BH, Plat. XI. fig. 1. which is the distance of the object from the lens, is encreased, the distinct picture will approach to the lens, by proposition 87; that is, when BH encreases EH decreases. Now if EH is the distance of the paper from the lens, and one way of keeping the picture distinct upon the paper is by moving the paper nearer to the lens as the picture moves nearer to it, yet this is not the only way: for as by moving the paper nearer to the lens the distance EH is diminished, so likewise without stirring the paper the distance EH may be deminished, if the lens is moveable and can be brought nearer to the paper. By the same means the picture may be kept distinct, when the object approaches to the lens: for as the distinct picture will in this case depart, the distance EH between the lens and the paper must be encreased till it is equal to the distance of the distinct picture; and this may be done not only by moving the paper farther from the lens, but by moving the lens farther from the paper without stirring the paper at all.

99. *When the object is very near the lens, though its distance is greater than the principal focal distance, yet, in order to make the picture distinct, the area of the lens must be very small.*

The picture is not distinct, unless all the rays, that come from one and the same radiant in the object, are united in one and the same point upon the paper, by proposition 86. Now if a , Plat. XI. fig. 2. is a radiant, and the lens is very near it, as suppose at np , and suppose np to be the diameter of the lens: then the cone of rays, the base of which is the lens, will be nap . Of these rays the extreme ones as an and ap diverge much more than the middle ones ad and ae : so that if the paper, upon which the focus of these rays is to be received after they have passed through the lens, is placed where the middle or less diverging rays are collected, the extreme rays, because they diverged more, will not be collected so near to the lens, and consequently, if they fall upon the paper, they will make the picture confused, as they are not brought to a focus. But if these extreme rays are excluded, or prevented from passing through the lens, then the picture will be distinct. The way to exclude them is to diminish the area of the lens, or of the hole where the lens is placed: for if the diameter of it was de instead of np , only the middle rays could pass through it, and the extreme ones would be stopped by the window-shutter. This would not be necessary, if the radiant was at a greater distance from the lens. Suppose the lens was at sr , then the extreme rays as an and ap pass along in the directions nb and nc , so that some of them will be above the lens and some below it: only the middle rays of the cone, as and ar which are ad and ae continued, and such rays as are contained between these, will pass through the lens, though the area of it is not diminished, or though the diameter is sr equal to np .

100. *When the distances of the object and picture and likewise the diameter of the object are given, the diameter of the picture will not be altered by altering the area of the lens.*

If GHK, Plat. XI. fig. 1. is the lens, and the hole in the window-shutter is big enough to leave the whole area of the lens open so that the rays may pass through it, then, supposing DF to be the length of the picture, no alteration would be made in the length of this picture, though the hole in the shutter was to be diminished, and by that means the whole area of the lens was to be covered, except a very small part in the middle of it at H. The height of the picture DF is the distance between the two extreme focuses D and F. One of these focuses is always in the axis AHF of that cone which comes from A one extreme point of the object; the other is always in the axis CHD of the cone which comes from C the other extreme point of the object, and these two axes go strait through the middle of the lens, as appears from what has been said in demonstrating

ting proposition 85. Since therefore DF the height of the picture is the distance between these two lines AHF and CHD, when they come to the paper, the height of the picture will be the same whether the whole area of the lens is open or only a small part of it in the middle at H: because these two extreme axes go strait through the middle of the lens and there cross each other; and consequently the distance between them, when they come to the paper, or DF, will be the same, whether the area of the lens is great or small.

101. *The brightness of the picture, when its distance from the lens is given, is directly as the area of the lens.*

The distinctness of the picture is not the same thing as its brightness: nor is the confusion of its parts the same thing as its obscurity. The picture may be distinct in all its parts, the rays, which come from one and the same point in the object, may be exactly collected into one and the same point upon the paper, and yet if but few rays should pass through the lens so that the space or area upon the paper, where the picture is painted, should be but faintly enlightened, this picture, though it is distinct, will be a faint or obscure one. Or, though the picture is confused either by the papers being placed at an improper distance from the lens, or by any other means, yet if many rays should pass through the lens, so that the space, where this picture is painted, should be strongly enlightened, this picture, notwithstanding its confusion, will be a bright one.

Now the brightness or faintness of the picture in every part, as we have here explained what is meant by brightness and faintness, must evidently depend on the number of rays that come to that part; the picture will be bright or faint in proportion as it is formed by more or by fewer rays. And the quantity of light or number of rays that pass from a given object into the room is greater or less as the hole, through which they pass, or as the area of the lens is greater or less. Therefore the degree of brightness, that the picture has, will be greater or less as the area of the lens is greater or less; that is, it will be directly as this area.

From hence it is plain that though the size of the picture is not altered by diminishing the area of the lens, as was proved in proposition 100, yet the brightness of it will be diminished. Thus whatever degree of brightness the picture DF, Plat. XI. fig. 1. may have when the whole area of the lens GHK is open; if the greatest part of this area was to be covered, and none but a small round hole at H was to be left open for the light to pass through, this would make the picture much fainter. For when

the whole area is open, the entire cone of rays AG, AK, and all the other rays included between those extreme ones, passes through the lens from the points A in the object: but when the area is diminished to a small hole at H, the greatest part of this cone is excluded, no rays but the middle ones AH and a few that are very near to AH can pass through the lens; and consequently the focus F where the point A is represented will be but faintly enlightened. The same may be said of every other cone of rays and of every other point in the picture. Therefore the whole picture will be made fainter or less bright by diminishing the area of the lens.

Here too we may observe farther, that though, when an object is very near to the lens, the picture is made more distinct by diminishing the area of the lens, as was shewn in proposition 99, yet by this means it will at the same time be made more faint: for by diminishing the area of the lens many rays are excluded, and the part of the paper, where the picture is painted, will consequently be less enlightened.

102. *The brightness of the picture, when the area of the lens and the distance of the object are given, is inversely as the square of its distance from the lens.*

When the area of the lens is given, we have seen in proposition 101, that the number of rays, which pass through it and form the picture, will be given too. Now the same quantity of rays, if they are spread over a large space upon the paper will not enlighten it so strongly, as they would a smaller space, where they are kept closer together and all of them fall within a narrower compass. That is to say, the brightness at the paper will be greater, as the space, upon which the rays fall, is less, and will be less as that space is greater; or the brightness will be inversely as that space. But the space upon which the rays fall, that produce the picture, is the area of the picture. Therefore the brightness of the picture will be inversely as its area. Now the area of the picture, when the distance of the object is given, is as the square of its distance from the lens, by proposition 96, and consequently its brightness is inversely as the square of its distance.

103. *When the area of the lens is given, the brightness of the picture will not be altered, let the distance of the object be what it will.*

It appears from proposition 102, that the brightness of the picture is inversely as the square of its distance from the lens; when this distance is varied by changing the convexity of the lens without altering the distance

stance of the object. And so likewise it would be when the distance of it is varied by changing the distance of the object, if such a change of the objects distance made no other alteration but in the distance of the picture. But there is another alteration made in this case, which will preserve the degree of brightness in the picture always the same.

Now the area of the picture is inversely as its brightness, as appears from what has been said in proving proposition 102. And the area of the picture is inversely as the square of the objects distance, by proposition 92. From hence it follows that the brightness of the picture and the square of the objects distance are varied in the same proportion, for they are both of them inversely as the area of the picture. But two quantities which are varied always in the same proportion are directly as each other. Therefore, if all other circumstances were equal, the brightness of the picture would be directly as the square of the objects distance from the lens. Light is propagated in right lines from every point of the object, by proposition 34. Therefore the quantity of light that falls upon a given space, or the quantity that passes through a lens the area of which is given, will be inversely as the square of the objects distance, by proposition 22 of mechanics. And consequently the brightness of the picture, if all other circumstances were equal, would be inversely as the square of the objects distance. We see therefore that upon altering the objects distance, the brightness of the picture is altered upon two accounts; first, it will be directly as the square of the objects distance upon account of the alteration that is made in its area; and secondly, it will be inversely as the square of the objects distance, upon account of the alteration that is made in the quantity of light, by which it is formed: or, calling the objects distance D and the square of its distance D^2 , the brightness of the picture will be as D^2 directly and as D^2 inversely, that is, it will be as $\frac{D^2}{D^2}$. But such a fraction will be equal to 1, since the numerator and denominator are the same. And 1 is an invariable or given quantity. Therefore the brightness of the picture, will be the same or invariable let the distance of the object be what it will.

We have not here taken notice that the rays of light are to pass through the atmosphere from the object to the lens, and that some of them will necessarily be stopped in their passage by the air and by such bodies as float in it. From whence it is evident that the more of the atmosphere the rays have to pass through the more of them will be stopped, and the less of it they have to pass through the fewer of them will be stopped.

And.

And consequently more rays will be stopped in passing from a remote object than from a near one. This will make the picture of a remote object something fainter than it would be if the object was nearer. So that this proposition is true only upon supposition that the rays pass freely from the object to the lens, and that none of them are intercepted in their passage.

104. *When all other circumstances are equal, the picture will be brighter in proportion as the object is brighter.*

The reader could scarce want to be reminded of this: for it is plain, that in equal circumstances, the more rays will pass through the lens from the object in proportion as the object is brighter: and consequently, since the brightness of the picture depends upon the number of rays that form it, the picture will be more or less bright in proportion as the object itself is more or less bright. Upon this account it is, that, when a house, or a man, or any other object abroad is represented upon the paper in a dark room, this picture will be less bright, when the sun does not shine upon the object, than when it does.

105. *If the hole in the window-shutter is very small, the inverted pictures of objects, that are abroad, will be represented upon a piece of paper in a dark room; though there is no lens in the hole.*

The picture in a dark room, when there is a convex lens in the hole of the window-shutter, is drawn upon a piece of white paper placed at the focus, because the rays, that come from one and the same point in the object, will be collected there so as to fall upon one and the same point of the paper, by proposition 84. Now if the hole is very small this may in a great measure be brought about, though there is no lens made use of: for the smallness of the hole will in some sort answer the purpose of a lens; because it will admit only a slender cone of rays from each point in the object: and when any one of these cones comes to the paper, all the rays of it will be so near to one another as to fall nearly upon one and the same point. Let *abc*, Plat. XI. fig. 3. be an object, *ml* a dark room, and *ik* the diameter of a very small circular hole in the window-shutter. Of all the rays that come from the point *a* only the slender cone contained between *aif* and *akf*, can be admitted into the room, on account of the straitness of the hole through which the light is to pass. The diameter of the base of this cone at the paper will be *f, f*, for all the rays of it are there included within this compass. And when the hole is very small this circular base will necessarily be so too: so that all the

the rays, which come from one and the same point a in the object, though they do not fall exactly upon a single point of the paper, fall upon only a very small round spot: and at this spot the picture of the point a will be represented in the same manner that such a point is represented in the confused picture described in proposition 86, where a lens is made use of. For the same reasons, the slender cone of rays that is admitted through the hole from the point b and falls upon the paper within the compass of a small round spot, the diameter of which is e, e , will paint the picture of that point. And the slender cone, that is admitted from the point c and falls within the compass of a spot having d, d , for its diameter will paint the picture of that point. Thus the picture of each point in the object, and consequently the whole object abc will be represented upon the paper. But then this representation will be confused: because the rays from each point of the object are not collected into a single point upon the paper, but cover a circular spot upon it: by which means the rays of one point will be mingled a little with those from another; the rays f, f , which come from the point a , with the rays e, e , that come from another point, and the rays d, d , with the same rays e, e .

This picture will be inverted in respect of the object. For the rays from a , which is the highest point, as they pass through the hole, will cross those from c , which is the lowest point: and therefore at the paper, after the rays have thus crossed, those which came from the highest point will fall lower than those which came from the lowest point. The rays must cross in this manner; because they describe right lines by proposition 34: and no two right lines can be drawn from a and c into the room ml through so small a hole as ik without crossing each other at the hole. That the rays do thus cross each other, or that those from the highest point a fall lower than those from the lowest point c , would be evident to any person standing within the room. For if his eye was at e, e , directly before the hole he could then see, through so small a hole, only the middle of the object about b : in order to see the highest point of it or a , that is, in order to receive rays from the highest point into his eye, he must sink his eye to f, f , lower than e, e , and in order to see the lowest point or c , he must raise his eye to d, d , higher than e, e .

106. *When there is no lens in the hole, through which the rays are admitted into a dark room, the diameter of the picture, if the distance of the object and picture are given, will be proportional to the diameter of the hole.*

It will be otherwise, as we have seen in proposition 100, if a lens is made use of: for then neither the height nor breadth of the picture will be

be varied by any alteration that can be made in the area of the hole. For when a lens is made use of the height of the picture will be the distance between the two extreme focuses DF , Plat. XI. fig. 1. and these focuses are always in the lines CHD and AHF , or in the axes of two cones of rays, one of which comes from C the lowest and the other from A the highest point in the object. And as these two axes pass through the middle of the lens, the distance between them, when they come to the paper, will be the same whether the area of the lens is great or small. But if there is no lens in the hole, the picture is terminated at the bottom, not by the middle ray or axis of the cone that comes from a , Plat. XI. fig. 3, but by af , which is the lowest ray of that cone; and it is terminated at the top, not by the axis of the cone that comes from c , but by cd , which is the highest ray of that cone. Now if the diameter of the hole ik was made less, it is plain that the lowest ray af would be excluded at k and some other ray, which passes nearer to i than this, would be then the lowest and would terminate the picture at the bottom: for the same reason the highest ray cd would be excluded at i and some other ray, which passes nearer to k than this, would terminate the picture at the top. Thus by diminishing the diameter of the hole the bottom and top of the picture will be brought nearer together, that is, the height of it will be diminished. And the same might be proved of the breadth or of any other diameter of the picture. In like manner by encreasing the diameter of the hole ik the height or any other diameter of the picture will encrease. For when i and k are farther asunder the lowest ray, that passes by k , will fall lower upon the paper than the ray akf , and the highest, that passes by i , will fall higher upon the paper than cid . But the height of the picture is the distance between the lowest and highest ray, when they come to the paper. Therefore the height of the picture is encreased by encreasing the diameter of the hole ik .

107. *The heat at the focus of a burning-glass, when the area of the glass is given, is as the square of the focal distance inversely.*

The burning spot, into which the rays of the sun are collected, is a picture of the sun, as was observed under proposition 84. And the distance of this picture will always be equal to the focal distance of the lens, upon account of the remoteness of the sun, which is the object. Now supposing the heat at the focus to be proportional to the number of rays or quantity of light that is collected there, since the brightness of the picture is likewise proportional to the quantity of light, it follows that the heat will be proportional to the brightness. The distance of the object is here

here given ; for the object is the sun ; and the area of the lens is supposed to be given. Therefore the brightness of the picture is inversely as the square of its distance from the glass, by proposition 102 ; and consequently the heat at the focus, which is always as the brightness of the picture, will be inversely as the square of the focal distance of the glass.

Upon this account of two lenses, which have equal surfaces, that lens will set fire to any thing the soonest which has the shortest focus. Thus if the focal distance of one lens was 1 inch and the focal distance of any other lens, that had an equal surface, was 4 inches, the squares of their focal distances are to each other as 1 to 16 ; and the heat at the focuses will be as those squares inverted, or as 16 to 1 ; that is, the heat at the focus of the former lens will be 16 times greater than the heat at the focus of the latter.

108. *The heat at the focus of a burning-glass, when the focal distance is given, is directly as the area of the glass.*

The heat at the focus is proportional to the brightness of the picture, by what has been said in demonstrating proposition 107. The distance of the sun, which is the object, is given, and the focal distance of the lens, or distance of the picture, is supposed to be given. But in this case the brightness of the picture is as the area of the lens, by proposition 101. Therefore the heat at the focus is likewise as the area of the lens. This is the reason why, if two lenses have equal focal distances, that lens which is the broader of the two will set fire the soonest to any thing that is placed at the focus.

109. *The heat at the focus of a burning-glass is to the common heat of the sun, as the area of the focus to the area of the glass inversely.*

The common heat of the sun is the heat of the direct rays before they are refracted. Now when the direct rays fall upon the glass, they are spread over the whole area of it ; and when they are collected at the focus they are reduced within the compass of the area of the focus. But the brightness arising from the same number of rays is inversely as the area over which they are spread, or the same number of rays when they are spread over a large surface will not make that surface appear so bright, as they would make a smaller surface appear if they were brought closer together and were all of them made to fall upon this smaller surface. Therefore the brightness at the focus is to the brightness upon the surface of the glass, as the area of the focus to the area of the glass inversely.

ly. But since the heat is proportional to the brightness, it follows that the heat at the focus will be to the heat at the glass, or to the common heat of the sun, as the area of the focus to the area of the glass inversely: or as much as the area of the burning spot is less than the area of the glass, so much greater the heat of that spot will be than the common heat of the sun.

Indeed the heat at the focus will be something less than in this proportion, because in proving the truth of the proposition we have supposed that there is the same number of rays at the focus, as there is upon the surface of the lens, that is towards the sun. But this supposition is not exactly true: for though the greatest part of the rays, that fall upon the glass, will pass through it to the focus, yet all of them will not; some few rays, that fall upon the surface of the glass, will be reflected from it, and will therefore never come to the focus.

CHAP. IX.

Of plane vision.

110. When the rays, that come from the several points in any object, enter the eye, they will paint an inverted picture of that object upon the bottom of the eye.

IN order to understand this proposition it will be necessary to give some account of the structure of the eye and of the principal parts that compose it. If an eye, when it is taken out of the head of an animal, is cut by a plane, that is parallel to the horizon and passes through the center of the eye; this plane will divide the sphere or globe of the eye into two hemispheres; and the flat side of either hemisphere is called a section of the eye, and will appear as in Plat. XI. fig. 4. In such a section we have at once a view of all the humours and other principal parts of the eye. ABBA is the white coat, which encompasses the whole globe of the eye, except the fore part of it between A and A. This coat is called the sclerotica. The fore part AA is covered by a transparent coat that is a little more protuberant than any other part of the eye and prevents the figure of it from being a perfect sphere. This transparent coat is called the cornea. The cavity of the eye is divided into two parts by a convex lens CC. This lens consists of a hard transparent substance, and is called the chrySTALLINE humour. It is kept in its place and fixed to the coats by certain ligaments all round it, as ee, which are called ligamenta ciliaria. Under the cornea, and at some little distance from it, is the iris

so,

fo, fo, which has a hole in it at *oo*. This hole is called the pupill of the eye. The iris is tinged with variety of colours; and from the difference of these colours some eyes are called blue ones, others brown, others black. It consists of muscular fibres, which can either contract or dilate the hole or pupill *oo*. So much of the eyes cavity as lies between the cornea and chrystalline humour is filled with a thin transparent fluid like water, which is called the aqueous humour. *eSN, eSN* is a white coat, that consists of the fibres of the optic nerve woven together like a net. This coat is called the retina. Between the sclerotica *ABBA*, and the retina *eSN, eSN*, is another coat, which is called the choroides. The cavity of the eye between the chrystalline humour and the retina is filled with a transparent substance, that is not so fluid as the aqueous humour, nor so hard as the chrystalline: it has much the same consistency with the white of an egg. This is called the vitreous humour. The optic nerve *MBB* is inserted at *N*.

Now the cavity of the eye may be considered as a dark room, and the pupill is the hole through which rays that diverge from the several points of any object *abc*, Plat. XI. fig. 5, pass into this cavity. The rays first pass through the cornea *ik*, which is a meniscus like a watch-glass; and therefore the effect of it upon the rays may be neglected. The fore part of the aqueous humour under the cornea is convex; and this humour being denser than the air will make the several diverging beams from *a* and *b* and *c*, and any other points of the object, diverge less. After the rays, that come from the several points of the object, have passed through the pupill, which to keep the figure distinct is not here represented; they are to pass through the chrystalline humour *gob*, which is a convex lens and will make them converge to as many points upon the retina *lem*, by proposition 83. And consequently at *df* or somewhere upon the white retina, as upon a piece of white paper in a dark room, there will be painted an inverted image of that object, by propositions 84, 85.

This theory is agreeable to fact. For when the sclerotica is carefully taken off, the back part of an animals eye *lem* is transparent. And if the eye thus prepared is placed in the hole of a window-shutter of a dark room with its fore part *ik* towards the objects abroad, a person in the room may see the inverted pictures of those objects through the transparent coats where they are painted.

III. *The optic axis is the axis of the chrystalline humour continued to the object, that we look at.*

The chryſtalline humour is a convex lens, and po , Plat. XI. fig. 5, is the axis of it, by propoſition 19. This axis po continued from o to any point b in the object that we look at is the optic axis.

112. *By the middle of the retina we mean that point of it, upon which the optic axis continued back to it would fall.*

If the optic axis bp , Plat. XI. fig. 5, was continued back to the retina it would fall upon the point e ; and this point we ſhall hereafter call the middle of the retina.

And here we may obſerve that N , Plat. XI. fig. 4, or the place where the optic nerve is inſerted is not in the middle of the retina but a little on one ſide towards the noſe.

113. *The pictures upon the retina are the cauſe of viſion.*

When the picture of an object is diſtinct upon the retina, the object is ſeen diſtinctly; and when the picture is confuſed, the object appears in confuſion. In like manner when the picture is bright, the object is clearly ſeen; and when the picture is faint, the object appears faint. Since therefore viſion ſo much depends upon theſe pictures, that the diſtinctneſs or confuſion of the picture will occaſion diſtinctneſs or confuſion in the appearance of the object, and the different brightneſs of the picture will make a proportionable alteration in the clearneſs or faintneſs of viſion; we may conclude that theſe pictures are the cauſe of viſion, or that by means of theſe pictures, the objects, which are painted upon the retina, are made viſible.

The only difficulty is, how we ſhall be able to know, when an object is ſeen by the eye, whether the picture on the retina is diſtinct or confuſed, bright or faint. For if the eye is in the head of an animal, ſo as to enjoy the uſe of ſight, we have no opportunity of making any obſervations upon the pictures on the retina. And if the eye is taken out of the head, ſo as to give us an opportunity of obſerving the pictures, then the eye has no viſion. So that when there is viſion, how ſhall we know whether the pictures are diſtinct or confuſed, bright or faint? and when we are able by obſerving the picture to know whether it is diſtinct or confuſed, bright or faint, how ſhall we be able to tell what ſort of viſion an animal would have in theſe circumſtances of the picture? Now to this we may anſwer that there is no occaſion to take the eye out of the head of the animal, and to remove the hinder coat, ſo as to ſee the picture upon the retina, in order to know whether it is diſtinct or confuſed, bright or faint. For ſince the picture on the retina is of the ſame ſort with that in

a dark room, and since we know what will make the picture in a dark room distinct or confused, bright or faint, we know likewise what will have the same effect upon the picture on the retina. Thus for instance, if an object is placed too near the lens, the picture upon a paper in a dark room becomes confused; but its distinctness will be restored, if the area of the lens, or hole through which the rays are admitted, is very much contracted, by proposition 99. The same cause will produce the same effect in the picture on the retina: if a book is held too near the eye, the pictures of the letters on the retina will be confused; but the distinctness of them will be restored by looking through a small hole made with a pin in a piece of paper, which is an artificial contraction of the pupill. By this way of reasoning we may find out in other instances when the picture on the retina is distinct and when it is confused. And as the letters are seen in confusion, or vision is confused, when the book is too near the eye, but are seen distinctly, or vision is distinct, if we look through the pin hole; so likewise we should find, in all other instances, that our distinct or confused vision depends upon the distinctness or confusion of the pictures on the retina. When the sun shines upon an object, the picture of it in a dark room, and consequently the picture of it on the retina, will be bright: but if the object is shaded, either of these pictures will for the same reason be faint or obscure, by propositions 104, 105. And we find by our own experience that vision is more clear, when the sun shines upon any object that we look at, and more obscure, when the object is shaded. The same holds good in all other instances. When we have found, by reasoning about the picture on the retina as if it was a picture in a dark room, whether it is clear or obscure, we shall always find by experience that our vision will be clear or obscure accordingly.

114. *That point in any object, towards which the optic axis is directed, is seen more distinctly than the rest.*

The truth of this is confirmed by every ones own experience: if we turn our eyes directly towards one particular part of an object so as to look steadily at it, we may indeed, if the object is not very large, see all the rest of it at the same time, but this part will appear more distinct than the rest: and turning of our eyes towards it or looking at it steadily is turning our optic axes towards it. And indeed when the optic axis *ob*, Plat. XI. fig. 5, is directed towards any point *b* in an object, so that this point is painted on *e* the middle of the retina; if this part of the picture is distinct, the other parts will be a little confused: because the retina in any other place, as *d* or *f*, is nearer to the lens or chrySTALLINE humour

mour than e is; therefore if e is at a proper distance from the chrystalline humour to make the picture distinct there, d or f will be rather too near, and upon that account the picture in these places will be a little confused, by proposition 86.

115. *Objects appear erect, when their pictures on the retina are inverted.*

This is one objection to the account which has just been given of the cause of vision. For if vision is owing to the picture on the retina, and this picture is inverted, we are apt to enquire why the object does not appear in the same situation with the picture. This objection would be of some weight if it was the picture that we saw and not the object. But the picture is not seen at all: it is the instrument by means of which the object is perceived; but it is not perceived itself. When we perceive an object by the sense of feeling, how do we know the situation of it? If for instance we were to shut our eyes and to lay hold on the middle of a stick, that is perpendicular to the horizon, how do we find out which end of the stick is the highest and which the lowest? The way of finding this out is by feeling along the stick, and we know that end to be the highest, which we feel by moving our hand upwards, and that to be the lowest, which we feel by moving our hand downwards. If we ask the same question about the sense of seeing, we may give the same answer. When we perceive an object by the sense of seeing, how do we know the situation of it? how do we find out, which end of it is the highest and which the lowest? The way of finding this out is by looking along the object, which is done by moving the eye and is a sort of feeling with the eye. For since, by proposition 113, that point of an object is seen the most perfectly which we look at directly, we know that part of an object to be the highest, which we see distinctly by turning our eyes upwards, and that part to be the lowest, which we see distinctly by turning our eyes downwards.

There is one remarkable instance of inverted vision, which seems to confirm this account, and cannot well be explained any other way. If any one holds a pin in an erect position so near to his eye that it appears confused; notwithstanding the confusion, the picture of the pin upon the retina is inverted, by proposition 85; and the pin though seen confused will appear erect. But if he turns his face towards a window in order to have a strong light, and holds a paper with a small hole in it, such as a pin-hole, at a little distance behind the pin, so as to be between the pin and the window, the pin will then appear more distinct, and will be inverted.

The difficulty here to be explained is why the pin should appear inverted; since certainly the putting a paper behind it will not alter the position

position of the picture upon the retina? Why therefore, as the pin appears erect, when there is not paper behind it, should it appear inverted, when there is? Let *fg*, Plat. XI. fig. 6, be the paper, *o* the hole, *psn* the pin, *rct* the eye, and *d* a candle, which, instead of the window, casts a strong light through the hole. The light, which comes through the hole *o*, diverges from thence afterwards as from a radiant, so that any two rays as *ao* and *bo*, when they have passed the hole, will diverge from thence in the lines *oe* and *ol*: but the crossing of these rays at the hole, since this happened before they came to the pin, will make no alteration in the position of the picture. But here it is to be observed, that the ray *oe*, which passes by *p* the head of the pin, came from the lowest part of the candle, and the ray *ol*, which passes by *s* the lowest part of the pin that can appear opposite to the hole, came from the highest part of the candle. And as the hole, though it is a small one, is very near the eye, the whole flame of the candle, or however the greatest part of it, will appear through the hole. But it is plain that if the eye is turned downwards to look at the lowest part of the flame, *p* the head of the pin will intercept the rays which come from that part, or will cover that part, and if it is turned upwards to look at the highest part, then *s* the lowest part of the pin will intercept the rays which come from thence, or will cover that highest part. Now the pin is no otherwise visible than as it intercepts the light of the candle, and so, properly speaking, the picture of it upon the retina is nothing but a shadow in the shape of the pin. And since the head of the pin intercepts the light, which comes from the lowest part of the candle, by turning the eye downwards we see the head; and since the lowest part of the pin intercepts the light, which comes from the highest part of the candle, by turning the eye upwards we see the lowest part of the pin. And for this reason the head of the pin will appear the lowest, because we turn the eye downwards to see it, and the bottom of it will appear the highest, because we turn the eye upwards to see it. What is here said of the flame of a candle may easily be applied to a pane of glass in a window, or to any light space beyond the hole: for the head of the pin will cover the bottom of the pane, and the lowest part of the pin will cover the top of it.

From this particular instance and from the common instances of plane vision we may conclude that the position, in which an object appears, does not at all depend upon the position of the picture upon the retina. For in this case the picture is inverted and the object appears inverted; in common cases the picture is likewise inverted, and the object appears erect. But the apparent position of an object, or what point of the object shall

shall appear the highest and what the lowest, depends upon the direction in which the eye must be turned in order to see that point to the best advantage.

116. *An object may appear single, though it is seen with both eyes at once.*

This is a second objection to the cause of vision, as we have just explained it. For since, when we look at an object with both eyes, there is a picture of it upon each retina, it may be asked, if the picture is the cause of vision, why is vision simple, ought we not to see as many objects as there are pictures? And since there are two pictures of every object why does not every object appear double?

In answer to this enquiry, we say that the same object may appear either double or single, and by observing in what position the two eyes are, when it appears double, and in what position they are, when it appears single, we may be led to the true cause of simple vision. If, in common language, both eyes are turned directly towards an object, or, in the language of optics, if the optic axes are directed towards it and concur or meet at the place of the object, the object appears single: but if, when we see an object, one eye only is directed towards it, and the other is directed another way, or if neither eye is turned towards it, but both of them are directed towards something else; in either of these cases the object appears double.

If both eyes are turned directly to the object C, Plat. XI. fig. 7, or, if the optic axes AC and AB concur in this object, it will appear single. But if, whilst one eye is turned towards C, the corner of the other is pressed with the finger so as to turn it another way, then the object will appear double. Or if, whilst both eyes are directed to C, there is any other object D placed at some distance beyond it; the object D will appear double, as the eyes are not directed towards it, but towards something else, which is placed nearer, at C. Or if two objects *c* and *d*, Plat. XI. fig. 8, were placed as before, and both eyes were directed towards the remoter of them *d*, then the nearer, which is at *c*, will appear double, since therefore an object appears single, when the optic axes concur in it, and double, when they do not, we may conclude that simple vision is owing to the concurrence of the optic axes. But in what manner it is owing to this cause is the question: as those persons see objects single, who know nothing of either optic axes or their concurrence. It is true that common people know nothing of these technical terms: but then they know something which is equivalent to what these terms stand for, something which is the true reason why such a concurrence of the optic axes should produce simple vision.

When

When one eye is turned towards an object, and the other is turned a different way, which is the case, if the corner of either is pressed with the finger; then if the same object is seen by both the eyes, it is seen by each in a different direction, or one eye sees it in one place and the other eye sees it in another: this is obvious to every man who has the use of his eyes; and it is as obvious that when an object is seen in two places at once it must appear double. But, if both eyes are directed the same way or to the same place, and that is the place of the object; then, though it is seen by each eye, so that two objects may be said to be seen, yet as both of them are seen in the same place they will appear but as one.

This is so certain, that even when there are really two objects, yet if they can by any contrivance be made to appear in the same place they will look like one. Let C and D, Plat. XI. fig. 9, be two candles. EH FG a piece of paste-board with a hole in it at K. Place the candles at such a distance from each other and from the paste-board that, when the eyes are at A and B, if the right eye B is shut the left eye A may see only the candle D, and if the left eye A is shut the right eye B may see only the candle C. From this supposition it is evident that if both eyes are open both the candles will be seen, one of them by one eye and the other by the other eye. And yet if both eyes are directed to the hole K so as not to attend to the distance of the candles from the hole, then both will seem to be at K, or both of them will be seen in the same place, and then these two candles will appear as if there was only one.

If the eyes were both of them fixed upon the same object C, Plat. XI. fig. 7, any other object as D, which stands beyond it will appear double. For since the eyes see the object D without being turned directly towards it, the place of D is quite indeterminate; only it appears on the right hand side of C to the right eye at B, and on the left hand side of C to the left eye at A: and therefore it appears double because it is seen on both sides of C or in two different places at the same time.

If the eyes were both of them fixed upon the same object *d*, Plat. XI. fig. 8, any other object *c*, which stands nearer than *d*, will appear double. For since the eyes see the object *c* without being turned directly towards it, the place of *c* is quite indeterminate; only it appears to the right eye somewhere in the line *bce*, and to the left eye somewhere in the line *acf*. And because the eyes are not directed towards *c* but towards *d*, no sort of judgment is made about the distance of *c*, but it seems to be somewhere in the same plane *odt* where *d* is. Now these two lines *bce* and

S s

acf

acf cross each other at *c*; so that the line *bce*, which is drawn from the right eye falls on the left hand side of *d* at the plane *ot*, and the line *acf*, which is drawn from the left eye, falls on the right hand of *d*. Therefore the object *c* is seen on the left hand side of *d* by the right eye, and on the right hand side of it by the left eye; and consequently since each eye sees it in a different place, it must necessarily appear double.

From these two last cases we might now draw two general conclusions about double vision; but as they will be of some use to us hereafter and will more easily be referred to, if we make distinct propositions of them, they shall be the subject of the two next propositions.

People, that squint, never direct both their eyes to the object they are looking at: and for this reason if they used both their eyes to look at it, they would see it double. But when one of their eyes is directed towards the object, the distorted eye is turned so far away from it, towards their nose commonly, that by this eye the object is not seen at all or else is seen too confusedly to be taken notice of: so that their single vision arises from their using but one eye at a time. A Person, who can squint a little when he pleases, by a voluntary distortion of his eyes, may easily try this. For if he distorts them just enough to appear to squint to any one that looks at him, but not enough to turn either eye quite away from the objects before him, he will find that these objects appear double.

117. *When an object is more remote than the concurrence of the optic axes, it appears double; and of this double appearance the left-hand one is seen by the left eye, and the right-hand one by the right eye.*

If the eyes are both of them directed to C, Plat. XI. fig. 7, that is, if C is the concurrence of the optic axes; any object D, which is farther off than C, will appear double. This was proved and the reason of it was shewn in proposition 116. Where we observed that D is seen by the left eye on the left hand of C, and by the right eye on the right-hand of C. And as this is the reason why D appears double, it follows that since in this double appearance the left-hand one is seen on the left hand of C it must be seen by the left eye; and for the same reason the right-hand appearance must be seen by the right eye. This will be made farther evident upon tryal. For if the left eye A was shut, the right one continuing open, then the left-hand appearance, would be seen no longer, but D would appear single: from whence it is plain that the left eye, when open, sees this left-hand appearance. In like manner if the right eye B was shut, the left one continuing open, then the right-hand appearance would

would vanish and D would appear single : from whence it is plain that the right eye, when open, sees this right-hand appearance.

118. *When an object is nearer than the concurrence of the optic axes, it appears double ; and of this double appearance the left-hand one is seen by the right eye, and the right-hand one by the left eye.*

If both eyes are directed to *d*, Plat. XI. fig. 8, or if *d* is the concurrence of the optic axes, then this concurrence is farther from the eyes than an object placed at *c*, and the object *c* will appear double ; as was shewn in proposition 116 : to the right eye *b* it will appear at *e* on the left-hand of *d*, and to the left eye *a* it will appear at *f* on the right hand of *d* : so that in this double appearance the left-hand one *e* is seen by the right eye, and the right-hand one by the left eye. This will be farther evident upon tryal. For if only the right eye *b* is shut the left-hand appearance at *e* vanishes ; and if only the left eye *a* is shut, the right-hand appearance at *f* vanishes.

119. *The choroides is the principal seat of vision.*

Let two small circles of white paper be fastened upon a dark-coloured wall at such a height as to be level with the eyes and at the distance of something less than two feet from each other. Then let the spectator place himself at a proper distance (what that proper distance is he will best find by trying.) And if he shuts his left eye and turns his right eye towards the paper on his left hand, he will not be able to see the right-hand paper at all. In like manner if he shuts his right eye and turns his left eye so as to look directly at the paper on his right hand, he will see nothing of the left-hand paper. In either of these cases, in the first of them for instance, when his right eye is directed towards the left-hand paper, notwithstanding the turn of his eye makes it very oblique to the right-hand paper, yet some rays from thence will enter it : for if this obliquity was great enough to prevent the rays from entering it, then certainly no rays could enter it from any other objects which are farther placed to the right than this paper is, and to which consequently his eye is more oblique. But such objects as are so placed will be visible at the same time that the paper is not visible. Therefore rays coming from those objects and consequently rays coming from the paper enter his eye. From whence we may conclude, that as the paper is invisible, though rays, which come from thence, pass into the spectators eye, those rays or the picture of the paper must fall upon such a part at the bottom of the eye as is insensible. The part where they fall is the point N, Plat. XI. fig. 4, where

the optic nerve is inserted. Now the retina, which is spread every where else over the bottom of the eye is not discontinued at N but is spread over that part as well as the rest. Therefore if the retina was the seat of vision the paper would be seen as well, when the picture of it falls upon the point N, as when it falls any where else. But in this place where the nerve is inserted, the choroides, which encloses and touches the retina in all other parts, does not touch it at the base of the optic nerve. From whence we may conclude that though the picture is painted upon the retina, yet the preception of this picture is in the choroides; for where this coat is close to the retina, and can receive impressions from it, there the picture produces vision; but notwithstanding a picture is painted where it is not close enough to receive those impressions, that picture produces no vision.

From hence we may see the reason why the optic nerve was not inserted exactly in the middle of the retina: for if it had, any picture falling upon the middle of the retina would have produced no sensation, and consequently that point of an object, towards which our eyes are immediately directed, instead of being seen distincter than the rest, would have been invisible.

120. *Objects, which are placed at different distances, could not be seen distinctly, unless some change was made in the distance between the chrySTALLINE humour and the retina, or else in the convexity either of the fore part of the eye or of the chrySTALLINE humour.*

If the figure of the eye and the situation of all its parts were to continue always the same at all distances of the objects that we are looking at; the picture of some objects upon the retina would be confused because they are too far off, and of others because they are too near.

For suppose an object was seen distinctly at the distance of 2 feet from the eye; then the picture of it, when it is at this distance, is distinct upon the retina; as was observed under proposition 112. But since, by proposition 111, this picture has the same properties with those that are painted upon a paper in a dark room; if the object was nearer than 2 feet, and the eye did not alter its figure, the picture upon the retina and consequently the appearance of the object would be confused. For, by proposition 87, as the object approaches the cornea, or fore part of the eye, the distinct picture will depart from it: and consequently if the retina does not move but continues at the same distance from the cornea, the picture of the object upon it cannot be distinct. But if either the retina could be moved farther from the cornea, and consequently farther
from

from the chryſtalline humour ; or if the chryſtalline humour was moveable and could be brought forwards ſo as to be farther from the retina ; that is, if the paper was moved farther from the lens, or the lens was moved farther from the paper, the picture, and, in conſequence of that, the appearance of the object would be diſtinct ; by propoſitions 87, 98. In like manner, if the object was at a greater diſtance from the eye than 2 feet, becauſe as the object departs the diſtinct picture approaches, the retina in order to preſerve the representation upon it diſtinct muſt be brought nearer to the chryſtalline humour, by propoſition 87, or elſe the chryſtalline humour muſt be moved nearer to the retina, by propoſition 98. Thus if either the chryſtalline humour or the retina or both of them are moveable, ſo that the diſtance between them may be varied ; objects, which are placed at different diſtances from the eye, may be ſeen diſtinctly.

The ſame advantage of diſtinct viſion, at different diſtances of the object, may be obtained another way, and that is by changing the convexity either of the chryſtalline humour or of the cornea. Where we are to obſerve that changing the convexity of the cornea will neceſſarily change the convexity of the fore part of the aqueous humour which is contained in this coat. The picture upon the retina is made diſtinct by either of theſe means, in the ſame manner as, in propoſition 97, the picture upon the paper in a dark room is preſerved diſtinct, when the diſtance of the object is altered, by changing the lens and uſing one either of a greater or of a leſs convexity, as the diſtance of the object requires. Thus if the object is remote the convexity either of the aqueous humour where the rays ſuffer the firſt refraction, or of the chryſtalline humour where they are refracted again, ſhould be diminished ; that the picture may be thrown back by thus diminishing the convexity, juſt as much as it is brought forwards by encreaſing the diſtance of the object. Or if the object is a near one, then the convexity of one or perhaps of both theſe humours may be encreaſed, that the picture may be brought forwards by thus encreaſing the convexity, juſt as it is thrown backwards by diminishing the diſtance of the object. By ſuch an alteration as this the picture may be preſerved diſtinct upon the retina, though the retina itſelf is immoveable.

Some of theſe changes are neceſſary. But which of them is the change that the eye has a power of making we muſt leave to the anatomists to determine.

121. *The defect of ſome eyes is of ſuch a ſort, that no objects will appear diſtinct unleſs they are very near.*

This

This is the defect in their eyes, who are said to be short-sighted. If an object is at such a distance, that a person with good eyes might see it distinctly, it will appear confused to them; but if the object is at a less distance or is almost close to their eyes, then they are able to see it distinctly. If the eye, either in the fore part of it or in the chrySTALLINE humour is too convex it will have this defect. For the greater the convexity is, where the rays are refracted, the nearer the picture will be to the fore part of the eye, when the distance of the object is given, as appears from what was said under proposition 95. Now if the distinct picture is nearer to the fore part of the eye than the retina is, then the representation upon the retina and consequently the vision will be confused; by propositions 86, 110, 113. But when the object is brought nearer, the distinct picture will be farther from the fore part of the eye, by proposition 87. And therefore, if the object is brought near enough, the distinct picture may by this means be thrown so far back as to fall upon the retina and produce distinct vision.

Some short-sighted persons in reading or in looking at any object hold the book or the object oblique to their eyes, and this helps them to see it more distinctly. For if they hold it directly before their eyes the picture will fall upon the middle of the retina about the point *e*, Plat. XI. fig. 5. But if they hold it obliquely it will fall upon the side of the retina between *d* and *l*, or between *f* and *m*, according as they hold the object on one side or the other. Now the middle of the retina is farther from the fore part of the eye than the side of it is. Therefore though the picture is so near to the fore part of the eye as to be confused, if it was to fall upon the middle, it may be distinct, when it falls upon the side.

Persons, who are not born with this defect, are frequently found to bring it upon themselves by the manner in which they use their eyes. This is remarkably the case of watch-makers, engravers, and painters in miniature, who are obliged to look very near to their work. For when the eye has been long used to that figure, which is proper for seeing near objects distinctly, it settles into that figure, and cannot, as a good eye may, be altered at pleasure so as to see any objects distinctly that are not near to it. For the same reason students, if they hold their book too close to their eyes, will be apt to grow short-sighted.

From hence we may observe that this defect does not always mend as a man grows older: for by looking long and frequently at very near objects, whilst he is young and his eyes are capable of being fixed in any shape by use and custom, he will keep himself short-sighted, if he was so at first, or will make himself short-sighted, though he had naturally good

good eyes. This observation will point out a way either to remedy this defect or to prevent it, if our employment is of such a sort as to allow us to follow it. If those, who are short-sighted, instead of using the artificial helps of glasses, will frequently look at remote objects, and in reading would hold their book as far from their eyes as they possibly can; the eye will change its figure by degrees and by custom will acquire the power of adapting itself so as to see distant objects more distinctly. And good eyes may be prevented from becoming short-sighted, if in reading we are careful not to hold the book too close to our eyes, and use ourselves to look at distant objects as well as at near ones.

122. *The defect of some eyes is of such a sort that no objects are seen distinctly, unless they are at some distance.*

Old people are most liable to this defect, and those more especially who have been used to look much at very distant objects, and seldom at near ones, such as travellers, sailors, and sportsmen. For as the eye grows flatter, either by age or by use, the rays of each pencil in passing through it are made to converge less, and the place of the distinct picture, if the object is near, will be more remote than the retina, and consequently the picture upon the retina will be confused; by proposition 87. But if the object is removed farther from the eye, this will bring the distinct picture nearer to the fore part of the eye; and by removing the object to a proper distance, the distinct picture may be brought near enough to fall upon the retina.

It is sometimes impossible to prevent this defect by any care whatever: for age will sometimes flatten the eye notwithstanding all our endeavours. But it may frequently be prevented by using ourselves to look often at near objects as well as at remote ones. For this will prevent the eye from settling to any particular figure. And indeed the muscles, which are used for changing the figure of the eye in order to see objects at different distances, will best retain their use in all instances by frequent exercise; so that by accustoming ourselves sometimes to look at very near objects and sometimes at very remote ones, we may commonly preserve our eyes, and neither make ourselves short-sighted, when we are young, nor be long-sighted, when we are old.

123. *An object may be either so near or so remote, that the best eyes would be unable to see it distinctly.*

Though the eye has a power of changing its figure so as to see objects distinctly at different distances; yet this power is a limited one. If
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the object is either too near or too remote, the eye cannot change itself enough to make the picture distinct upon the retina. And for this reason near objects or very remote ones will appear a little confused to the best eyes. To see an object very distinctly the distance of it should not be less than about 7 or 6 or at the least 5 inches.

124. *The optic angle in viewing any object is an angle at the eye subtended by the diameter of the object.*

The angle aoc , Plat. XI. fig. 5, which in viewing the object abc is subtended by ac the diameter of this object, is called the optic angle.

125. *The apparent diameter of any object is proportional to the diameter of that object's picture upon the retina.*

The apparent diameter of an object is a quantity of visible extension, that lies between the two extremities of that object. Thus the apparent diameter of an object abc , Plat. XI. fig. 5, to an eye placed at o is that visible extension, which lies between the two extreme points of it a and c . But if two lines ao and co are drawn from these points so as to cross one another at the eye, these two lines will be the axes of the extreme pencils, one of which comes from a , and the other from c ; and each point will be painted upon the retina in these two lines continued thither, as was observed in proving proposition 85. From hence it follows that the apparent diameter of the object is a visible extension contained within two right lines ao and co , which are drawn from the extremities of the object and cross each other at the eye; and df the diameter of the picture upon the retina is likewise contained between the same right lines continued to the retina. Therefore the angle aoc , which is subtended by the object, is equal to the angle dof , which is subtended by the picture: for they are vertical angles made by two right lines crossing each other. Euc. b. I. prop. 15. But if these two angles are always equal, one of them can neither encrease nor decrease but the other will encrease or decrease in the same proportion: thus if aoc was to encrease so as to be double what it is, dof would likewise be doubled, if aoc was to be tripled, dof would be tripled; or if aoc was to be diminished one half, dof would be halved at the same time, if aoc was reduced to $\frac{1}{3}$ of what it is, dof would be reduced to $\frac{1}{3}$. Now the visible extension contained between ao and co will be greater as the distance between these lines is greater, or less as the distance between them is less; that is, the apparent diameter of the object will be proportional to the angle, which these two lines make with another at the eye, or to the angle aoc . And the diameter of the picture
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upon the retina which is contained between do and fo will be greater as the distance between these lines is greater, and less as the distance between them is less; that is, the diameter of the picture will be proportional to the angle which these lines make with each other at o , or to the angle dof . But the angle aoc is always equal and consequently proportional to the angle dof . Therefore the apparent diameter of the object is proportional to the diameter of its picture upon the retina.

126. *When the diameter of an object is given, its apparent diameter is inversely as its distance from the eye.*

The diameter of the same object does not appear equal, when it is viewed from different distances; but the apparent length of it will be inversely as the distance of the eye from it, greater in proportion as this distance is less, and less in proportion as this distance is greater. For the apparent diameter of an object is proportional to the diameter of its picture upon the retina, by proposition 125. And when the diameter of the object is given, the diameter of the picture, like that of a picture in a dark room will be inversely as the distance of the object, by propositions 91, 113. Therefore the apparent diameter of the object is likewise inversely as the objects distance.

What has here been proved of the apparent diameter or apparent length of an object is applicable to any other apparent length whatsoever. Thus if AE and BF , Plat. XI. fig. 10, are two parallel rows of trees; to a spectator standing at G the two opposite trees E and F at the farther end of the row will seem nearer to one another than the two opposite trees A and B , which are at the end next to him. For the apparent lengths AB , CD , EF , or the apparent distances of any two opposite trees, will be inversely as the spectators distance from them. And upon this account the two rows, though they are parallel, will seem to converge. In like manner if AE was the cieling and BF the floor of a large room, the distance between the floor and the cieling will appear to be less at E and F than at A and B , if the spectator stands at G . Or, if GH an imaginary line is conceived to be drawn parallel to both the floor and the cieling; the cieling will converge downwards and the floor upwards towards this line. Now this apparent rising of the floor towards the line GH would evidently be the same whether there was any cieling or not. Let us therefore suppose that BF , instead of being the floor of a room, was the surface of the sea; this surface would apparently rise towards the imaginary line GH . And this is the reason why to a spectator upon the shore the surface of the water, as he looks out to sea, seems gradually

to ascend, and the ships out at sea seem to be higher than the land that he stands upon. Or if BF, instead of being a horizontal plane such as the floor of a room or the surface of the sea, was the tower of a church or any high building perpendicular to the horizon; then if a spectator stood at the bottom or at G, and we conceive an imaginary line GH drawn from his eye parallel to the building, however upright the building may stand, it will apparently converge to this imaginary line, and therefore will seem to lean towards that side, where the spectator is.

From this proposition it follows that the apparent diameter of an object when we see it with the naked eye, may be magnified in any proportion that we please. For, since the apparent diameter is inversely as the distance of the eye, in whatever proportion we have a mind to encrease the apparent diameter we need only diminish the distance of the eye from the object in the same proportion. Thus suppose there is an object AB, Plat. VIII. fig. 1, which to an eye at E appears under or subtends the angle AEB; we may magnify the apparent diameter in what proportion we please by bringing our eye nearer to it. If for instance we would magnify it in the proportion of FG to AB, that is, if we would see the object under an angle as great as FEG, or would make it appear of the same length that an object as long as FG would appear, this may be done by coming nearer to the object. For the apparent diameter is as the distance inversely; therefore, if CD is as much less than CE as FG is greater than AB, by bringing the eye nearer to the object in the proportion of CD to ED, the apparent diameter will be magnified in the proportion of FG to AB. So that the object AB to the eye at D will appear as long as an object FG would appear to the eye at E. In the same manner we might shew that the apparent diameter of an object, when seen by the naked eye, may be infinite. For since the apparent diameter is reciprocally as the distance of the eye, when the distance of the eye is nothing, or when the eye is close to the object at C, the apparent diameter will be the reciprocal of nothing or will be infinite.

There is indeed one great inconvenience in thus magnifying an object without the help of glasses only by placing the eye nearer to it. The inconvenience is, that we cannot see an object distinctly unless the eye is about 5 inches from it: therefore if we bring it nearer to our eye than 5 inches, however it may be magnified, it will be seen confusedly. Upon this account the greatest apparent magnitude of an object, that we are used to, is the apparent magnitude, when the eye is about 5 inches from it. For though by bringing it within 1 inch of our eye the apparent diameter would be 5 times as great, or at the distance of $\frac{1}{4}$ inch would be

20 times as great, or at the distance of $\frac{1}{10}$ inch would be 50 times as great; yet since in these near approaches of the eye the object will be seen in great confusion, we never place an object so near us in order to look at it: because though it might be magnified by this means, the confusion would prevent us from making any use or advantage from seeing the object so large. The size of any object, when it is viewed through a convex lens, seems to be extraordinary; not because it is impossible to make it appear of the same size to the naked eye, but because in so near an approach of the naked eye, as would be necessary for this purpose, it would appear very much confused; for which reason we never do bring our eye so near to it: and consequently, as we have not been commonly used to see the object of this size, we think it an extraordinary one.

127. *The apparent diameter of an object is directly as its real diameter, when its distance from the eye is given.*

The apparent diameter of an object is proportional to the diameter of the picture of it upon the retina, by proposition 125. And the diameter of the picture upon the retina, when the distance of the object is given, is proportional to the real diameter of the object, by propositions 93, 113. Therefore the apparent diameter of an object, whose distance is given, is proportional to its real diameter.

128. *The apparent diameters of different objects at different distances from the eye will be equal, if their real diameters are proportional to their distances.*

If a man 6 feet high is at 40 yards distance from us, and a boy 3 feet high is at only 20 yards; then their heights are proportional to their respective distances for 6 is to 3 as 40 is to 20. In this and in all other instances of the same sort, their apparent heights will be equal. This follows from propositions 94, 125. For the diameters of their respective pictures upon the retina will be equal; and their apparent diameters are proportional to the diameters of these pictures.

129. *The apparent diameter of an object seen obliquely, is proportional to the apparent length of a subtense of the optic angle perpendicular to the optic axis.*

We have hitherto supposed that the object is seen directly, or that the optic axis *ob*, Plat. XI. fig. 5, is perpendicular to the object *ac*. But let the object *AB*, Plat. XI. fig. 11, be seen obliquely, that is, let it be inclined to the optic axis *OC*; and then if *AOB* is the optic angle the ap-

parent diameter of the object will be proportional to the apparent length of DB, which is a subtense of AOB perpendicular to OC the optic axis. For since the apparent diameter of the object is the visible extension which lies between A and B, or which is included between AO and BO the two legs of the optic angle, by proposition 125; and since the subtense DB is included between AO and BO; it follows that the visible extension or apparent length of this subtense is proportional to the apparent diameter of the object. Or otherwise. If AB is an object seen obliquely, and DB is an object seen directly; that is, if AB is oblique to the optic axis and DB is perpendicular to it, then supposing both of them to subtend the same angle AOB, their pictures upon the retina and consequently their apparent diameters will be equal, by proposition 125. That their pictures upon the retina will be equal is evident: for the same right lines BO and ADO continued to the retina determine the length of both their pictures, as appears from what was said in proving proposition 125.

From hence we see that an object appears shorter for being seen obliquely. For though the object AB is longer than DB, yet because AB is seen obliquely and DB is seen directly, AB appears no longer than DB. If AB was more oblique it would appear still shorter. And if it was so oblique as to lie strait out from the eye, so that the end B was towards the eye and the end A was in the same right line directly from it, the apparent length of BA would then be nothing. For in this case a line AO drawn from A the farther end of the object to O the eye, and a line BO drawn from the nearer end B to O would coincide, so that the angle AOB contained between them, and consequently DB the subtense of that angle, would be nothing.

130. *When equal objects are seen very obliquely, the apparent lengths of them are inversely as the squares of their distances from the eye.*

Let the eye be at O, Plat. XII. fig. 1, and any where in the line BC at a considerable distance from the eye take equal spaces DF and df . The apparent lengths of these equal spaces DF and df will be inversely as the squares of their distances, that is, they will be as the reciprocals of the squares of those distances. Thus if Of is double OF, or if the distances are as 2 to 1, the apparent length of df will be $\frac{1}{4}$ the apparent length of DF. If Of was to OF as 3 to 1, the apparent length of df would be $\frac{1}{9}$ that of DF. If their distances were as 4 to 1, their apparent lengths would be as $\frac{1}{16}$ to $\frac{1}{1}$.

The apparent length of DF is proportional to the apparent length of GD, by proposition 129. For DF is seen obliquely, OR is the optic axis,

axis, DOF is the optic angle, and GD is a subtense of this angle perpendicular to OR. For the same reason the apparent length of df is proportional to the apparent length of gd . But if Of is double OF than gd is $\frac{1}{2}$ GD, if Of is triple OF, gd is $\frac{1}{3}$ GD, and universally, gd is to GD as Of to OF, or as the distances of f and F from the eye inversely. In order to prove this, we must observe that GD is the subtense not only of the angle DOF, but likewise of the angle GFD; and gd is the subtense not only of the angle dOf , but likewise of the angle gfd . Now if Of is double OF, the angle OfF is $\frac{1}{2}$ the angle OFB. For since OF f is a plane triangle the sides are to one another as the sines of the opposite angles, by proposition 32, or Of is to OF as the sine of the angle OF f to the sine of the angle OfF . But OF f and OFB have the same sine, because one of these angles is the complement of the other to two right ones. Therefore Of is to OF as the sine of the angle OFB to the sine of the angle OfF or OfB , since OfF and OfB are one and the same angle. Now Of , by the construction, is double OF, and consequently the sine of the angle OFB is double the sine of the angle OfB . But small angles are to one another nearly as their sines, by proposition 26. Therefore if these are small angles OFB will be double OfB ; and the subtense GD will be double gd , or GD will be to gd as 2 to 1; that is, since OF is to Of as 1 to 2, GD will be to gd as OF to Of inverted; or the subtenses to which the apparent diameters of DF and df are proportional will be inversely as the distances of DF and df from the eye, upon account of the different obliquities at which the eye sees these two equal spaces DF and df . But then besides this, without considering the difference of obliquity, or if their obliquities were the same, the apparent length of DF is to the apparent length of an equal line df , as their distances inversely, by prop. 126. Since then the apparent lengths of DF and df are inversely as their distances upon account of their different obliquities; and are likewise inversely as their distances upon account of their different distances; it follows that upon both accounts together they will be in a ratio compounded of the inverse one of their distances, or will be inversely as the respective distances multiplied into themselves, or inversely as the squares of their respective distances. Thus if Of is twice OF or if Of is to OF as 2 to 1, the apparent length of df will be to the apparent length of DF in the compound ratio of $\frac{1}{2}$ to $\frac{1}{1}$ and $\frac{1}{2}$ to $\frac{1}{1}$ or as $\frac{1}{4}$ to 1. For df appears but $\frac{1}{2}$ as long as DF upon account of their different obliquities, and but $\frac{1}{2}$ as long upon account of their different distance, therefore, the apparent length of df being twice halved in respect to DF, upon both accounts together it will be only $\frac{1}{4}$ of the apparent length of DF.

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From whence we see the reason, why in walking through a double row of trees, the apparent distance between the two nearest trees in the same row encreases much faster than the distance between the two opposite trees in the different rows. Thus, Plat. XI. fig. 10, if trees were planted in a row along the line ACE, and other trees along the line BDF, any person walking between the rows in the direction GH will see the apparent distance DF between the two trees D and F, that are nearest to one another in the same row, encrease much faster than the apparent distance EF, which lies between two opposite trees in the different rows. For the distance EF is seen directly and consequently the apparent length of it encreases, when the spectator approaches it, in the same proportion that his distance from it decreases; for the apparent length of it is inversely as his distance, by proposition 126. But, by the proposition now before us, the apparent length of DF which is seen obliquely encreases in the same proportion that the square of the spectators distance decreases, or is inversely as the square of his distance. Therefore since the square of the spectators distance decreases faster than the distance itself, the apparent length of DF will encrease faster than the apparent length of EF. If the reader is not aware that the squares of a quantity decrease faster than the quantity itself, he may see it in the following instance. Suppose a spectator was walking in the direction GH, his distance then from EF and from DF would keep decreasing: suppose him to view EF and likewise DF at three several distances, and let these distances be to one another as 3, 2, and 1; then the squares of these distances will be 9, 4, and 1: and it is evident that these latter numbers decrease faster than the former. Now the apparent length of EF is as his distances inverted; and the apparent length of DF is as the squares of his distances inverted. Therefore at the distances 3, 2, and 1, the apparent length of EF will encrease in the proportion of 1, 2, and 3; and the apparent length of DF will encrease in the proportion of 1, 4, and 9. But these latter numbers encrease faster than the former; and consequently the apparent length of DF will encrease faster than the apparent length of EF.

131. *The apparent diameter of an object is not at all changed either by contracting or dilating the pupil.*

When the distance of the object is given, the diameter of its picture upon the retina will be the same, whether the area of the pupill is large or small, by proposition 100. And consequently the apparent diameter of the object will be the same, whatever is the area of the pupill, by proposition 125.

132. *If all other circumstances are the same an object will appear larger when it is seen confusedly than when it is seen distinctly.*

When an object is seen confusedly the picture upon the retina is confused, by proposition 113. But the confused picture is bigger than the distinct picture of the same object would be, when all other circumstances are the same, by proposition 89. And the apparent magnitude of the object is always proportional to the magnitude of its picture upon the retina. Therefore any object, when it is seen confusedly, will appear bigger than when it is seen distinctly.

This is the reason why any object, as a man, or a house, or a tree, in very misty weather should seem larger, than they do when the weather is clear. For when the rays that are to make the objects visible are to pass through so many small drops of water, as they do in misty weather, such a great variety of surfaces will so differently refract them, that those of the same pencil will not be collected into one and the same focus upon the retina; and this will make the picture at the retina a confused one.

Upon this principle any luminous body, when the picture of it upon the retina is confused or imperfect, should appear larger, than when that picture is distinct or perfect. And the less perfect the picture is, so much the larger the body ought to appear. Now when the sun or the moon are in the horizon, that is, when they are either rising or setting; all the rays, which come from them, pass along very near the surface of the earth, through all the vapours that float in the lower part of the atmosphere; whereas when they are in the meridian or at their greatest height, the rays come more directly to the spectators eye without passing so close to the earths surface, or without passing through so many vapours or so much mist. For this reason the picture of the sun or moon upon the retina, when they are in the horizon, will be less perfect and consequently will be bigger, than when they are in the meridian; and this makes them appear larger at rising and setting than when they are at their greatest height. The apparent magnitude of the sun or moon is the same, when they are in the horizon, as when they are in the meridian, if they are seen through a telescope: for the picture, which is imperfect when they are viewed by the naked eye, is made a perfect one by the refraction of a telescope: and if the difference of apparent magnitude in the horizon and meridian was owing to the imperfection or confusedness of the picture, then, by making the picture perfectly distinct, the apparent magnitude ought to be the same. From hence we may see the reason, why the horizontal moon should at some times appear much larger than

at others : for the more imperfect the picture is the larger the moon will appear ; and the picture will be more imperfect, when the rays have a thicker mist to pass through, that is, when the lower parts of the atmosphere are filled with a greater quantity of vapours. Whether the cause here assigned is the only one, upon which the difference of apparent magnitude of the sun or moon in the horizon and in the meridian depends, is more than I will venture to affirm : but perhaps we may find as we go on that the principal causes which have commonly been assigned, are at least as unsatisfactory.

133. *Bright objects seem to be larger than obscure ones, though they are really of the same size.*

This is remarkably the case in the appearance of the moon, just before and just after the new. At that time the whole circular face or disk of the moon is visible ; but only one part of it, which is the crescent or horns, is bright ; the rest of the circle has barely light enough to make it visible and is very faint in comparison of the crescent. Now though the crescent and the faint part of the disk are portions of the same circle, yet the crescent seems to be a portion of a greater circle than the faint part. There are many instances of the same sort : and it is well known that a white or light-coloured object will appear larger than a black or dark-coloured one, more particularly if they are at some distance from the eye.

When an object is remote enough from the eye to make the picture of it upon the retina a little confused, the rays of each pencil, instead of being collected into a single point upon the retina, are scattered over a small circular spot upon it ; and a picture, that is made up of such spots, will be something larger than one, that is made up of exact focuses or points, as we observed in proposition 89. Upon this account, by proposition 130, all objects whether bright or obscure will appear the bigger for being seen a little confusedly. But we are to assign a reason why this should be more perceivable in bright than obscure ones. Now the edges of a confused picture will always be fainter than any other part of it. Thus for instance, if the picture *df*, Plat. XI. fig. 5, was confused, the ends of it at *d* and *f* would be fainter than the middle of it at *c*. The rays indeed of the middle pencil, which should be all of them collected into one and the same point at *e*, are spread over a small circular spot there of which spot *e* is the center : and for this reason any part of that spot will be less enlightened, and consequently will be fainter than the point *e* would have been, if they had all been collected into that single point. But then

as the rays of this pencil are thus scattered, so likewise are the rays of each pencil all round it; so that some rays of all these pencils, when they are thus spread upon the retina, will fall upon the same part of it with those of the middle one: and consequently what the middle spot would want in brightness, if it had been enlightened by no rays but those of the middle pencil, will be made up by the other rays, which are extended from the neighbouring pencils and fall upon the same place. This is equally true of any other part of the picture as well as of the middle, except only its edges. For at the two ends *d* and *f*, or any where else at the edge, the rays of each pencil are scattered, and spread themselves over a small circular spot, instead of being collected into a single point. The picture extends no farther upon the retina than the extreme focuses, when it is distinct; but it is extended as far as the extreme spots reach, when it is confused. But as all the rays of any pencil will enlighten a single point more, than they will a circular spot which is bigger than a point, the edges of a confused picture will necessarily be fainter than the edges of a distinct one: unless where the rays of the neighbouring pencils are spread over the same space with those of the extreme ones; which, as we have already seen, is the case in the middle of the picture. But at the edges this is impossible: because there are no other pencils beyond the extreme ones. Therefore the edges of the confused picture will be fainter than the middle or than any other part of it. Now if the object is a bright one, the edges of the confused picture, though they are faint, will however be bright enough to be perceived; and consequently the object will appear larger, than if the picture had been distinct. But if the object is very obscure, the edges of the confused picture will be too faint to be perceived, and the object will scarce appear larger than if the picture had been distinct. Therefore a bright object will seem bigger, when seen a little confusedly, than an obscure one of the same size.

This is no objection to the account that was given of the horizontal moon. For though the moon, when in the horizon, is not so bright, as when in the meridian; yet in the horizon it is bright enough for the whole confused, or rather imperfectly distinct, picture upon the retina to be perceived.

134. *Objects would appear equally bright at all distances, if no rays were intercepted in their passage from them to the eye.*

This follows from propositions 103, 113. For the brightness of vision depends upon the brightness of the picture upon the retina: and this picture would be equally bright at all distances of the object, if no rays were intercepted in coming from the object to the eye of the spectator.

The sun and the moon appear fainter when they are rising or setting, than they do when they are at their greatest height. This faintness is not owing to the remoteness of either of them, as has been already proved. But when they are in the horizon, the light that comes from them passes along near the surface, where the vapours are thick and will intercept many of the rays: whereas when they are in the meridian, the light comes to the eye more directly, and does not pass so far through the lowest, which are also the densest, parts of the atmosphere, and consequently fewer rays will be intercepted.

Some have imagined that this faintness of the moon in the horizon may be the reason, why it should seem bigger, when it is there, than it does, when it is in the meridian. Because when an object is faint, the pupill in looking at it is dilated, and this, as they affirm, will make the picture upon the retina so much the bigger. But this has already been shewn to be a mistake, since we have proved, in proposition 131, that the apparent diameter of an object is not changed either by contracting or dilating the pupill. To this indeed it is replied, that the merely contracting or dilating the pupill will not alter the diameter of the picture upon the retina, and consequently will not alter the apparent diameter of an object; yet when the pupill is dilated the consequence is that the ligaments *e* and *c*, Plat. XI. fig. 4, are drawn outwards, or towards *A* and *A*, by which means the chrystalline humour will be brought forwarder or will be removed farther from the retina; and when the pupill is contracted this humour will be removed nearer to the retina: and the farther the chrystalline humour is from the retina so much bigger the picture will be, or the nearer this humour is to the retina, so much less the picture will be, by proposition 95. Now if this was true, if a dilated pupill made the picture larger and a contracted one made it smaller, all faint objects, which are looked at with a dilated pupill, ought to appear larger than bright objects of the same sort, which are always looked at with a more contracted one: but since we have seen, in proposition 133, that just the reverse of this happens, we may conclude that it is not the contraction or dilatation of the pupill, nor any consequences of such contraction or dilatation, which makes the moon appear bigger in the horizon and less in the meridian.

135. *The judgment that we make about the distance of objects does not depend upon any single principle.*

The distance of an object is the extension that lies between the eye and the object, or is the length of a line drawn from the eye to the object. Thus

Thus, Plat. XI. fig. 10, the distance of EF the end of a room from the spectator at G is the extension that lies between G and EF or is the length of the imaginary line GH. Now this line is invisible in itself, or if it was not so but was a real line, such as a pack-thread, extended from G to H, yet the length of it would be invisible from its situation, by proposition 129; because the end of it or G is at the eye and it is drawn directly from thence in the direction GH. Therefore the distance of EF, if we look at EF directly, without considering any thing else, is not to be seen or is not the object of sense. But though GH is invisible, yet BF the floor of the room, or AE the cieling of it, or the walls on each side, are any of them visible extensions, which lie between the eye at G and the end of the room EF, therefore by looking at any of these we may judge of the distance of EF or may make this distance in some sort the object of sense. By the same method we perceive or judge of the distance of other objects in many instances. The visible length of road, which is between us and the place we have in view, is our apparent distance from that place. The visible length of the fields, hedges, woods, hills, and buildings, between us and a house at the end of the prospect, is the apparent distance of that house from us.

This method of estimating apparent distances will frequently fail us and some other must be made use of. When the distances of objects are very great, the farther parts of these distances are invisible. The apparent magnitudes of the remoter parts of long lines decrease very fast, by proposition 130; in the line BC, Plat. XII. fig. 1, the apparent magnitudes of DF and *df* are inversely as the squares of their respective distances from the eye; and in all other cases the apparent length of any given part in the same line BC continued will decrease as fast as the square of its distance from the eye encreases. But since this quick decrease of apparent magnitude makes the remoter parts of long lines invisible on account of the smallness of the angles which they subtend; it follows that though two remote objects are at different distances, yet the eye cannot perceive this, because the difference of their distance may subtend too small an angle for the eye to discern it. This is the reason why a person, who is unused to a sea-prospect, and has not by habit acquired some other method of judging how far an object is from him, besides this of looking at the apparent space that lies between him and the object, will be much mistaken in his judgment of distances. A ship that is eight or ten miles from the shore, will scarce seem to him to be a mile off. For at land, when we have an extensive prospect we commonly stand upon some eminence, and the country about us from the valley beneath or from the foot of the eminence rises gradually; so that there the variety of

of visible objects rising one above another in the prospect helps us to perceive and observe the intermediate extension. But upon a continued uniform level, as the sea is, we have no such helps: and if we attend only to the length of a visible space lying between us and the object, we shall be misled in our judgment: because the same distance, when seen upon an uniform level, will appear much shorter than it used to do at land. This method of judging how far an object is from us will likewise fail us at land, when there are hills between us and the object, which do not rise one above another, so that neither the valleys between them are seen, nor most of the hills which lie beyond the first of them. If at the end of a flat prospect there are two hills one much farther off than the other but of nearly the same apparent height, and a wind-mill stands upon each of them; the eye could not perceive at once which of the two wind-mills was the most remote: for the space that lies between them cannot be seen, because the nearer of the two hills will hide it. In any of these cases we find out the distance of an object, or rather the difference between the distances of different objects, by reasoning upon it. How we reason about it and form our judgment will best appear from considering what answer a man would make, who can tell which of two ships out at sea or which of two wind-mills in the situation last described appears to be the farthest off, if he was to be asked why he thinks so. He might say that one of the ships seems less than the other, and that he judges that to be the nearer of the two, which appears the bigger. This would be a rational answer. But the reader must observe that this principle of judging might fail or might mislead the spectator: for if the ships were of different sizes, and the remoter of the two was the larger, it might appear as big or bigger than the nearer, by propositions 126, 127, 128. There is another thing worth taking notice of in this principle of judging, and we should be led to it by asking the spectator why he thought either of the two ships was very far from him. His answer would probably be that even that which appears the bigger of the two seems to be very small. From this answer it is evident that he has in his mind a notion of the real magnitude of a ship, and that he has been used to see it at different distances: so that since it appears much smaller than it really is and than he has been used to see it, when it was near to him, he concludes that it must be at a great distance. Upon this account he will make false judgments of the distance of such objects, as he is before unacquainted with and has not frequently viewed at different distances, if he depends wholly upon this principle. In travelling towards a large city, or a castle, or a cathedral church, or a mountain larger than ordinary, if he has not been used to see such large buildings

buildings or so large a mountain, he will think himself nearer to them than he is. But when this principle fails, or even in instances when we might make use of it, we sometimes have recourse to another. For if the spectator was to be asked, in the case just now proposed, why he thinks that one of the wind-mills is nearer to him than the other, it would be a very rational answer to say he can see all the parts of one more distinctly than of the other; and since by experience he has found that remote objects appear more confused than such as are nearer, he concludes that the more distinct one of the two is nearer than the other. If the spectator was to change his place he might judge which of the two was the more remote, and likewise whether any single object was near to him, or far from him, by the parallax of the object, that is, by the change of the objects apparent place, whilst he has been changing his station. Let L, Plat. XII. fig. 2, be one object and H another object more remote than L from the spectator. Then if the spectator is at A the remoter object H will be seen in the line FA, and the nearer object L will be seen in the line DA: and supposing that AD is on the left-hand side of AF the nearer object L will appear on the left-hand side of the remoter H. But if the spectator changes his place and removes to B, then the nearer object seen from this station will appear in the line BC. and the remoter in the line BE; so that the nearer one L will be on the right-hand of the remoter one H. To those, who are used to judge of distances, this change of the apparent place of objects, when viewed in different stations, is as sure a principle as any: for such objects, as by altering the spectators station change place the most, are always the nearest, and such as change it the least are always the most remote. There is yet another principle, besides those already mentioned, by which we sometimes make a judgment of the different distances of remote objects; and that is their apparent faintness. When a man sees two mountains at a great distance from him, though he cannot measure with his eye the visible extension, that lies between him and either of them; though he is unacquainted with their real magnitude and therefore cannot judge of their distance by their apparent magnitude; though they both of them appear confused; and though he has no opportunity of changing his station so as to try which has the greater parallax; yet if one of them is so faint as to look almost like a blue cloud in the horizon, and the other is something brighter, he would scarce be at a loss to determine which is the remoter of the two.

All these principles, except one, are observed in painting a landscape. The parts, that are to appear the farthest off, are drawn of a less size,
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are made more confused and more faint, than those that are to appear nearer. But where objects at different apparent distances are to be represented upon the same plane surface of the canvas, it is impossible to give them a different parallax, or to make those, which are nearer, change their apparent place in respect of those, which are more remote, from the spectators changing his station. And indeed as a person looking at a landscape does not change his station much, such a parallax in the several parts of the picture is neither expected nor wanted.

Now we have explained the several principles, by which we judge of distances, when the objects are very remote, we may the better understand how we judge of distances when the objects are nearer. I chose this method, because when objects are very remote we have not such frequent occasion to judge of their distances, as we have when they are nearer. Upon this account we are not much practised in judging of the distances of remote objects, and therefore the mind proceeds more slowly for want of frequent exercise and habit, the several steps that it takes are more evident, the principles, upon which it reasons, are more easily discerned, and the conclusion of the judgment can seldom be mistaken for the perception of sense. Whereas, when objects are nearer to us, though we judge of their distance by nearly the same principles, yet by frequent exercise the mind is too quick in the several steps that it takes for those principles to be discerned and attended to: so that the conclusion of the judgment being made at once is commonly imagined to be a perception of the senses; we imagine that we immediately see the distance of an object, when in fact we have determined it by a deduction of reasoning; but by such a deduction as frequent practice has made so obvious and familiar to us that we have no sooner turned our eyes towards an object, but we can tell at the first view whether it is near to us or far from us.

If an object is at no great distance from us, and we attend to the visible extension between us and that, such as the floor of a room, the walk of a garden, the breadth of a bowling-green, we may then in some sort be said to see the objects distance. But if it is held before the eye in such a manner as to be detached from all visible extension between the eye and that, if we look so directly at the object as either not to see or not to attend to any thing else, then we must judge of its distance upon some other principles. We judge of the distance of objects by their apparent confusion, when they are near enough to appear confused. When we know beforehand that two objects are very remote from us, and one appears more confused than the other, we conclude that the more confused

fed one is the more remote one of the two. But if this principle is made use of in one case, it seems likely that we should make use of it in another. And when we know beforehand that an object is near to us, and by that means are convinced that its apparent confusion cannot be owing to its remoteness but must be owing to its nearness, we have a good reason for concluding, that those near objects, which appear most confused, are at a less distance than those, which are seen more distinctly. We judge of the distance of remote objects from their parallax, or their change of apparent place by the spectators changing his station. Upon the same principle we may judge of an objects distance when it is nearer to us, without moving out of our place. For if the distance between the two eyes bears any proportion to the distance of the object, then one eye views it from one station and the other views it from a different station; so that by opening each eye alternately a parallax or change of apparent place will be produced in the object. If *c*, Plat. XI. fig. 8, is an object at such a distance from us that the distance between the two eyes *a* and *b* bears some proportion to it, and if *d* is an object a little farther off, then by shutting the left eye at *a* the object *c* will appear to the right eye in the line *be*, on the left-hand side of *d*, and by shutting the right eye at *b* it will appear to the left eye in the line *af* on the right-hand side of *d*. This change of apparent place, when each of the two eyes views the object *c* from a different station, is a certain mark that the object is near to us, and so much the nearer as this parallax is greater. Indeed we seldom proceed in this method when we judge of an objects distance; for some time will be taken up in making this tryal: the mind must therefore proceed slowly in determining, and the conclusion of our judgment could not be mistaken for the perception of our senses. But though we do not use this method yet we proceed upon the same principles: for where the same principles for judging of distances, that are applyed in one case, can be applyed in another, there seems to be little reason to doubt of their being made use of in both. It is this parallax or difference of apparent place to the two eyes which makes an object seem double, unless we turn both eyes directly towards it, as has been already shewn in propositions 116, 117, 118. And when to avoid this double appearance the eyes are both of them directed to the object, if the distance of it is small, they are turned very much towards each other, and if its distance is greater they are turned towards each other in a less degree. Thus in looking directly at the nearer object *c*, or in directing the optic axes towards *c* the eyes must be turned so much towards each other as to look along the lines *ac* and *bc*; but in directing them to a remoter object *d*, they are less turned towards each

each other; for then they look along the lines *ad* and *bd*. And as this different turn of the two eyes is necessary to see objects single at different distances, when we use both eyes at once, it is by experience and habit made a settled mark for determining the distance of an object. We no sooner find it necessary to turn our eyes very much towards each other in order to look at an object, but the mind concludes the object to be near; and by the degree in which they are thus turned determines how near. After we have long used ourselves to judge of the distance of an object by this method, in which both eyes are necessary, we should be at a loss for some time, if we were to lose one of our eyes, to make any exact judgment of distances, till by use and experience we had settled some other method and had made it habitual. For this reason a person, who loses an eye, is at first apt to mistake the place of an object, that is not far from him, as will appear in his snuffing a candle or pouring liquor out of one vessel into another. A person, who plays well at tennis, will find himself subject to the same mistakes, the first time that he plays with one eye hood-winked. But in either instance experience and use will prevent these mistakes. Those, who cannot apply the principles of judging of distances by the help of both eyes, will by practice establish a method of judging with one eye only, upon the principles of apparent magnitude. For these, which are principles of judging of distances, when objects are remote, will likewise be made use of, when the objects are near; where it is out of our power to make use of such as are better. But from the mistakes that are made by a man, when he first loses the use of one eye, it is evident, that this is not the first principle that we go upon in forming our judgments, for if it had, he would determine the place of an object immediately with one eye as readily and as certainly as he used to do with both. From hence we may see the reason, why in looking through a lens either convex or concave, we should commonly judge of an objects distance, that is seen through it, by the apparent magnitude only. For the object is seen through the lens detached from every thing else, so that no extension between the eye and the object can be seen: and since we seldom make use of more than one eye in any experiments of this sort, we have no other way to judge of the objects distance but by its apparent magnitude. Therefore when it is proved that in these experiments we always do judge of distances upon this principle; yet this is very far from proving that this is either the only principle, or the principle that is in all cases most attended to: all that such proof amounts to is this, that we make use of this principle where it is not in our power to make use of any other. For even in looking through a lens we may find

find by a very easy experiment that the apparent distance of an object is not always determined by its apparent magnitude. Take a convex lens so large that you may look through it with both eyes at once, and hold a long slender object, such as an ivory folding knife, behind it in such a manner that half the knife may appear through the lens, and the other half may be seen below the lens by the naked eye: let the knife be so far behind the lens that the part, which is seen through it, may appear distinct. In this situation, that part, which is seen through the lens, will be magnified, and the rest of the knife, which is seen by the naked eye, will not. And if the spectator looks directly with both his eyes at these two parts of the object, first at the magnified part and then at the other; he will find, after having with some attention compared the apparent distances of the two parts, that the magnified part, which is seen through the glass seems to be farther off than the other, which is seen below the glass and is not magnified. At least I know that they appear so to me. Whereas, if he had judged of distance from apparent magnitude only, just the reverse must have happened, the part which appears to be the larger of the two must likewise have appeared to be the nearer. When an object is very remote from us, we sometimes judge of its distance, when compared with that of other objects, by the apparent faintness of it. But in near objects this principle of judging is of little or no use. For the brightness of an object is no otherwise altered by the greater distance of it, than as the rays have so much more of the atmosphere to pass through, by proposition 134: and this will make but little difference, unless when objects are very remote and their respective distances differ very much from one another. Indeed in objects, that are near us, there are innumerable accidents which may vary their apparent brightness: for which reason the idea of a different distance is not so steadily connected by use and habit with the idea of a different brightness, as to make us conclude at once, when an object appears faint, that it is therefore at a great distance from us. Nay even in the case of very remote objects, this conclusion is not made at once without some previous doubt; and an object is not judged from the faintness of it to be at a very unusual distance from us, unless it appears fainter than almost any accident can make an object appear, when it is nearer to us. This principle is commonly the last we have recourse to, and is scarce ever made use of, where we can possibly form a judgment upon any other. This apparent faintness is indeed imitated in painting, but it is done rather to preserve a likeness between the picture and the real appearance of things, than to suggest to us the idea of a greater distance of those parts which are drawn

the most faint. For though remote objects always appear fainter than near ones, and therefore, in imitation of nature, those parts of the landscape, which are designed for the most distant ones, should be made less bright, than those, which are nearer; yet faint objects do not always appear the most remote: and consequently though the parts of the picture which appear the most remote are made the most faint, this apparent remoteness is not so much owing to their faintness as to their smallness and apparent confusion. When the moon or the sun are in the horizon, they appear more faint, than when they are in the meridian. But this does not make either of them appear more remote. For notwithstanding they are fainter in the horizon, yet even there they are either of them as bright, and the sun in particular is more bright than most other objects are, which are very near to us.

The heavenly bodies, the sun, the moon, the other planets, and the fixed stars, though they are at very different distances from us, seem to be all at the same distance and appear as if they were in one and the same concave surface of the sky. For here all our principles of making any quick judgment of distances fail us. There is no visible extension, whose length we can judge of, between them and us. The apparent confusion is the same in all of them. Their real magnitudes are unknown to us, and therefore cannot be compared with their apparent magnitudes. The parallax of those, which have any, is not obvious, nor can it be found without taking up some time. And their apparent faintness is not sufficient to shew us at once that they are very remote from us, and much less is it sufficient to shew us that one of them is more remote than another. Therefore, as our judgment cannot help us to distinguish their several distances, they all seem to be at the same distance from us.

Indeed the moon or the sun or any other of the heavenly bodies, will seem farther from us, when it is in the horizon, than when in the meridian. And the concave surface of the sky in an open horizon does not seem a hemisphere, but the parts that are directly over our heads seem nearer than those parts which are directly before us about the horizon, where the sky all round seems to touch the surface of the earth. So that this concave surface appears like a less segment of a sphere than a hemisphere is. And an arc drawn from the point L, Plat. XII. fig 3, in the horizon through R to the opposite point H will not appear to be a semicircle as LRM having its center at S, which is the place of the spectator, but will seem to be a segment of a circle having its center below the horizon, and so will be less than a semicircle. For the only visible extended line between the spectator and the apparent sky, if his face is towards L, will be SL,

a line drawn along the visible plane of the horizon, and by this visible line he is to estimate, as well as he can, both the distance of the point L from him and of all the other points between L and R. Now when he looks directly before him towards L, he sees this line to the greatest advantage as to the apparent length of it, that he possibly can; for if he raises his eye to B this common measure SL will be seen more obliquely; and consequently, when it is referred to the point B as the measure of its distance from him, it will appear shorter than when referred to L; upon which account he will suppose the point B nearer to him than the point L. In like manner, when he would judge of the distance of the point C from him, his common measure SL is seen still more obliquely, and therefore will appear still shorter, or the distance of C will seem to be less than the distance of B. And when this common measure is referred to R it is seen most oblique of all; so that R will appear nearer than any other point of the sky. And since, as the eye rises from L towards R, every point appears nearer and nearer; the whole apparent figure of this quarter of the sky will be *Lbcr*; and for the same reasons the whole apparent figure of the other quarter will be *rmH*.

136. *In viewing objects that we are much used to, if they are at such distances as we can readily allow for, the judgment of the mind about their magnitude is commonly more attended to than the perception of the eye.*

If a man 6 feet high is 40 yards from us, and a boy 3 feet high is 20 yards from us, their apparent heights, as perceived by the eye, will be equal, by proposition 128. And yet when we look at them, we are apt to think otherwise; the man seems to be as much bigger than the boy, as he would if they were both at the same distance. The two objects are such as we see frequently, we have often viewed them at different distances, and know by experience how much an encrease of distance will diminish their apparent magnitudes: the distances, at which they are supposed to be placed, are likewise such as we are used to, and therefore are such as we can readily allow for. And by a quick and imperceptible act of the judgment we determine which of the two is the more remote and what is the real magnitude of each. And having thus at once found out their real magnitude we attend so much to this, as to neglect the apparent magnitude perceived by the eye, and then think that we see each of them of the same size that it really is, and of the same size that they would appear if they were less remote.

The reader will do well to observe here, that this can never be the case unless the objects are placed at such distances as we are used to and

can readily allow for, and unless the objects themselves are such as have been familiar to us, such as we have frequently viewed at different distances, till we are well acquainted with their real magnitudes. Some of the principal instances to support these observations, I shall borrow from the ingenious and truly learned doctor Jurin. The use, that is to be made of the observations themselves, will be seen hereafter.

First; in distances that we know not how to allow for readily, the perception of the eye is all that we attend to in judging of the magnitude of an object. "Children of three or four years of age, not having yet learned how to allow for the lesser apparent magnitude of distant objects, take every thing seen at a distance to be less than it really is. If they see a man or a woman thirty or forty yards off, they take the one for a boy and the other for a girl. And though experience teaches them in a little time to correct this error in objects seen upon the level, or from a moderate height, yet if they happen to look upon them from any high building, they are apt to make the same mistake as before. Let a boy, who has never been upon any high building, go up to the top of the monument, and look down into the street; the objects seen there, as men and horses, will appear so small as greatly to surprize him. But ten or twenty years after, if in the mean time he has used himself now and then to look down from that and other great heights, he will no longer find the same objects to appear so small. Because now, having had frequent experience of the lessening of the apparent magnitude by the encrease of distance, he does by a quick and imperceptible judgment conceive the objects to be of the same magnitude as if they were less remote: which judgment of the mind not being distinguished from the perception of the eye, he thinks that he actually sees them larger than before. And if he were to view the same objects from such heights, as frequently as he sees them upon the same level with himself in the streets, I suppose, they would appear to him just of the same magnitude from the top of the monument, as they do from a window one story high. For in the street a man at a hundred yards distance appears of the same height as another at ten, though the picture of the former in the eye is but a tenth part of the length of the other, the smallness of the picture being compensated by the knowledge of the greatness of distance." But though we think that a man at a hundred yards distance seems as big as if he was only ten yards from us, yet in greater distances than these we should think otherwise. A man at the distance of a mile seems less than at the distance of ten yards: and as children three or four years old, who are unable to estimate and allow for smaller distances, take a man at thirty yards distance

stance for a boy; so when the distance of an object is too great, as by proposition 130, it soon may be too great, for the eye of a grown person to estimate it, if he does not fall into the same mistake that children do, yet the greatness of the distance, because it is unknown, cannot compensate for the smallness of the picture upon his retina: he cannot therefore persuade himself by any quick and imperceptible judgment, that the objects at such great distances appear of the same magnitude, as if they were less remote; since for want of a ready way to estimate and allow for those distances, he has no opportunity of forming such a judgment. And thus, though in travelling upon the road, a man ten yards before us does not seem larger than another a hundred yards before us; yet a milestone only ten yards from us, if we can see the next beyond it at the same time, will seem much larger than that which is a mile and ten yards from us. A church, at the distance of a furlong may seem no less than at the distance of half a furlong, because these distances are small enough for the eye to estimate and allow for readily; the visible extension between us and the object is the measure of its distance. But if we were to see two churches of nearly the same size one at five miles distance and the other at ten, and were asked which of those two was the farthest off, and the reason why we think so; the answer would shew that they do not seem to be of the same size; for it would most probably be that we think one of them farther off than the other, because it seems to be less. And I fancy, no one imagines, that either of them seems to be as big as it does, when he is within half a furlong of it. The reason is obvious. For in very great distances the length of the visible extension between us and the object, decreases too fast, by proposition 130, for the eye to make any estimate of it; and upon this account our judgment of the objects distance is commonly made by attending to its apparent magnitude as perceived by the eye; and when we have attended enough to this apparent magnitude to determine the distance of the object, we cannot immediately allow for the distance thus estimated, forget the apparent magnitude as perceived by the eye, and then attend only to the dictate of our judgment about the real magnitude. If we attend only to this dictate of the judgment, we cannot estimate and allow for the objects distance; if we do estimate and allow for the distance, we must attend to the apparent magnitude as our senses perceive it.

Secondly; unless the objects, that we are viewing, are such as have been familiar to us, such as we have frequently viewed at different distances, till we are well acquainted with their real magnitudes, the perception of the eye is all that we attend to in judging of their magnitude.

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“Tho’ some objects, when seen at a distance, are by a quick and imperceptible judgment conceived to be of the same magnitude as if they were less remote; yet the famous monsieur Perrault seems to be greatly mistaken, when he applies this principle to overthrow an opinion, which has generally been received by all architects and sculptors since the time of Phidias, that statues placed upon very high buildings ought to be made of a larger size, and to be heightened with bolder strokes than those, which are to be seen at a nearer distance. He says the judgment made by the eye is so perfect and so accustomed to make the proper allowance for the distance of the object, that it appears of the same magnitude to us whether at a greater or a lesser distance. In things very familiar to us he is certainly right: but statues upon very high buildings are not so constantly seen as men and horses in the streets; and therefore we cannot make so just an allowance for the distance in one case as in the other. Very few spectators are able to make a near estimate of the length of the dragon upon Bow-steeple, or of the height of the St. Michael at Brussels, except such as saw those figures before they were put up.” It is for want of being acquainted with the real magnitude of the object, for want of having frequently viewed it at different distances, that a person, as has been already observed in proposition 135, when he is travelling towards a large city, or a castle, or a cathedral church, or a mountain larger than ordinary, is not able to allow in his mind for the distance and to determine the real magnitude; but attending only to the apparent magnitude, as his eye perceives it, judges of the distance from thence and concludes himself to be nearer to them than he finds himself upon tryal.

We may now be able to guess whether the moon, when it is in the horizon, appears to be bigger, than when in the meridian, upon account of its apparent distance being greater, when in the former situation, than when in the latter. If this is the true reason of this remarkable appearance, we must explaine it in the following manner. The moon is really at the same or at nearly the same distance from us in either place; and consequently the picture of it upon the retina, and the apparent magnitude, as perceived by the eye, is the same [in both, by propositions 127, 125. But the apparent distance of the moon, when in the horizon, is greater than when in the meridian. Therefore the mind by a quick and imperceptible act allows for this apparently greater distance and concludes that the moon, which at this greater distance seems as big to the eye, as when it is in the meridian at an apparently lesser distance, must necessarily be bigger in itself. And when this conclusion is once made, we attend to it more than to what the
eye

eye perceives, and fancy that it looks bigger in the horizon than it does in the meridian.

It is, upon these principles, a sort of perspective view of the moon: for as in a picture, if two objects seem at different distances, and are drawn equal to each other, the imagination will immediately conceive that object to be the larger of the two, which appears to be at the greater distance, and this dictate of the imagination will be more attended to than the preception of the eye; so if the moon seems to be more remote in the horizon than in the meridian, and the apparent magnitude, as perceived by the eye, is the same in both situations; the imagination will immediately conceive that the moon is larger in the horizon than in the meridian; and our attention to this conclusion of the mind is all that we attended to, the perception of sense serves only to lead us to this conclusion and then is forgotten. Thus suppose that the eye is at *S*, Plat. XII. fig. 3, the moon in the horizon at *H* and in the meridian at *M*; so that the angle, which it subtends at *S*, when in either place, is equal. If the moon in the meridian was to appear as if it was at *M*, that is, if its apparent distance *SM* when in the meridian, was equal to *SH* its apparent distance when in the horizon, then in both situations it would appear of the same size. But the moon in the meridian does not seem to be at the distance *SM* but at a less distance *Sm*. Now since a moon at the greater distance *SH* must be larger in itself, than one, which at the lesser distance *Sm*, appears of the same size, as much larger as *H* is than *m*; and since the eye does perceive the moon of the same size when it appears at *m* as when at *H*; therefore the mind by a quick and imperceptible act conceives it, when at *H*, to be bigger, than when at *m*, and we attend so much more to this conclusion of the judgment, than we do to the perception of sense, as to think, not only that it is bigger, but that we see it bigger in the horizon at *H* than in the meridian at *m*.

Now here we may apply the two foregoing observations. First; this conclusion cannot be made, except in instances, where the mind can readily allow for the apparent distance of the object. The conclusion of the judgment is never substituted instead of the perception of sense, except where this conclusion can be made at once. But the mind cannot act in so quick and imperceptible a manner, unless where the eye can easily estimate the different distances of the object. And though the moon does seem farther off, when in the horizon, than when in the meridian, yet both these apparent distances are much greater than the eye can easily estimate and readily allow for. If in a prospect from any eminence we could see two buildings of the same size, one at 5 miles distance and the other

other at 10; though we might perceive a long tract of country lying between one and the other, and so might be sure that one is more remote than the other; yet we should never fancy that one seemed to be as large as the other. This is only done, where the objects are near, and the eye can, not only perceive that they are at different distances, but can likewise readily estimate that difference and allow for it at once. But in such remote objects, the difference of distance, though it is perceived, can be estimated no otherwise than by attending to the apparent magnitude of the objects; and when this is attended to in estimating the distance, it is impossible immediately to allow for that distance, forget the apparent magnitude, as perceived by the eye, and substitute instead of it an imagined real magnitude, as dictated by the judgment. Though the distance of the moon in the horizon is apparently greater than in the meridian, yet as both of them are too great for the eye to estimate, we can form no conclusion of judgment to strike our imagination and take off our attention from what the eye perceives: and if the apparent magnitude as perceived by the eye, is the same in either situation, we cannot fancy that it seems bigger when in the horizon, than when in the meridian.

Secondly; the conclusion of judgment, that we have been speaking of, cannot be made, except in such objects as are familiar to us, and have been often viewed at different distances. In objects that we are strangers to, or that we have never seen but at one and the same distance, we attend only to the perceptions of sense; and instead of judging what their real magnitudes are from their apparent distances, we estimate their distance from their apparent magnitudes as perceived by the eye. This is the case of the moon; it has never been viewed by us at more distances than one; its real magnitude is what common spectators have no conception of, and we cannot tell by an alteration in its distance, how much or how little its apparent magnitude will be altered: so that though its apparent distance, is greater in the horizon than in the meridian, the mind has not been used to make allowance for difference of distance and consequently has never acquired such an habit as is requisite, where, in looking at an object, a conclusion of judgment is substituted in the place of the perception of the sense.

We may now examine the instance of a picture in perspective, which we before made use of in explaining the opinion, that we are now endeavouring to confute. In a perspective picture, the apparent distance of the several parts depends upon the magnitude of them compared with one another: so that one part will appear remoter than the rest by being made smaller

Fig. 1.

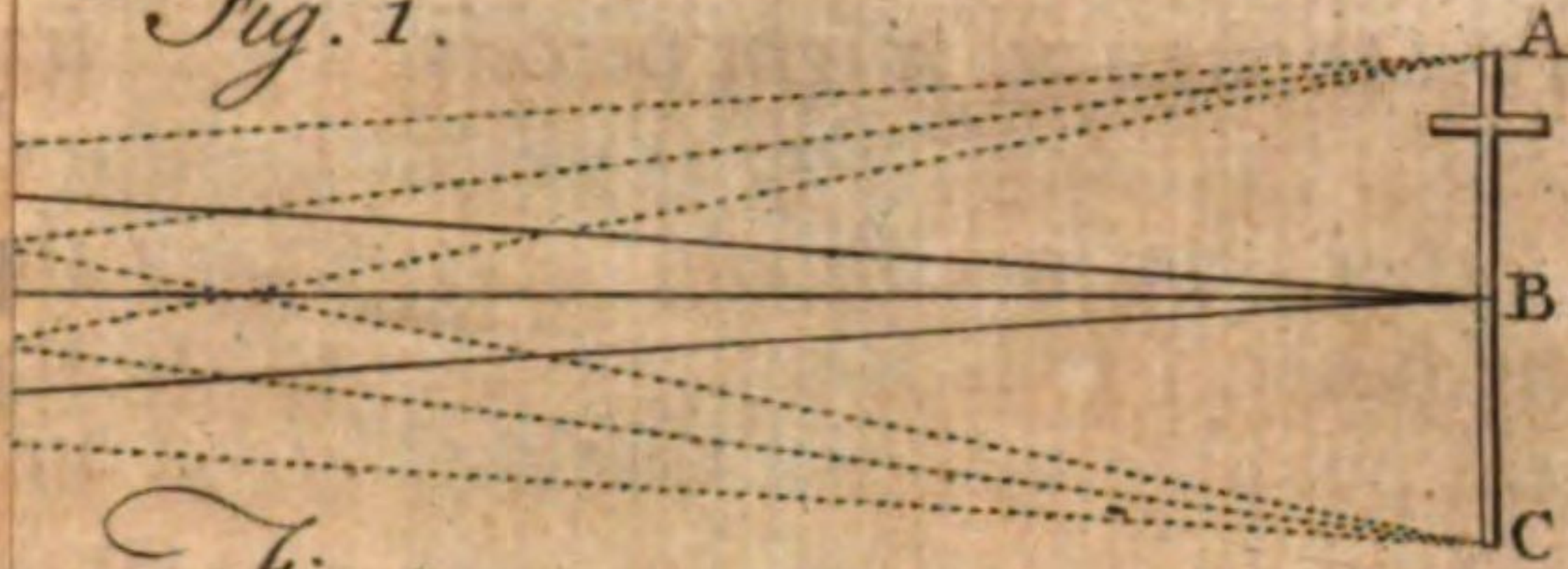


Fig. 2.

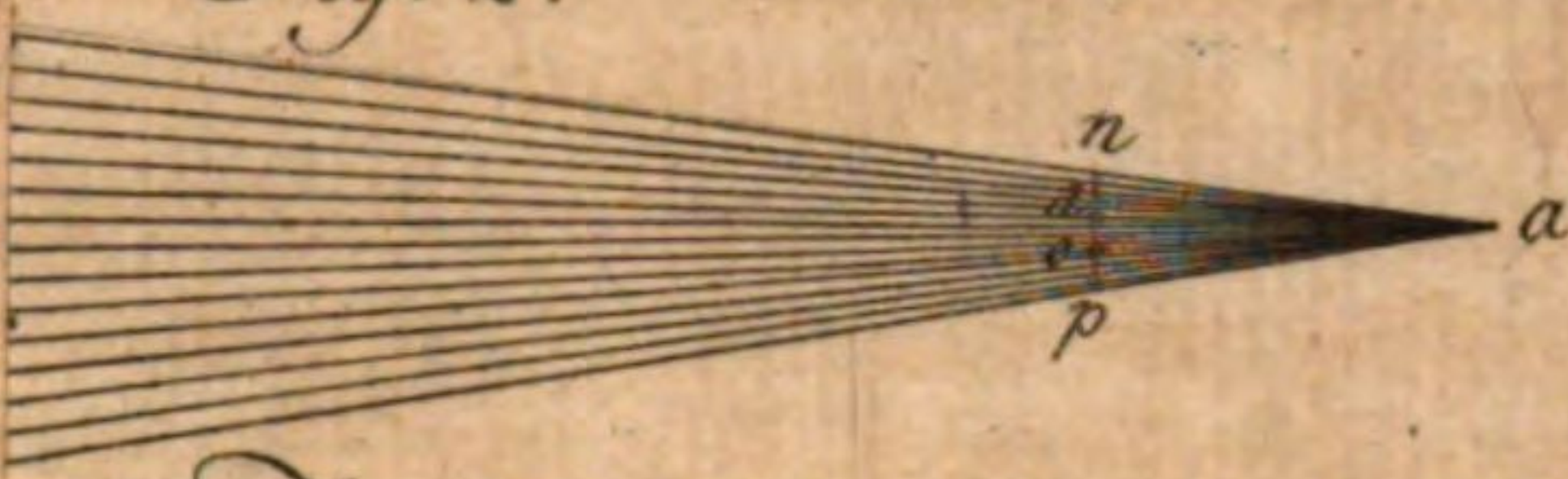


Fig. 4.

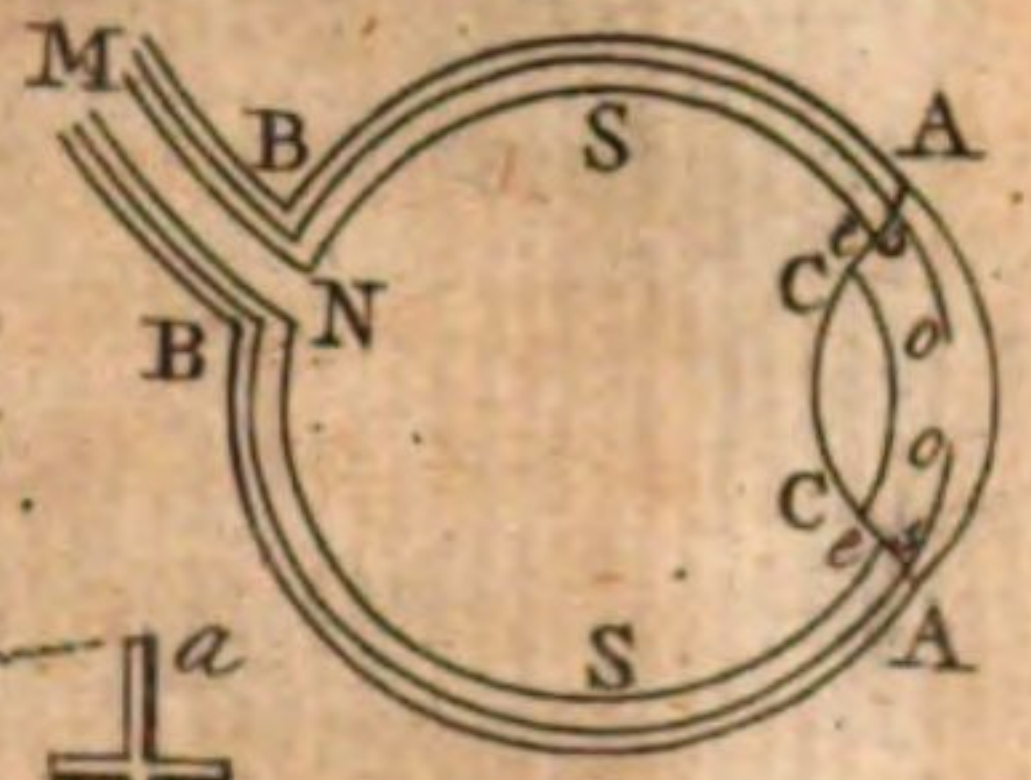


Fig. 3.

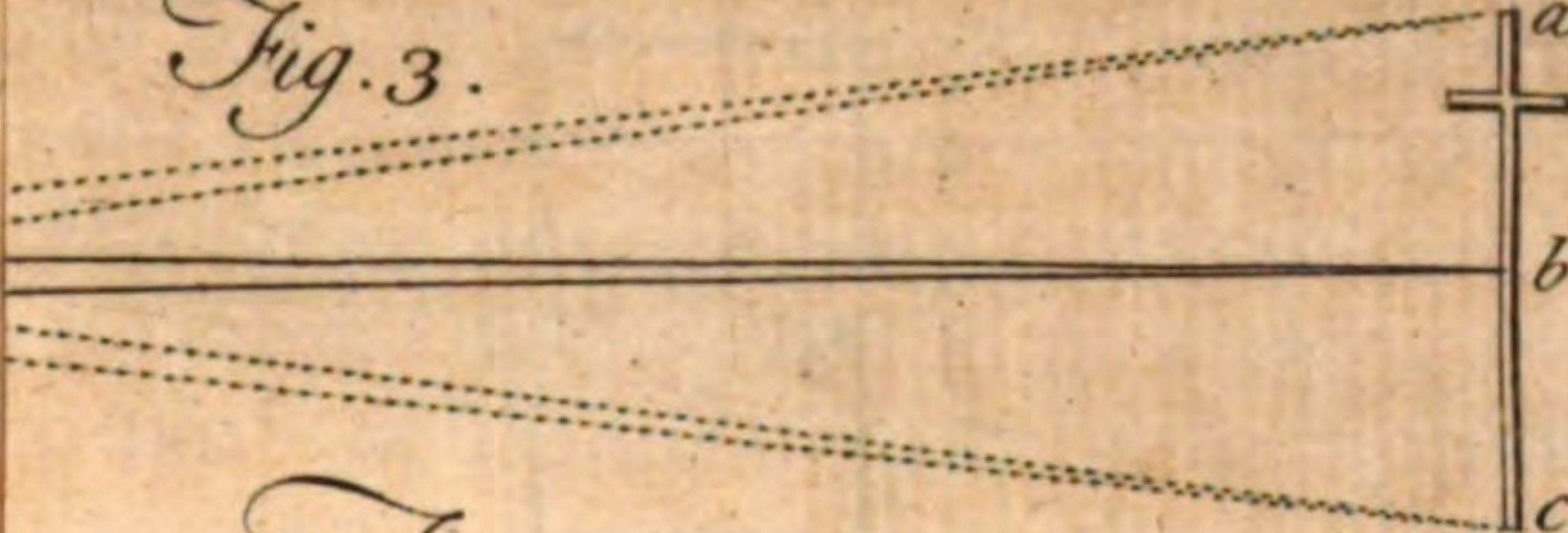


Fig. 5.

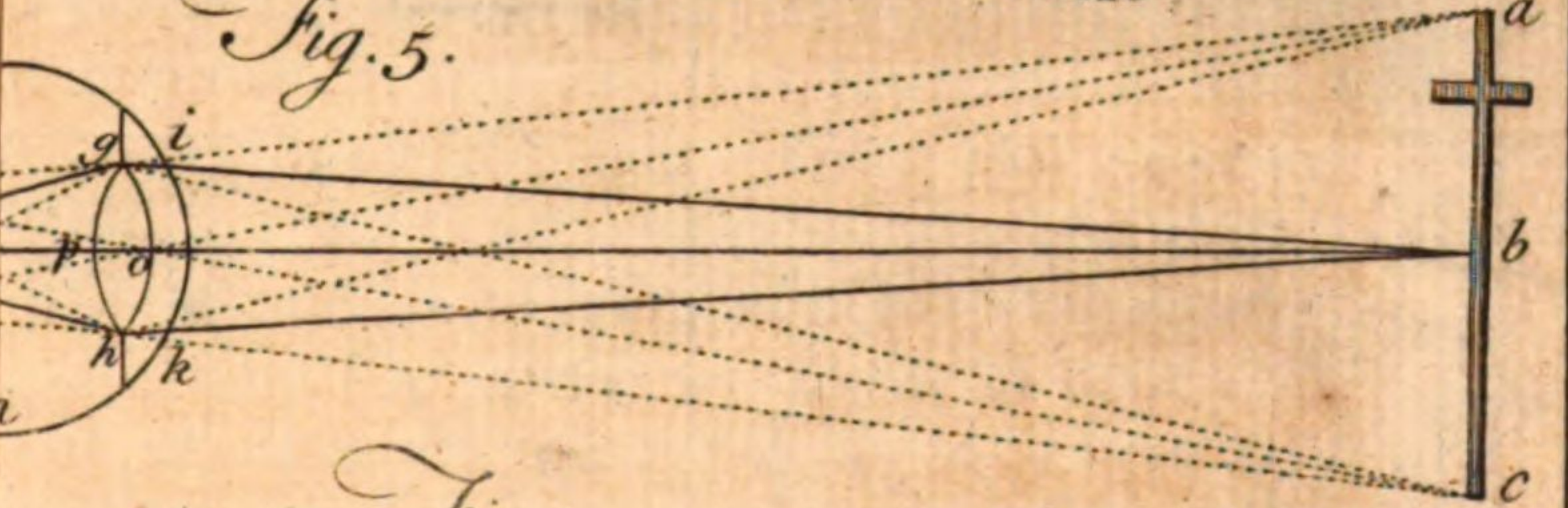


Fig. 7.

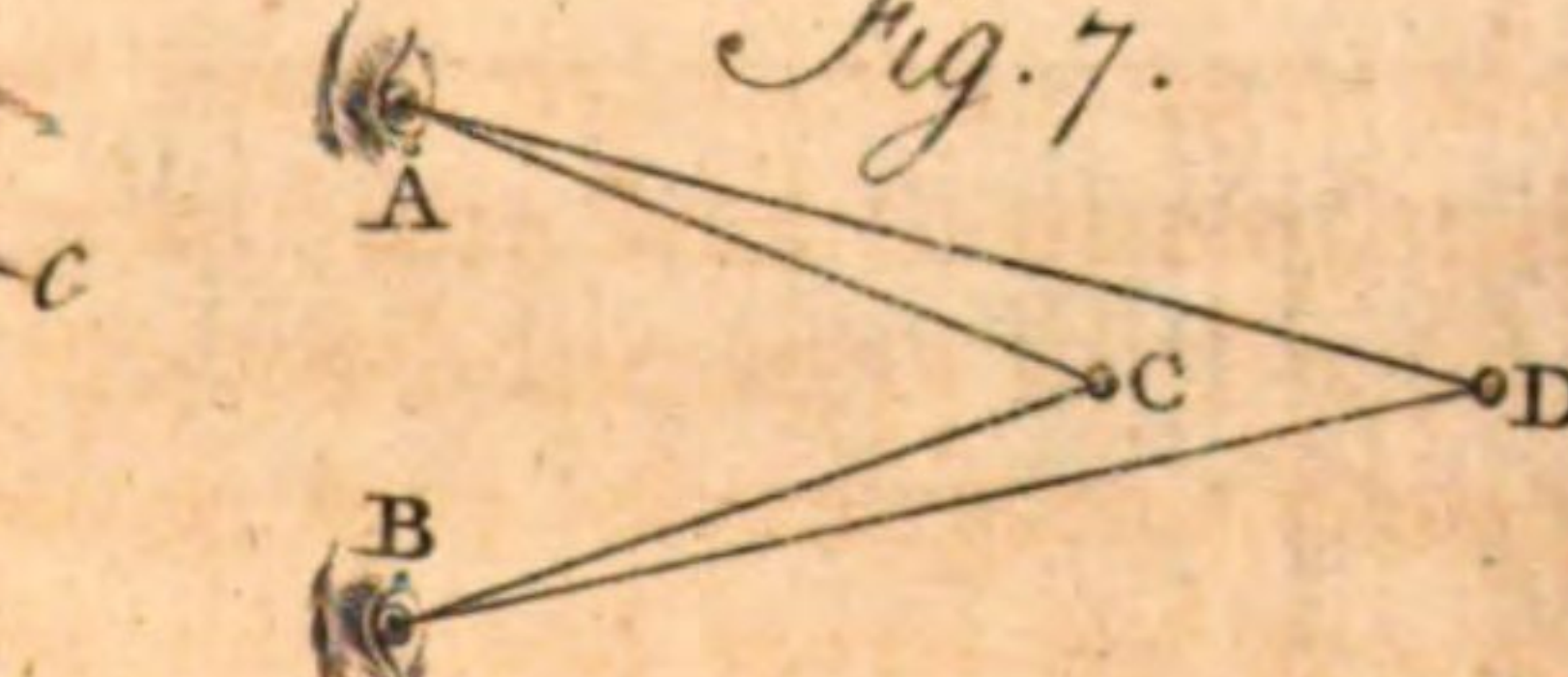


Fig. 8.

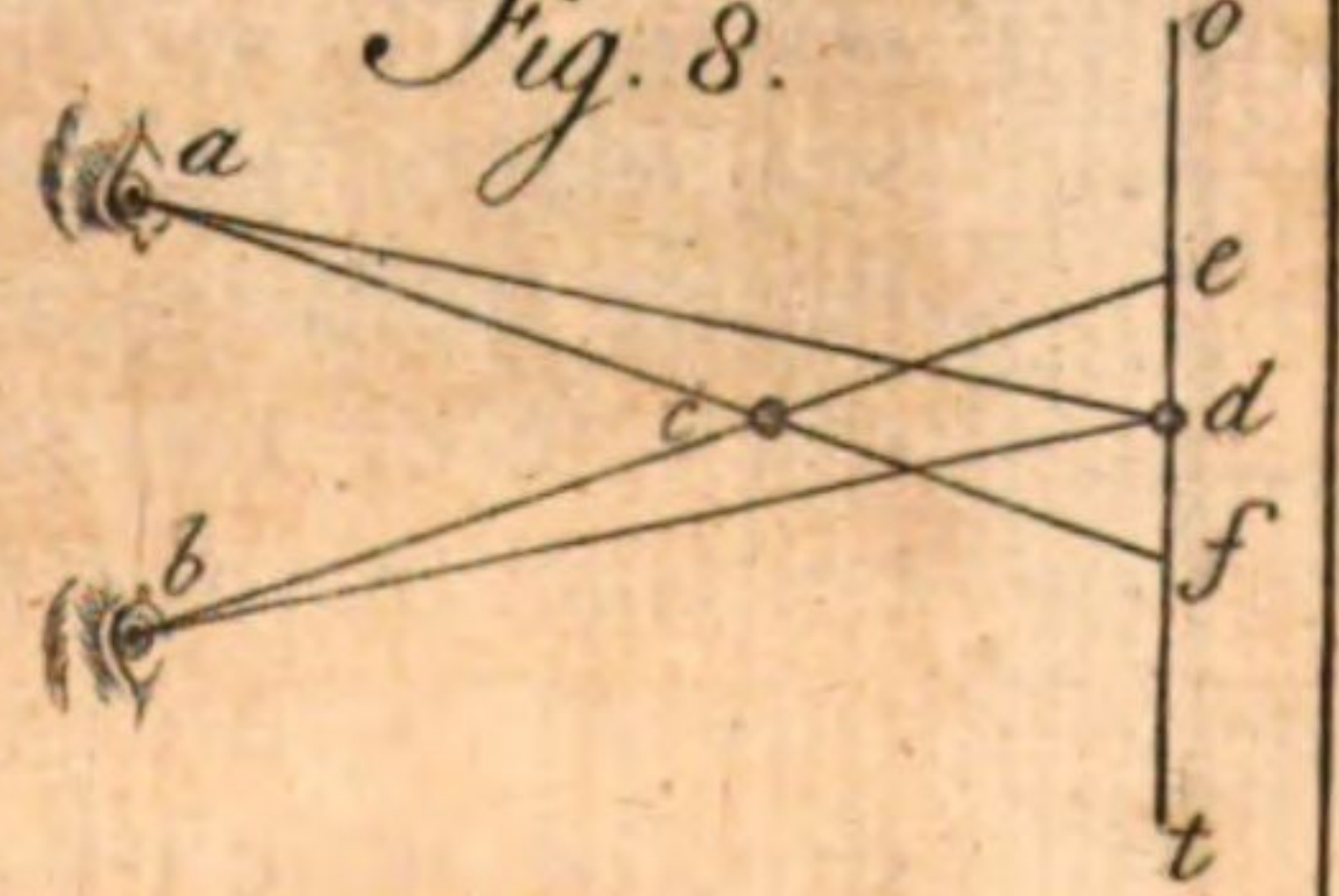


Fig. 10.

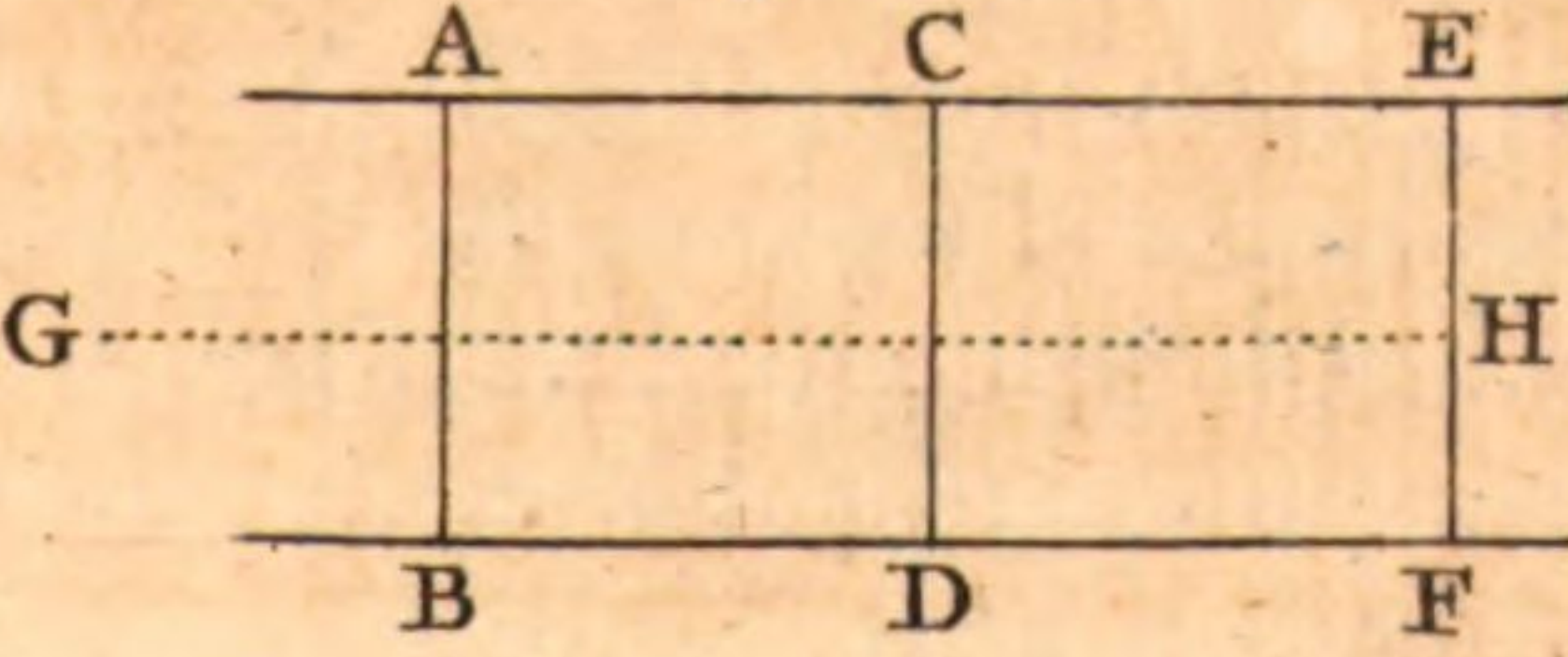
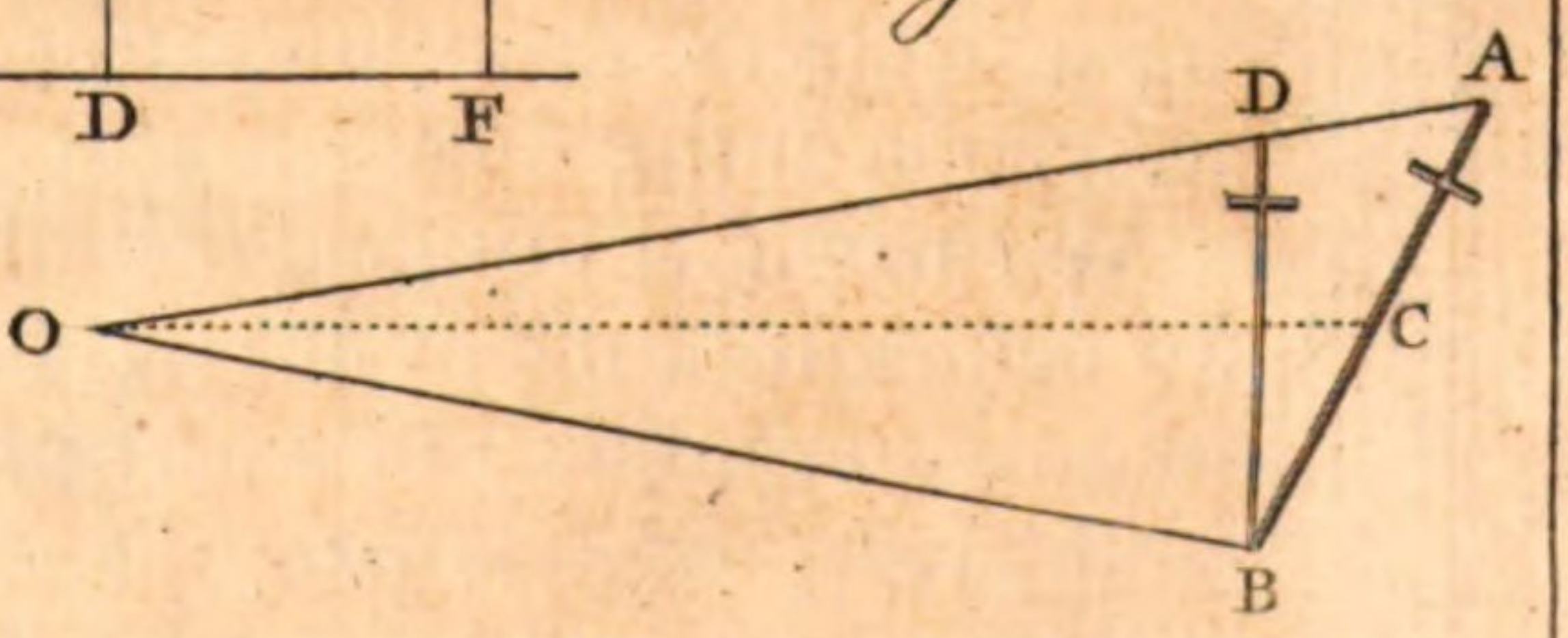
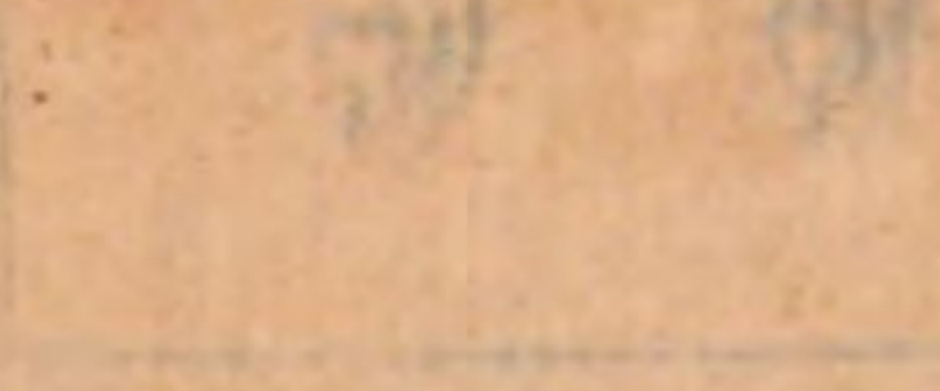


Fig. 11.



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smaller than the rest; that is, the apparent distance is judged of by the apparent magnitude: which agrees very well with what has been proved in proposition 135, that in very remote objects the apparent magnitude is the leading principle, by which we estimate the distance. But if we consider the moon in the horizon and the moon in the meridian as two perspective views of an equal object at different distances, the judgment is formed just the contrary way. Of two objects, that are equal as to sense, we fancy that appears bigger which seems to be the most remote: so that in this instance there could be no perspective view, unless the mind in very remote distances is able to judge of magnitude from apparent distance. How consistent this is with the principles laid down in proposition 135, how in very great distances, such as we cannot estimate without attending to the objects apparent magnitude, we can judge of the real magnitude from the apparent distance, let the reader determine. But let this be as it will. In a perspective picture the fancy determines the apparent distance of the several parts from the eye by the comparative magnitudes of those parts; and if in a perspective view of a portico, all of it was to be covered except one pillar, for want of the rest to compare it with, the spectator would not be able to judge whether that pillar was a near one or a remote one. In like manner in the perspective view of the moon, where the fancy is supposed to determine the magnitude from the comparative distances of the moon in the horizon and the moon in the meridian, if those distances are hid, as they will be if we look through the tube of a short telescope with the glasses taken out of it, for want of an opportunity of comparing the different apparent distances, the spectator ought not to see the moon at all larger in one situation than in the other. Whereas upon tryal he will find the appearance just the same as if he had made use of no tube at all.

C H A P. X.

Of refracted vision through single glasses.

137. *When any small object or any point of an object is seen by refracted light, it appears in the direction of that line, which the rays describe after their last refraction.*

IF the rays, that come from any small object D, Plat. XII. fig. 4, pass through a glass prism, the end of which is ACB, the ray DE will be refracted towards a perpendicular, when it enters the prism, and will describe the line EF, and when it goes out of the glass, it will be refrac-

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ted from a perpendicular into the line FG; which line FG is the direction of it after its last refraction, and the object D will be seen in the direction of the line FG, and will appear to be at L instead of D. For in this and in all other cases of the same sort, the picture of the object upon the retina will be in the same place, that it would have been in, if the naked eye had been looking at an object really placed at L; for the refraction gives the rays the same direction, as if they had come originally from L.

From hence we may understand why an object seen through a multiplying glass or through a glass that is cut into different surfaces inclined to one another, appears at one view in many different places. If the object *f*, Plat. XII. fig. 5, is seen through the glass *abcd*; by the ray *fo* that passes through the surface *cb* the object by the eye at *o* will be seen at *f*: the ray *fx* passes through the surface *cd*, and, when it is refracted, comes out in the direction *xo* as if it had come from *l*; therefore through the surface *cd* the object appears at *l*: and for the same reason, through the surface *ab* it appears at *m*. Consequently there must be the appearance of as many objects as there are such surfaces on the glass; for each of them shews the same object in a different place.

Hence it is that if such a glass as *abcd*, Plat. XII. fig. 5, or a lens, such as *gil*, fig. 7, 8, was to be shaken before the eye, the objects on the other side would appear to shake: for the rays from the same point of the object, in different situations of the glass, will fall with different obliquities upon its surface, and therefore will be differently refracted. Now as the glass is shaken, its situation is constantly changing, upon which account the refracted rays will be shaken, and the apparent place of the object will by this means be constantly varied. In viewing an object through the plane glass AB, fig. 6, it will be otherwise: for in all situations of the glass, the rays fall in the same manner on any part of that plane surface which is towards the object, and the refracted rays come out in the same manner from the other plane surface which is towards the eye: so that by shaking the glass, the refracted rays will not be made to tremble, and consequently the object seen through it will not vary its apparent place, or will not appear to shake.

138. *In refracted vision it is not the object itself that we see, but the last image of it, which consists of all the imaginary radiants, from whence the rays appear to diverge after their last refraction.*

It will be necessary to describe particularly what is here meant by the last image. Plat. XII. fig. 7. Let *ac* be an object nearer to a convex lens than

than its principal focus; the rays that diverge from any point b in this object, will by passing through the lens be made to diverge less, and the imaginary radiant will be more remote than the real one, by propositions 67, 28. Thus the rays bg , bl , when they have gone through the lens will not proceed strait forwards in the lines gk , lp , but will be refracted into the less diverging directions gm , ln , as if they had come from the imaginary radiant e , which is more remote than the real one b . The same will happen to the rays, which come from a or c or any other point in the object, so that there will be somewhere behind the lens, as at df , as many imaginary radiants as there are real ones in the object, and these imaginary radiants taken altogether compose the last image. And since all the rays fall upon the eye as if they had diverged from this last image, the eye will be affected by the object abc just in the same manner, when it looks through the lens, as it would be without the lens by an object in all respects like def , or as it would be by the last image, if without the lens the last image could be made visible: and because the eye is affected when it looks through the lens as if def was the object and not abc , therefore we say that it is not the object itself that we see but its last image def . This is universal, in refracted vision the eye is affected by the rays of light after refraction as if they had come, not from the object itself, but from its last image, which consists of all the imaginary radiants from whence the refracted rays appear to diverge at the time they fall upon the eye. We will apply this to other particular instances as we have occasion.

139. *The apparent place of a star, when it is near the horizon, is higher than its real place.*

If a , Plat. XII. fig. 9, is a star not very high above the horizon; any ray such as ab , which falls obliquely upon the atmosphere, because it passes out of a vacuum into the air, will be refracted towards a perpendicular, by proposition 37, and will not go strait forwards, but will describe the line bc . Now the atmosphere is not so dense at the top as it is near the earths surface, and the density of it keeps encreasing from the top down to the earths surface: upon which account this ray as it goes farther into the atmosphere keeps passing constantly into a denser medium, and will be constantly refracted towards a perpendicular. Thus in passing through the first lamina it described bc , in passing through the next below it, which is something denser, it will describe cd , in the next lamina below that it will be refracted into the line de , and in the lowest, where the spectators eye is at f , the direction of the ray will be ef . But the star

will be seen, by proposition 137, in the last direction of the refracted ray, or in the line gf so as to appear at g , instead of a : and because the ray in its descent through the atmosphere, has been constantly refracted towards a perpendicular, its last direction ef continued back, or fg will be nearer perpendicular to the surface of the atmosphere than ba , that is, g the apparent place of the star will be higher, or nearer to the zenith, than its real place a .

From hence it is plain, that a star may be seen before it is above the plane of the horizon: for when its real place is below the horizon, its apparent place by the refraction may be above it.

140. *The nearer a star is to the horizon the more its apparent place is changed by the refraction of the atmosphere.*

If ABC, Plat. XII. fig. 10, is the earth's surface, LDM the surface of the atmosphere, T the common center both of the earth and of the atmosphere, that surrounds it; when a star Z is in the zenith of a spectator at B, or is directly over his head, a ray as ZD coming from the star to his eye will be perpendicular to the atmosphere, for its direction ZD continued passes through T the center of the atmosphere, therefore this ray, by proposition 41, will not be refracted, and consequently the apparent place of a star in the zenith will not be altered by the refraction or does not differ from the real place. But when the star is at G in the horizon, as far off from Z as it can be, if it is seen at all, the incident ray GP will be as far from a perpendicular OT as it can be, and consequently as the angle of incidence GPO is there the greatest, the refraction will be the greatest, by proposition 40. And therefore the star's apparent place will be more altered when it is at G than when it is any where else between G and Z. Thus for instance when the star is at H a ray HR is less oblique to the surface of the atmosphere, than the ray GP when the star was at G, or the angle of incidence HRN, is less than the angle GPO. For since, when it is at Z, this angle made with the perpendicular, or angle of incidence, is nothing, the nearer it approaches to Z, so much the less this angle will be, and the less the angle of incidence is the less will be the refraction, and consequently the less likewise will be the change of apparent place.

Indeed when the star is at G, or any where else, as well as when at Z, there will be a ray coming from it, which will fall perpendicularly on the surface of the atmosphere; but this perpendicular ray, as GT passes towards the earth's center; and therefore is not one of the rays which makes the star visible to a spectator at B on the earth's surface.

141. *The*

141. *The sun, or the full moon, appears of an oval figure, when they are in the horizon.*

As the apparent place of a star, by proposition 139, so likewise the apparent place of any point in the sun, will by the refraction of the atmosphere be higher than its real place. Now suppose a diameter drawn across the disk or face of the sun perpendicular to the horizon; the lowest point of this diameter will be apparently raised by the refraction, and so likewise will the highest point of the same diameter. But then as the apparent place of a star, by proposition 140, is more raised the nearer it is to the horizon, so for the same reason the lowest point of this perpendicular diameter will be apparently more raised than the highest point of it, because the lowest point is nearer to the horizon than the highest. Therefore the two extremities of this diameter will be made to appear nearer to each other, or the perpendicular diameter will be apparently shortened by the refraction. The diameter, which crosses this at right angles, or which is parallel to the horizon, is no otherwise affected by this refraction than that both ends of it are equally raised higher. So that since the perpendicular diameter of the sun is shortened and the parallel one continues the same; this will apparently change the circular figure of the sun into an oval or elliptical one, the shorter diameter of which will be perpendicular to the horizon.

142. *A vessel seems to be shallower, when it is full of water, than when it is empty.*

When the vessel is full of water the bottom of it is seen by refracted light: for the rays, which make it visible, pass out of the water into the air, before they come to the eye. Therefore, by proposition 138, it is not the bottom itself that is seen, but its last image. And since, by proposition 45, the imaginary radiants will be nearer to the eye than the real ones, the last image consisting of these radiants, which is seen when the vessel is full, will be nearer than the bottom itself, which is seen when the vessel is empty.

143. *In vision through any glass the object will appear erect if the object and its last image are on the same side of the glass; but inverted if they are on contrary sides.*

Since the last image is what we see in refracted vision instead of the object, by proposition 138, an object seen through a glass will appear erect, if its last image is so. And the image will be erect, or in the same position with the object if they are both on the same side of the glass, but it will be inverted, or the object and image will be in contrary positions,

tions, if they are on contrary sides. If abc , Plat. XII. fig. 7, is the object, and the glass, through which it is viewed by the eye at o , is a convex lens gil ; then suppose the object to be nearer to the lens than its principle focus, and all rays which diverge from any one point in it as b , will diverge less after refraction, and the imaginary radiant e will be more remote than the real one b , by proposition 67. However, all the rays, which diverged from one and the same point b before refraction, will apparently diverge from one and the same point e after refraction. So likewise all the rays which diverged from the same point a in the object, will diverge from the same point d in the image; and all, which diverged from c , will after refraction diverge from f . Now ai is the axis of the beam that comes from a the highest point in the object, and this axis passes through the lens unrefracted, by proposition 59. But the ray ai before refraction diverged from a as well as any other ray of the beam, and consequently after refraction will appear to diverge from the same point d with the rest of the rays. If therefore it passes straight through the lens and yet appears on the other side to come from d , the point d must be in the same right line with a , or dai , which is drawn from the highest point in the image through the highest point in the object to the vertex of the lens, is always a right line. In the same manner we might prove that fci , which is drawn from the lowest point in the object through the lowest point in the image to the same vertex, is likewise a right line. Now these right lines cross each other in the lens and no where else. And since the highest point both of the object and image is somewhere in the line di , and the lowest point of both is somewhere in the line fi , therefore if the object and its last image are on the same side of the lens, that is, if the lens is not between them, these extreme axes will not cross each other between the place of the object and the place of the image, but will be in the same situation in respect of each other at both places, dai , which is the highest at the object, will be the highest at the image, and fci , which is the lowest at the one will be the lowest at the other: and consequently the point a , which is highest at the object, appearing at the image in the line dai , will be the highest there, and the point c , which is the lowest at the object, appearing at the image in the line fci , will be the lowest there; that is the image will be erect or in the same position with the object.

This may be made universal. For by what has been said it appears that the highest point of an object, when seen by refraction, is always seen somewhere in the axis of the beam, which comes from the highest point; and the lowest point is always seen somewhere in the axis, which comes from the lowest point. Therefore if these two axes have not crossed each other

other between the object and its last image, the highest point in the object will appear highest in the image and the lowest point of the object will appear lowest in the image; that is, the object and its last image will be in the same position.

In a concave lens *gil*, Plat. XII. fig. 8, if the object is *abc* the rays coming from every radiant in the object are made to diverge more, and the imaginary radiants or last image will be nearer than the object, by proposition 76. The highest point *a* will appear at the image somewhere in *ai*, or in the axis of the cone of rays which comes from that point; and the lowest point *c* will appear in the image somewhere in the axis *ci*. Therefore if these axes *ai* and *ci* have not crossed between the object and the image, that point which is the highest or the lowest in one will appear the highest or the lowest at the other. But they cross no where except in the lens at *i*; and consequently if the object is at *abc*, and the last image at *def* or any where else on the same side of the lens with the object, so that the lens where the axes cross is not between them, they will both of them be in the same position.

But if the object *abc*, Plat. XII. fig. 11, is more remote than the principal focus of the convex lens *gsl*, there would be a distinct picture of the object formed somewhere on the other side of the lens, by proposition 84. Suppose then that, if a paper was held at *fed*, this distinct picture would be painted there. Now if there had been no paper, though the rays of the several beams would still be collected into their respective focuses at that place, yet they would not stop there, but would proceed straight forwards: the rays, that had converged to *d* in the lines *gd*, *ld*, after they had met there, would diverge from thence in the lines *do*, *dx*. And the rays, that had converged to any other points *f* or *c*, would proceed on and so diverge from thence, by proposition 15. And thus every focus in the picture becomes a radiant: for which reason, if the eye is at *o*, the points from whence the rays last diverge, when they fall upon it, are the several focuses in the distinct picture. Therefore the distinct picture is in this case the last image. And we have already proved, in proposition 85, that the distinct picture is inverted in respect of the object. Therefore when the object is on one side of the lens and the last image is on the contrary side, the last image will be inverted.

144. *In all lenses, the diameter of the object is to the diameter of the last image as the distance of the object from the lens to the distance of the last image from the lens.*

This has been proved already of the distinct picture, in proposition 88, therefore the proposition is true, when the object and last image are on
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contrary sides of the lens, as they are, Plat. XII. fig. 11. But in the concave lens fig. 8, or the convex one fig. 7, where the object and image are on the same side, the highest point of both will be somewhere in the strait line dai , and the lowest point of both will be somewhere in the strait line fci ; by what has been proved in proposition 143. Now if ac and df or the object and image are parallel to one another, the triangles aic and dif are similar: for the angle at i is the same in both of them; and because df and ac are parallel, the external angle will be equal to the internal opposite angle on the same side; Euc. b. I. prop. 29; that is, iac is equal to idf , and ica to ifd . Therefore the sides of these two equiangular triangles are proportional, Euc. b. VI. prop. 4, ac to df as ai to di , or the height of the object is to the height of the image, as the distance of the object from the lens to the distance of the image from the lens. The same might be proved in the same manner of their respective breadths or of any other corresponding diameters.

145. *The object itself is the object of plane vision, the last image is the object of refracted vision.* This follows from proposition 138.

146. *The apparent magnitude of an object seen through any lens is proportional to the apparent magnitude of its last image.* This likewise follows from proposition 138.

147. *The apparent magnitude of an object seen through a lens of any sort is equal to its apparent magnitude seen without the lens, if the lens is close either to the eye or to the object.*

First; suppose the eye to be close to the vertex i either of the convex lens gil , Plat. XII. fig. 7, or of the concave lens gil , fig. 8, the object seen through the lens will appear of the same size that it would do to the naked eye in the same station, or to the eye, if the lens was out of the way. For the diameter of the object of refracted vision df is to the diameter of the object of plane vision ac , as ei to bi ; or their diameters are as their respective distances from i the vertex of the lens, by proposition 143. Therefore their apparent diameters are equal, when they are seen from i , by proposition 128. Now the last image or object of refracted vision is what we see through the lens, and the object itself or object of plane vision is what we see when the lens is taken away from the eye: and consequently the apparent diameter of what we see, when we make use of the lens, upon supposition that the eye is placed close to the lens, is equal to what the naked eye in the same station would see, if the lens was taken from before it.

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Secondly; the apparent magnitude of an object is not altered by being seen through a lens, if the lens is close to the object. For when the real radiant is close either to a convex or a concave lens, the imaginary radiant will likewise be close to it, by propositions 73; that is, if the object, which consists of real radiants, touches the lens, the last image, which consists of imaginary ones, will touch it. And when both of them are close to the lens their diameters will be equal, by proposition 144. Therefore whether we use the lens and so see the last image, or from the same distance look at the object itself without the lens, their apparent magnitudes will be the same.

148. *If an object, when it is seen through a convex or plano-convex lens, is nearer to the lens than its principal focus, it will appear brighter than to the naked eye, and will be distinct and erect.*

When any object *ac*, Plat. XII. fig. 7, is nearer to the lens *gil* than its principal focus, the rays, that come from *b* or any other point in it, will diverge the less or will be the closer together for being refracted, by proposition 67, and consequently more of them will enter the eye; so that the object upon account of this refraction will appear the brighter, by propositions 104, 113. The rays of each pencil diverge after refraction, and therefore come to the eye in the same manner that they would if the object was at a moderate distance and no lens was used. But when the naked eye sees an object at a moderate distance, it appears distinct. Therefore, when an object, that is nearer to a convex lens than its principal focus, is seen through the lens, it will appear distinct. The refracted rays of each beam diverge less than the incident ones, by proposition 67; so that all the imaginary radiants are more remote than the real ones, by proposition 28, or the last image is farther from the lens than the object; and consequently, since the object is beyond the lens, the last image must necessarily be so too; or the last image is on the same side of the lens with the object. Therefore the last image, which is the object of refracted vision will be erect, by proposition 143.

149. *If an object, when it is seen through a convex or plano-convex lens, is farther from the lens than its principal focus, it will appear brighter than to the naked eye, and will be confused and erect, provided the eye is nearer to the lens than the place of the distinct picture.*

If *gsl*, Plat. XII. fig. 11, is a convex lens and *abc* is an object placed at a greater distance than the lens's principal focus, there is a place somewhere on the other side of the lens, suppose at *fed*, where the picture

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of this object would be distinctly painted upon a paper. This, whether a paper is held there or not, is called the place of the distinct picture. Now if the eye is any where between e and s , the object will appear brighter through the lens than it would to the naked eye. For the rays of each beam would diverge, if there was no lens; but the lens makes them converge, by proposition 63: upon that account they are kept closer together, and consequently more of them will enter the eye. The object seen in this manner will be confused, for when an object is seen by the naked eye, the rays of each beam diverge, if it is near, or are parallel, if it is very remote; no rays that come from one and the same point in an object naturally converge. Therefore since the rays of each beam by the refraction of the lens are made to converge, they come to the eye in such a state as the eye is not used to; and it has not a power of changing its figure so as to have distinct vision by rays that are in such a direction as no rays are naturally. But since there is no picture formed till the rays come to the retina, the picture there, though it is a confused one, will be inverted, and consequently the object will appear erect, by proposition 115.

150. *If an object, when it is seen through a convex or plano-convex lens, is in the lens's principal focus, it will appear brighter than to the naked eye, and will be distinct and erect.*

If the object abc , Plat. XII. fig. 7, was in the principal focus of the lens gil , the rays of each particular beam, such as eg , ei , el , will be parallel after refraction, by proposition 62: this makes them closer together than if they diverged, as they would do if the lens was taken away; and consequently more of them will enter the eye, which will make the object appear the brighter. The rays from very remote objects are nearly parallel, by proposition 30. Therefore an object placed in the lens's principal focus, because it is seen by parallel rays, will be as distinct as remote objects usually are to the naked eye. The last image def , because the refracted rays, which diverge from the several points of it, are parallel, will be very remote; and consequently it will be on the same side of the lens with the object. Therefore, by proposition 143, the object will appear erect.

152. *When an object seen through a convex or plano-convex lens is nearer than the principal focus it is magnified; unless the eye touches the lens, or the lens touches the object; and as the eye departs from the lens, its apparent magnitude will decrease.*

The apparent magnitude of df , Plat. XII. fig. 7, is greater, than that of ac , if the eye is at o , or if it is any where else, except at i ; that is, unless

less the eye touches the lens, the object of refracted vision will appear bigger than the object of plane vision. If the object abc continues in the same place or does not change its distance bi from the lens, then the last image will likewise be always at the same distance ei , by proposition 68. Therefore, by proposition 144, the real diameter of the last image will be invariable, wherever the eye is placed. And if the station of the eye is at the vertex i of the lens, the apparent diameter of the last image seen through the lens is equal to the apparent diameter of the object seen by the naked eye in the same station, by proposition 147. But if the eye withdraws from the vertex to any other station o , the apparent diameter of the last image, which is the object of refracted vision, will decrease, by proposition 126, because the eye is at a greater distance from it. By the eyes withdrawing itself from the vertex to any other remoter station o the apparent diameter of ac , when seen by the naked eye, will likewise decrease. Since then either df or ac have the same apparent diameter, when the eye is at i close to the lens, and since the apparent diameter of both decreases, when the eye withdraws from thence to o ; why should df appear larger than ac to the eye at o ; why should the apparent diameter of the object of refracted vision be greater than the apparent diameter of the object of plane vision, or why should the object, when seen from the station o through the lens, seem bigger than when seen from the same station by the naked eye? The reason is, that though the apparent diameter both of the last image and of the object are diminished by the departure of the eye from them, yet they are not diminished in the same proportion; the apparent diameter of the last image decreases in a less proportion than the apparent diameter of the object. For when the eye has withdrawn from i to o the apparent diameter of the last image decreases in the inverse proportion of oe to ie ; and the apparent diameter of the object in the inverse proportion of ob to ib . But oe bears a less proportion to ie , than ob bears to ib . Therefore the apparent diameter of the last image decreases in a less proportion than the apparent diameter of the object. That oe bears a less proportion to ie than ob to ib , may be easily proved, whatever is the length of io or whatever is the distance of the eye from the lens. For ie and ib are two unequal quantities of which ie the distance of the image is always the greater, by propositions 67, 28. And when the eye departs from the lens io is a given quantity added to both of them. Therefore both of them are equally encreased, but not proportionally: for a given addition when made to a greater quantity will encrease it less in proportion than when made to a less quantity. Thus particularly, let ie be to ib as 2 to 1, and let $io=1$, then $ie+io=2+1=3$;

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and

and $ib + io = 1 + 1 = 2$. Therefore, by the addition of io , ie is increased only in the proportion of 3 to 2, and ib by the same addition is increased in the proportion of 2 to 1. Since then the apparent diameter of the image is equal to that of the object, when the eye is close to the lens, and since though they both decrease as the eye departs from the lens, yet the apparent diameter of the image decreases the least of the two, it follows that, when the eye is at any distance from the lens, the object of refracted vision will be bigger than the object of plane vision; or the apparent magnitude of the object seen through the lens will be greater than its apparent magnitude seen by the naked eye.

The latter part of the proposition, that the apparent diameter of the object seen through the lens will decrease as the eye withdraws itself, has been proved already in proving the first part. All that we need observe upon this head is, that when the eye departs to an infinite distance from the lens, the apparent diameter of the object seen through it will be infinitely small. The use of this observation will be seen hereafter.

There is one thing more that will be worth our notice before we take leave of this proposition, which is, that though we say an object seen through the lens, when the eye is at o , appears to be magnified, yet by that we mean only that the apparent size, if seen through the lens, when the eye is there, is greater than it would be, if seen by the naked eye at the same station o . For the apparent magnitude of it seen through the lens when the eye is at o , is not greater than the same object might appear to the naked eye, if the naked eye was nearer to it. Nay the eye may be so near as to see the object bigger without the help of a lens, than it sees it at the station o with this help. Thus particularly, by the second part of this proposition, the object seen through the lens, when the eye is at o appears less than it would, if the eye was at i . But if the eye was at i , the object would appear no bigger with the lens than without it, by proposition 147. Therefore the object with the help of the lens does not seem so big, when the eye is at o , as it would without the lens, if the eye was at i or in the station of the lens.

Therefore the chief uses of a convex lens in vision may be reduced to these two. First, if we want to look at an object, and the eye must necessarily be placed at a certain distance from it; we shall see it larger at that distance by the help of a lens than we could without one. Secondly, if we want to see an object of a certain size, and the naked eye, in order to see it of that size, must come so near to the object as to render it confused; by the help of a lens we may see it of that size and yet see it distinctly.

153. *If an object seen through a convex or plano-convex lens is more remote than the principal focus, and the eye on the other side is nearer than the place of the distinct picture, the object appears magnified, unless the eye touches the lens, and the apparent magnitude of it will be inversely as the eyes distance from the distinct picture; so that as the eye withdraws from the lens towards the picture the apparent magnitude of the object will encrease.*

Let *gsl* be the lens, *abc* an object farther off from the lens than its principal focus, and *fed* the distinct picture received upon a paper. In order to prove the proposition, we will shew that wherever the eye is between *s* and *e* or between the lens and the picture, if it turns itself one way and looks through the lens at the object *abc* or turns itself the other way and looks at the picture upon the paper, the apparent diameter of the object will always be equal to the apparent diameter of the picture. Suppose the eye was at the side of the lens *g*, then it would see both the object and the picture without looking through the lens, and since the real diameter of the object is to the real diameter of the picture, as their respective distances from the lens, or from the eye when it is at *g*, by proposition 88, consequently their apparent diameters will be equal, by proposition 128. Now if the eye, instead of being at *g*, was close to the lens at *s*, and was to look through it, the apparent diameter of the object would not be altered, by proposition 147; and since its distance from the picture *fed* is the same whether it is at *g* or at *s*, it follows that if the eye is close to the lens, whether it turns itself one way and looks through it at the object, or turns itself the other way and looks at the picture upon the paper, the apparent diameters of the object and picture are equal; and either of them appears of the same size that the naked eye in the same station would see the object. But suppose the eye was close to the paper at *f* or *e* or *d*, the apparent diameter of the picture would then be infinite, by proposition 126. And if the eye when close to the paper was to turn itself and look at the object, the apparent diameter of that would likewise be infinite. For suppose it was at *f*; all the rays, which enter it, are those collected there; and they are only such as come from the point *c*: therefore nothing but the point *c* will appear diffused over the whole surface of the lens though that surface was ever so large. The same would be true of the point *b*, if the eye was at *e*; and of the point *a*, if it was at *d*. Therefore in this station of the eye every point in the object is infinitely magnified, and consequently its apparent diameter seen through the lens is infinite. Since then the object seen through the lens, and the picture seen upon the paper appear equal, when
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the eye is close to the lens, and appear equal again, and are both of them infinite, when the eye is at the paper; it follows that they must both of them have apparently encreased, whilst the eye departed from s to e , or from the lens to the paper: and they must both of them have encreased equally; because they were equal at first and are equal at last. But since they have encreased equally, wherever the eye was between s and e , the apparent diameter of the object seen through the lens must always have been equal to the apparent diameter of the picture seen upon the paper. From hence we may prove the three parts of the proposition.

First, the object will be magnified, if the eye does not touch the lens, for it appears of the same size either with the lens or without it, if the eye is at s , and as the eye withdraws towards a the apparent magnitude of the object encreases.

Secondly; the apparent diameter of the object seen through the lens is every where equal to the apparent diameter of the distinct picture; and the apparent diameter of the distinct picture is every where inversely as the distance of the eye from the picture, by proposition 126. Therefore the apparent diameter of the object seen through the lens is inversely as the distance of the eye from the distinct picture.

Thirdly, that as the eye withdraws from the lens towards the distinct picture, the apparent magnitude of the object seen through the lens will keep encreasing has been proved as we went along.

154. *If an object seen through a convex or plano-convex lens is placed in the principal focus, it will be magnified; unless the eye touches the lens; and the apparent magnitude of it will not be altered by the eyes withdrawing from the lens.*

If the object abc , Plat. XII. fig. 7, was in the lens's principal focus, the refracted rays of each beam would neither diverge, as they are there represented, nor would they converge, as in fig. 11, but would be parallel, by proposition 62. But since the refracted rays are parallel, if we consider them as diverging ones, the imaginary radiants or the last image must be at an infinite distance, and if we consider them as converging ones the focuses or the distinct picture must be at an infinite distance, by propositions 30, 31. Now when the eye departs from a lens, if the apparent magnitude of an object seen through the lens decreases, it is because the eye departs to a greater distance from the last image, as in proposition 152; or if the apparent magnitude of the object encreases, it is because the eye by departing from the lens gets nearer to the place of the distinct

distinct picture, as in proposition 153. But when the object is in the principal focus, there is neither last image nor distinct picture, except at an infinite distance. Therefore the apparent magnitude of the object, as the eye departs from the lens, will neither decrease nor encrease; since no finite departure of the eye can either remove it farther from the last image or bring it nearer to the distinct picture. From hence it follows, that when an object is placed in the lens's principal focus, its apparent magnitude seen through the lens is the same whether the eye touches the lens, or is withdrawn to any finite distance whatever. Now if the eye touches the lens, the object is not magnified, but appears of the same size through the lens that it would appear of without it to the eye in the same station, by proposition 147. But since, let the eye be where it will behind the lens, the object appears of the same size as when the eye touches the lens, why should we say that the object is not magnified in this station of the eye but will be magnified in any other? The answer is, that if in any place as *o* behind the lens, the object *abc* appears as big as the naked eye could see it at the station *i*, then at the station *o* the object seen through the lens is said to be magnified, not because it appears bigger than the naked eye could see it in any station, but because it appears bigger than the naked eye could see it in that particular station *o*.

In the three last propositions we have considered what alterations are made in the apparent magnitude of an object seen through a convex or plano-convex lens, by the motion of the eye. In the two following ones we will consider what alteration it will undergo by the motion either of the lens or of the object.

155. *When the eye and object are fixed in their stations, as a convex or plano-convex lens moves from the eye the apparent magnitude of the object keeps encreasing, till it has got to the middle between them; and then as the lens moves farther, the apparent magnitude keeps decreasing, till it has got close to the object: provided that in this whole motion of the lens the eye is never more remote from it than the place of the distinct picture.*

When the lens is at either extreme, either close to the eye or close to the object, by proposition 147, the object is not magnified. But when the lens is any where between the two extremes, by propositions 152, 153, 154, the object is magnified. Therefore it will be most magnified, when the lens is equally removed from the two extremes, or when it is in the middle between the eye and the object. Now since the object is not magnified, when the lens is close to the eye, but is most magnified
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when the lens is half way between the eye and the object, it follows that, as the lens moves from the eye towards that middle station, the apparent magnitude must keep encreasing. And since the object is magnified, when the lens is in that middle station, but is not magnified at all, when it gets to the other extreme and is close to the object, it follows that, as the lens moves from that middle station towards the object, the apparent magnitude must keep decreasing.

156. *When an object seen through a convex or plano-convex lens departs from the lens; if the eye is in the principal focus, the object's apparent magnitude is not altered; if the eye is more remote than the principal focus, the apparent magnitude is encreased; if the eye is nearer than the principal focus the apparent magnitude is diminished.*

When an object departs from a convex lens, its last image on the same side of the lens with itself departs at the same time, by proposition 72, and consequently the apparent magnitude is diminished, so that if the object was to depart to an infinite distance its apparent magnitude would be infinitely small, by proposition 152. But whilst the object and its last image depart on one side, the distinct picture will approach on the other side where the eye is, by proposition 71: and if the distinct picture was to approach till it came to the place where the eye is, the object would be infinitely magnified, by what was said in proving proposition 153. Now if the eye is supposed to be in the principal focus of the lens, the distinct picture cannot come to the place of the eye, unless the object departs to an infinite distance, as appears under proposition 107. Therefore when the eye is in the principal focus, if the object departs to an infinite distance, the apparent magnitude is infinitely diminished upon one account and infinitely encreased upon another; infinitely diminished, because the last image is at an infinite distance; and infinitely encreased, because the distinct picture will be at the same place with the eye. Consequently, since in an infinite departure of the object from the lens, the apparent magnitude is encreased and diminished equally, it must have kept decreasing just as much upon one account, whilst the object was departing, as it has kept encreasing upon the other; and therefore upon both accounts together, it must have been always the same.

If the eye was more remote than the principal focus; then the apparent magnitude, as before, would not be infinitely diminished, but by an infinite departure of the object. But since, when the object is at an infinite distance, the distinct picture will be in the principal focus, and since the eye is farther from the lens than the principal focus, the distinct picture

picture will be at the place where the eye is, before the object has departed to an infinite distance. Consequently a finite departure of the object will magnify it infinitely. But if the apparent magnitude may upon one account become infinitely great by the objects departing only to a finite distance, but cannot upon the other account become infinitely small but by departing to an infinite distance; it follows that in this station of the eye whilst the object is departing, its apparent magnitude must encrease faster than it decreases, and therefore it will encrease upon the whole.

If the eye was nearer to the lens than the principal focus, the apparent magnitude would, as before, be infinitely diminished by the objects departing to an infinite distance. But it cannot be infinitely encreased, unless the distinct picture is at the same place with the eye. Now when the object has departed to an infinite distance, the distinct picture is no nearer to the lens than its principal focus. Therefore, since the eye is supposed to be nearer, an infinite departure of the object will not bring the distinct picture to the place of the eye, and consequently will not infinitely encrease its apparent magnitude. But since an infinite departure of the object will infinitely diminish its apparent magnitude upon one account, but will not infinitely encrease it upon the other account, if the eye is nearer than the principal focus, it follows that the apparent magnitude in the departure of the object must decrease faster than it encreases, and consequently must decrease upon the whole.

157. *We commonly make a wrong judgment of an objects apparent distance, when we see it erect, through a convex or plano-convex lens.*

The object of refracted vision is always more remote from the lens than the object of plane vision. For if the refracted rays diverge less than the incident ones, as in proposition 148, then the imaginary radiants or the last image, will be more remote than the real ones, or than the object, by proposition 67. Or if the refracted rays either are parallel or converge, as in propositions 149, 150, then the imaginary radiants are at an indefinite distance, and the place of the last image though indeterminate will be at a greater distance than the place of the object. And yet in any of these cases, the object, when seen through the lens, seems to be nearer, than when seen by the naked eye. For in looking through a lens we seldom use more than one eye at once; and there is commonly nothing seen besides the object, so that there is no visible extension between the eye and the object by which we may estimate its apparent distance. In these circumstances the apparent magnitude is the principle, by which we judge of its distance: and consequently, by proposition 135, since

the lens encreases its apparent magnitude, we shall judge the distance to be less than it is. But if the object is placed at the end of a ruler, which is parallel to the horizon, and the lens is large enough for us to see the ruler through it; there is then a visible extension between the lens and the object. And by seeing the ruler through the lens its apparent length, which is the apparent distance of the object at the end of it, will be planely encreased. Or if the lens is large enough to use both eyes at once and the object is placed, as in proposition 135, so that part of it appears through the lens and part of it may be seen by the naked eye at one side of the lens, we have then another principle to judge by; and the part which is seen through the lens will seem at a greater distance than the other part, which is seen by the naked eye.

158. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye on the other side is nearer than the distinct picture, the place of the object of refracted vision is at a greater distance from the lens than the place where we look for it.*

In this situation the object seen through the lens appears double; and if the right eye is shut, the right-hand object vanishes; or if the left eye it shut, the left-hand object vanishes. Therefore the place of the object seen by refraction is beyond the concurrence of the optic axes, or at a greater distance than the place where we look for it, by proposition 117.

159. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye is at a greater distance from it on the other side than the place of the distinct picture, the object appears fainter than to the naked eye, it is inverted, and may be seen distinctly.*

If the object *abc*, Plat. XII. fig. 11, is at a greater distance from the lens *gsl* than the lens's principal focus, and the eye at *o* is at a greater distance than *fed* the place of the distinct picture; then the last image or object of refracted vision is the distinct picture itself, by proposition 138. For after the rays that come from the several points in the object, have met in as many focuses, at the place of the distinct picture, they will diverge last from the focuses, when they come to the eye at *o*. Now as the distinct picture is nearer to the eye than the object, the rays coming from that picture will diverge so much, that many of them will be too far asunder to enter the eye: and upon this account the object of refracted vision will appear faint. The object and the distinct picture, which is now the object of refracted vision, are on contrary sides of the lens: there-

therefore the object will appear inverted, by proposition 143. The rays diverge from the several imaginary radiants in the picture *fed* just as they would from the several real radiants of an object in the same place ; and consequently if the eye is too near it the appearance will be confused, just as it would be in too near an approach of the eye to a real object. But if the eye is removed to a moderate distance it will appear as distinct as any real object would at the same distance.

160. *When an object is more remote from a lens than its principal focus, and the eye is more remote than the distinct picture, the apparent diameter of the object is directly as the picture's distance from the lens, and inversely as its distance from the eye.*

In this position of the object and eye the distinct picture is the object of refracted vision, by proposition 138. And the apparent diameter of the object seen in this manner is proportional to the apparent diameter of that picture. Now the apparent diameter of the picture, like that of any real object is directly as its real diameter and inversely as its distance from the eye, by propositions 127, 126. But the real diameter of the picture is directly as its distance from the lens, by proposition 88. Therefore the apparent diameter of the picture, and consequently of the object, is directly as its distance from the lens, and inversely as its distance from the eye.

161. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye is farther from it than the distinct picture ; the place of the object of refracted vision is nearer to us than the place where we look for it.*

We look at the lens, and the object of refracted vision is nearer to us ; for it is at the place of the distinct picture, which is between the lens and the eye. And accordingly we find that the object thus seen appears double ; and in this double appearance the right-hand one is seen by the left eye, and the left-hand one by the right eye. Therefore the place of the object seen by refraction is nearer to us than the concurrence of the optic axes or than the place where we look for it, by proposition 118. But suppose the object that we are looking at to be a candle, and let a pencil or any other slender body be held at the place of the distinct picture as a guide to our eyes ; if we look directly at the pencil we may see the candle at the same time ; but then it will appear double no longer. For the optic axes concurring in the pencil, concur, by the supposition, in the place of the candle seen by refraction : therefore the candle ought to appear single, by proposition 116.

162. *When an object is farther from a convex or plano-convex lens than its principal focus, and the eye is farther from it than the distinct picture; if the lens is moved towards the right hand or towards the left, the object will appear to move in the same direction; but if the eye is moved towards the right hand or towards the left, the object will appear to move in the contrary direction.*

First; from any point as b in the object abc let a line bse be conceived to be drawn through the middle of the lens to the object of refracted vision fed . This line is the axis of the rays that come from b . Now, since the object is supposed immovable, one end of this line is fixed at b , and the other end will always be at that point e in the distinct picture, where the rays that come from b are united in their focus. But if the lens gsl is moved towards the left hand the middle point s moves the same way with the whole lens. And since the line bse always passes through the middle of the lens; this line whilst the lens is moved towards the left-hand will turn round upon b as upon a center and will move towards the left hand with the point s . But the other end of it is always at e in the distinct picture. Therefore as this line turns round upon b and moves towards the left hand, the point e in the distinct picture will move the same way. This might be proved of every other point in the picture; and consequently the whole picture or the object of refracted vision moves in the same direction with the lens; towards the left hand, if the lens moves towards the left; or towards the right hand, if the lens moves towards the right. So that, when the lens is moved, the apparent motion of the object seen through it arises from the real motion of the picture, which is the object of refracted vision.

Secondly; suppose the eye to be at o directly before s the middle of the lens; and from s through e the middle of the distinct picture to the eye at o conceive the line seo to be drawn. If the object and lens are both fixed, the distinct picture fed and the middle point of it e will be immovable. Now the line seo passes through this middle point. Therefore if the eye o or one end of the line seo is moved towards the left hand, it will turn round upon e as a center, and the other end of it next the lens will move the contrary way or towards the right; that is, towards l . The eye refers the distinct picture to the lens and determines the apparent place of it by the part of the lens upon which it appears, by proposition 161, and consequently the end of the line seo which is at the lens determines the apparent place of the distinct picture. But if the eye moves towards the left hand, that end of the line seo which is next the lens moves towards the right; and upon that account the picture or object
of

of refracted vision appears to move upon the lens towards the right: or if the eye was to move towards the right hand, the picture would seem upon the lens to move towards the left. So that, when the eye is moved, the apparent motion of the object seen through the lens arises from an apparent motion of the picture or last image in reference to the lens.

The many cases of vision through convex or plano-convex lenses is owing to the many different directions, in which the refracted rays may come to the eye. When the object is nearer than the principal focus, the refracted rays diverge, or if it is in the principal focus, they are parallel, when they come to the eye, wherever the eye is placed. When the object is more remote than the principal focus if the eye is nearer than the distinct picture, the rays are converging, when they fall upon it; if the eye is at the place of the distinct picture, they are met in their several focuses; if it is more remote than this, they have passed their focuses and have begun to diverge from thence. But the cases of concave or plano-concave lenses are very few; for wherever the eye is, the refracted rays, when they fall upon it, always diverge from an image on the same side of the lens with the object.

163. *When an object is seen through a concave or plano-concave lens, it seems fainter, than when seen by the naked eye; it appears erect and may be seen distinctly.*

If the object *abc*, Plat. XII. fig. 8. is seen through the concave lens *gil*, the rays that come from the point *b* or any other points in the object are made to diverge more by passing through the lens than they would have done, if the object had been viewed with the naked eye, by proposition 76. But the more they diverge, the farther they are asunder, when they come to the eye at *o*; and consequently fewer of them will come within the compass of the pupill and enter the eye. Therefore the lens will make the object seem fainter. The refracted rays diverge from imaginary radiants *d*, *e*, *f*, &c. which are behind the lens, by proposition 77. Therefore the object and the last image are on the same side of the lens, and the object seen through the lens will appear erect, by proposition 143. If the eye is very near to the lens, so as to receive the extreme rays *em*, *en*, of the cone coming from the imaginary radiant *e*, and the extreme rays of all other refracted cones coming from the several imaginary radiants *d*, *f*, &c. which compose the last image; these extreme rays will diverge too much and the picture upon the retina of most eyes will be confused, by proposition 99. But there are two ways of ren-

rendering it distinct. One way is to have a piece of paper with a pin-hole in it close to the lens; for by this means the extreme rays will be stopped and only the middle rays of each cone, which are less diverging, will enter the eye. The other way requires no such paper; for if the eye only withdraws itself to a moderate distance from the lens, the extreme rays will be so far asunder, as not to enter the pupill, only the middle or less diverging rays will enter it, and then the picture upon the retina and consequently the vision will be distinct, by propositions 99, 113.

164. *The apparent magnitude of an object seen through a concave or plano-concave lens is diminished, unless the eye touches the lens or the lens touches the object; and as the eye departs from the lens, its apparent magnitude decreases.*

If the eye touches the vertex of the lens at i , the apparent diameter of the last image df is the same, that the apparent diameter of the object itself ac would be to the naked eye in the same station, by proposition 147. Now as the eye withdraws from the lens to any point o , its distance both from the last image and from the object encreases, and consequently the apparent magnitude either of the last image seen through the lens from the station o or of the object seen by the naked eye at the same station will be less, than when seen from the station i , by proposition 126. But as the eye withdraws from the lens, its distance from the last image, which is the object of refracted vision, encreases faster in proportion than its distance from the object itself, which is the object of plane vision. Therefore since, by proposition 126, the apparent diameter decreases as the distance of the eye encreases, the apparent diameter of the object of refracted vision decreases faster than the apparent diameter of the object of plane vision. Their apparent diameters were equal when the eye was at i . But the apparent diameter of that will be the least, when the eye is at o , which in this departure of the eye has decreased the most; and this is the apparent diameter of the object of refracted vision. Therefore when the eye is at o the object of refracted vision will be apparently less than the object of plane vision; or the object, when seen through the lens by the eye at o , will seem less, than when seen by the naked eye at the same station. That the distance from e the last image encreases faster than the distance from b the object, whilst the eye departs from i to o , is evident from what has been shewn under proposition 152. For the distance of the image from the lens is always less than the distance of the object: ie is less than ib : and as the eye departs from

from i to o , both these quantities encrease by the addition of one and the same quantity io made to each of them. But if the same quantity is added respectively to two unequal quantities, the lesser of the two will be encreased more in proportion by this addition than the greater is; so that oe will bear a greater proportion to ie , than ob bears to ib .

The second part of this proposition, that the apparent magnitude of the object seen through the lens will decrease, as the eye departs from the lens, has been sufficiently proved in proving the first part.

165. *When the eye and object are fixed in their places, as a concave or plano-concave lens moves from the eye the apparent magnitude of the object keeps decreasing, till it has got to the middle between them, and then, as the lens moves farther, the apparent magnitude keeps encreasing, till it has got close to the object.*

When the lens is at each extreme, when it is either close to the eye or close to the object, the apparent magnitude of the object is the same, when seen through the lens, as when seen with the naked eye, by proposition 147. In all other stations of the lens, the object seen through it appears to be diminished, by proposition 164. But since the object is not at all diminished, when the lens is at either extreme, and is diminished, when the lens is in any intermediate station; it will be most of all diminished, when the lens is equally removed from each extreme or is in the middle between the eye and the object. And since the object is not diminished, when the lens is close to the eye, and is most diminished, when it is in the middle station, it follows, that, as the lens moves from the eye to this station, the apparent magnitude of the object must keep decreasing: so likewise since the object is most diminished, when the lens is in the middle station and is not diminished at all, when it is close to the object; it follows, that, as the lens moves from this station to the object, the apparent magnitude must keep encreasing.

166. *The apparent magnitude of an object seen through a concave or plano-concave lens decreases as the object departs from the lens.*

For as the object departs, the last image will depart with it, but slower, by propositions 78, 138. And consequently as the object departs the diameter of the last image, which is the object of refracted vision, will decrease on account of the objects distance from the lens being encreased, by proposition 144.

167. *We commonly make a wrong judgment of an objects apparent distance, when we see it through a concave or plano-concave lens.*

We

We seldom use both eyes in looking through a concave lens, and there is seldom any thing else to be seen through it besides the object. Therefore we judge of the objects distance by its apparent magnitude: and as the lens diminishes the object, we conclude it to be farther off, when seen through the lens, than when seen without it. This is certainly a wrong judgment: for the object of refracted vision is nearer to the eye than the object of plane vision, by prop. 76, 28. And this wrong judgment of the apparent distance, which is owing to our want of all other principles to go upon, except the apparent magnitude, may be corrected, as in proposition 157. For if the object is placed upon a ruler, which is parallel to the horizon, so that we may see the ruler through the lens, at the same time that we see the object; then if we judge of the objects distance by the visible extension of the ruler, we shall see that it is nearer to us, when we look at it through the lens, than when we look at it with the naked eye. Or if we place the object in such a manner as to see one part of it through the lens, and another part of it by the naked eye, and the lens is broad enough to see through it with both eyes together; upon comparing the apparent distances of the two parts, we shall find, that the part, which we see through the lens, appears nearer than the other.

168. *Short-sighted people see objects distinctly at moderate distances by the help of concave lenses.*

The defect of such persons eyes as are short-sighted has already been explained, in proposition 121. Their eyes are too convex and will bring the several pencils of rays to their respective focuses, or will form a distinct picture, before the rays come to the retina, unless the object is brought very near to the eye. Now all that is done by bringing the object near to the eye may be done by looking through a concave lens, whilst the object remains at a moderate distance. Bringing the object very near makes the rays diverge so much the more, by proposition 28. And the refraction of a concave lens will do the same, by proposition 76. So that when we look through a concave lens at objects, which are not near us, the rays come to the eye, as they do when we look at very near objects without a lens. Therefore such persons, as see very near objects distinctly with the naked eye, will be able to see remoter objects as distinctly by the help of a concave lens.

Though concave glasses help short-sighted people for the present, yet the constant use of them will make their eyes worse. For when an object is seen through a concave lens, the rays, that come from it, diverge as much as they would if that object was very near. So that the consequence
of

of using a concave lens constantly is the same as if a man was only to look at near objects. The eye has no occasion to strain itself; but is always kept and by custom is confirmed in that very figure, which is fit only to see near objects, and which is the occasion of short-sightedness. And though the persons, who have this defect in their eyes, find that it strains their eyes to look at distant objects without their glass, yet this straining would be of service to them; it might gradually change the figure of their eyes, and either remedy the defect, or at least prevent it from encreasing.

169. *Old people see objects distinctly at moderate distances by the help of convex lenses.*

The defect of old peoples eyes has been explained, in proposition 122. Their eyes are too flat and would not refract the rays enough to bring them to their respective focuses, or to form a distinct picture upon the retina, unless the object was removed to a considerable distance from the eye. Now removing an object far from the eye only makes the rays of each beam diverge the less, by proposition 28. And though the object is near, yet the rays will be made to diverge less, if they pass through a convex lens, by proposition 67. Therefore they, who can see only distant objects distinctly with the naked eye, will be able to see near ones as distinctly, if they make use of convex spectacles. For the rays come to the eye from near objects, when we look through a convex lens, just as they do from remote ones, when we look at them with the naked eye.

Persons, that have good eyes, and when they are employed in fine work, frequently use convex spectacles only for the sake of magnifying their work, will soon be unable to see distinctly any thing, that is near them, without such spectacles. For the rays come to the eye through convex lenses in the same manner that they would do from remote objects: and consequently the eye, when we look through such lenses, puts on the figure that fits it for seeing remote objects distinctly. Therefore if by using spectacles we keep it constantly of this figure, it will by degrees fix itself too much to be altered; and then we shall be unable without spectacles to see any objects distinctly but remote ones. The work might be magnified, as much as there is occasion for, by looking very near to it without spectacles: and though this may be found to strain the eyes, yet such straining will seldom do them any great harm: the only harm it can do them is to make them short-sighted; and the good it may do them is to prevent them from ever having occasion for convex spectacles.

170. *Objects seen through a plane glass are erect, nearer, and seem brighter and larger, than when they are seen by the naked eye.*

If GL, Plat. XII. fig. 6, is a plane glass the object PR, when seen through it will appear erect. The axes of the several beams, that come from any points P, Q, R, in the object, are the rays, which pass through the glass without being refracted: and these rays, by proposition 41, must be, PA, QC, RB, perpendicular to the surface of the glass. But lines perpendicular to a plane surface are parallel to one another. Euc. b. XI. prop. 6. Therefore the axes PA, QC, RB are parallel. Now all the rays of each beam diverge from one and the same point before refraction, and they diverge likewise from one and the same point after the two refractions, which they suffer in passing through the glass, by proposition 45. But the axis of any beam is one of the rays. Therefore the axis of any beam both before and after refraction diverges from the same point with the other rays of the same beam. But the axis itself is not refracted at all; and consequently the point from whence the other rays diverge both before and after refraction must be somewhere in the axis. Thus for instance, the rays of the highest beam, and PA amongst the rest, diverge before refraction from the point P. Suppose that all the other rays diverge after refraction from the point p , then this point p must be somewhere in the axis PA; for since PA passes strait forwards and yet after all the other rays are refracted is to diverge from the same point with them, this point must be somewhere in the line PA, or else the ray, whose direction is in the line PA could not diverge from it. The same may be said of the rays that come from the lowest point R; they will diverge after refraction from some point in the axis RB. From hence it follows that to the eye on the other side of the glass, the highest point P of the object will appear in the line Pa, and the lowest point R will appear in the line Rb, by proposition 137. Now Pa and Rb are always parallel, and consequently cannot cross one another between the eye and the object. Therefore that axis, which is highest at the object, will be highest at the eye, and that, which is lowest at the object, will be lowest at the eye. And since the highest point of the object appears in the highest axis or line aP, and the lowest point appears in the lowest axis or line bR, and the middle point Q appears for the same reason in the middle axis or line cQ; every point appears in its proper place, or the object is erect.

The last image is nearer to the glass than the object: for every imaginary radiant is nearer than every real one. If abkl, Plat. IX. fig. 9, is a plane glass and c is a radiant, the rays cd, ce, which diverge from it will be

be refracted, when they enter the glass, into the lines dg , en , and again, when they come out of it, into the lines gi , no , parallel to cd , ce , by proposition 47. But if from g and n lines as gf , nf , are drawn parallel to cd , ce , they will meet at f . Therefore the rays after their last refraction appear to diverge from the imaginary radiant f , which is nearer than the real one c . In like manner every imaginary radiant will be nearer than every real one in the object PQR , Plat. XII. fig. 6, or the last image pqr , which is the object of refracted vision, will be nearer than the object itself.

If gn , Plat. IX. fig. 9, is the diameter of the pupil the rays gi and no will enter it, which, if there had been no glass between the radiant c and the eye, would have passed straight on in the lines cd , ce continued, and when they came to the place of the eye would have been too far asunder to enter it. Therefore since more rays from each radiant enter the eye, when it looks through the glass, than would have entered, if there had been no glass, each point of the object will appear the brighter.

The last image is equal to the object. For the object PQR is contained between the extreme axes PA , RB ; and so is the last image pqr . And these axes are parallel, as has been proved already. Therefore they are just as far asunder at the place of the image as they are at the place of the object; and consequently the height of the image will be equal to the height of the object. But then the image is nearer to the eye than the object; and for that reason will appear larger, by proposition 126.

Common window-glass is too thin to make any alteration in the apparent distance or apparent magnitude of objects, which are seen through it. For it is plain that if the glass $abkl$, Plat. IX. fig. 9, had no thickness, or if the sides ak and bl were close together, then dg and en would have no length, so that g would coincide with d and n with e . But gf and dc are parallel. Therefore if g and d are close together gf and dc would coincide, and so likewise would nf and ec , that is, f and c would be at the same place. Now if the imaginary radiant is at the same place with the real one, the last image will be at the same place with the object and consequently will appear at the same distance and of the same size. This would be the case if the glass had no thickness at all: from whence it is evident that the less the thickness of the glass is the less alteration will be made in the apparent distance and apparent magnitude of the object. Therefore in such thin glass as common window-glass the alteration is too small to be perceived.

Of telescopes and microscopes.

171. *An astronomical telescope consists of two convex lenses; whose distance from each other is equal to the sum of their principal focuses: that lens, which is towards the object, is called the object-glass, and that, which is next the eye, is called the eye-glass.*

IF *nl*, Plat. XIII. fig. 1, is one convex lens, whose principal focal distance is *mf*, and *bd* is another convex lens, whose principal focal distance is *cf*; let these two lenses be so placed that the distance between them may be equal to *mf* added to *cf*, that is, let their distance from each other be *mc*, and in this situation they make an astronomical telescope.

172. *Very remote objects seen through an astronomical telescope appear distinct and inverted.*

If *zxy*, Plat. XIII. fig. 1, is a very remote object, the rays that come from one and the same point in it will be parallel to one another, by proposition 30. Therefore let the parallel lines *pn*, *pm*, *pl*, be the rays, which come from *x* the middle point in the object, let *an*, *am*, *al*, be those, which come from *z* the highest point, and *gn*, *gm*, *gl*, those, which come from *y* the lowest point. These rays, and likewise the rays which come from all the other points, because they are parallel to one another, will be collected into their several focuses at *g*, *f*, *e*, the principal focus, by proposition 60, and will there paint the distinct picture of the object. This distinct picture is the last image, and therefore, by proposition 145, is the object of refracted vision. But by the construction of the telescope *gfe* is in the principal focus of the eye-glass, for *fc* is the eye-glass's principal focal distance. Consequently the rays which diverge from any point *g* in this picture, will become parallel to their axis, after they have passed through the eye-glass, by proposition 61. So that if *gc* is the axis of the beam that proceeds forwards and diverges from the point *g*, this axis goes strait through the lens, but any other ray, as *gd*, will be refracted, and will come out of the eye-glass in the direction *do* parallel to *gc*. The same may be proved of the rays of any other beam. Therefore if the eye is at *o* or any where on the other side of the eye-glass, the last image *fed* or object of refracted vision may be seen as distinctly as any object is by the naked eye, when that object is so remote that the rays of

Fig. 1.

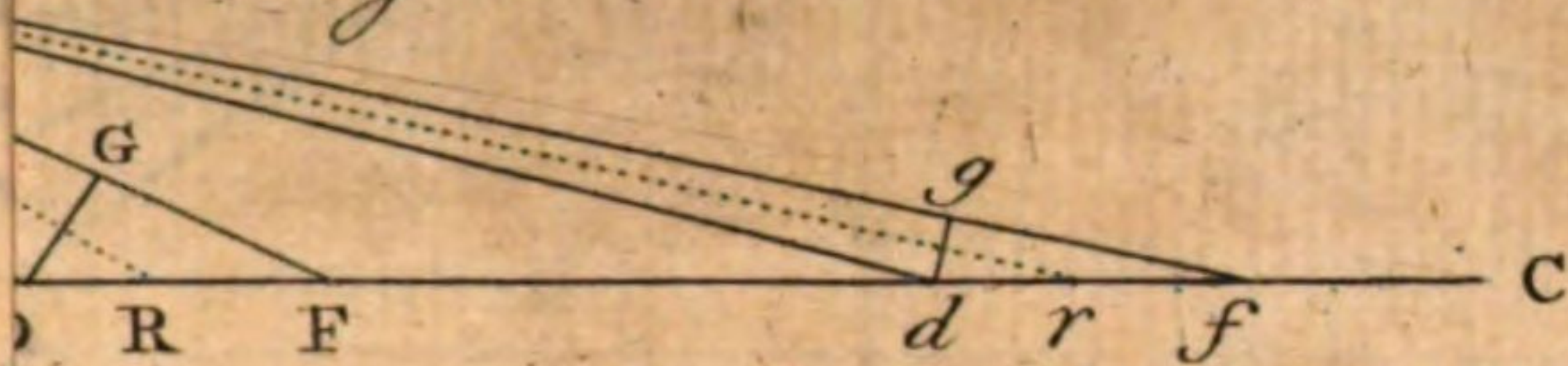


Fig. 3.

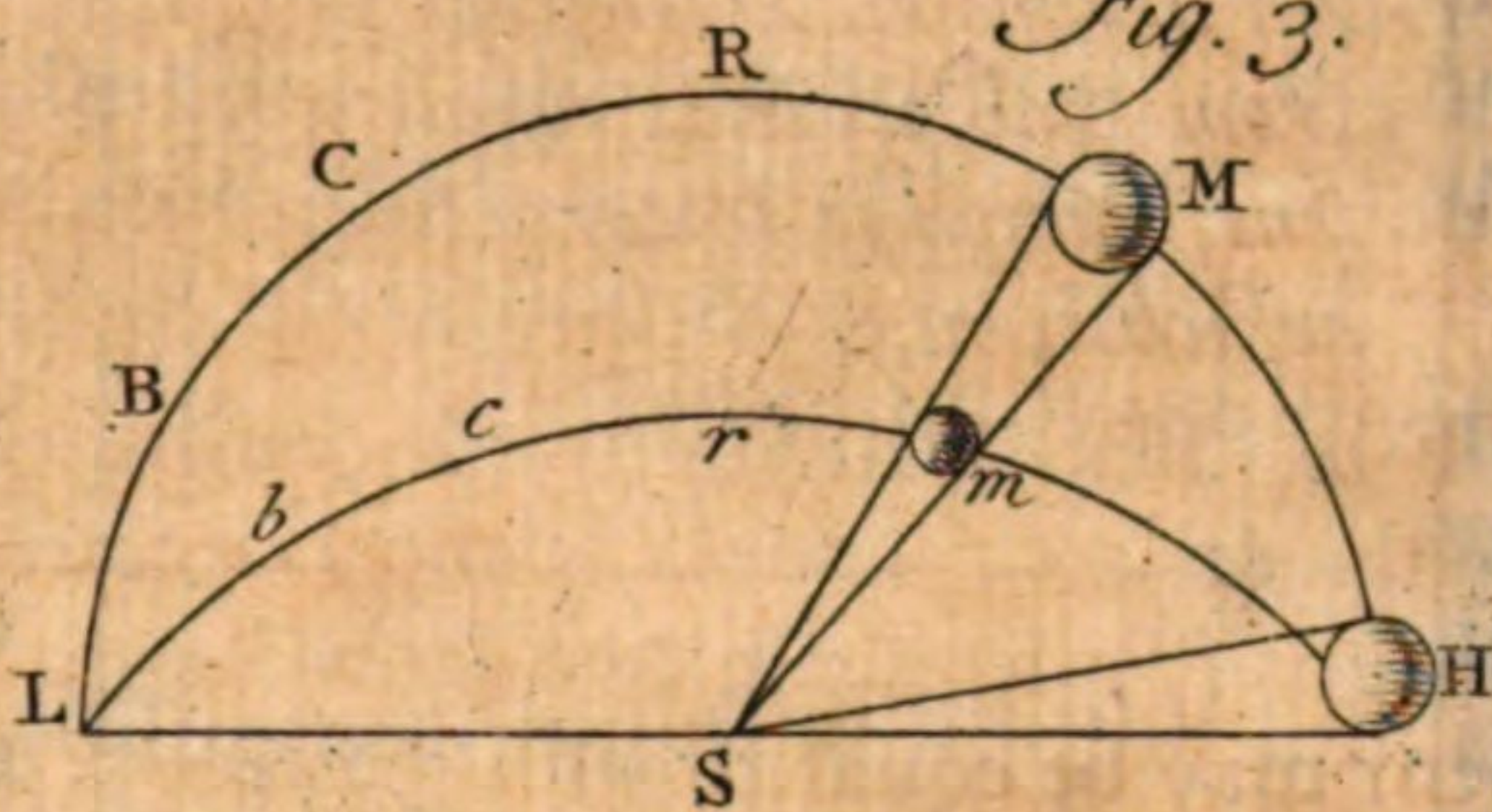


Fig. 5.

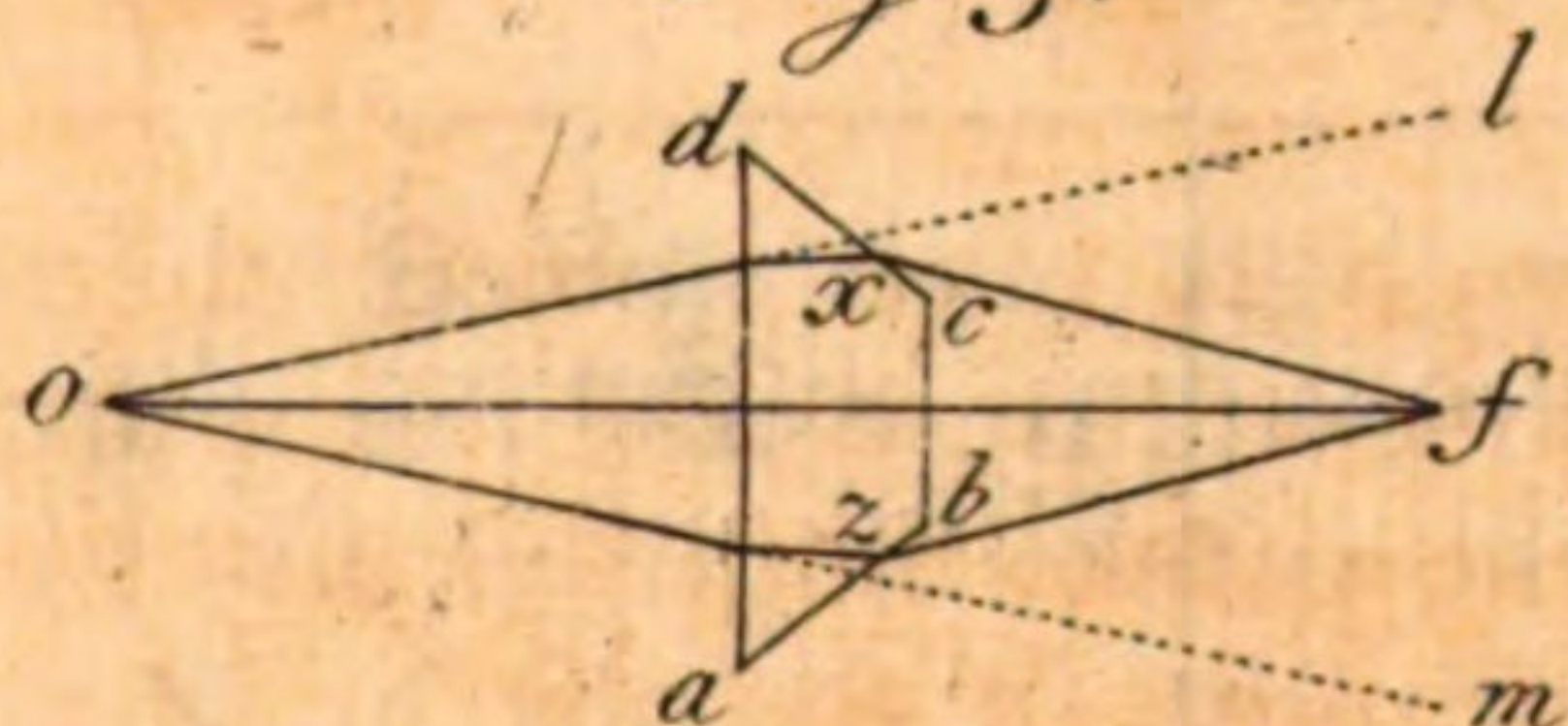


Fig. 6.

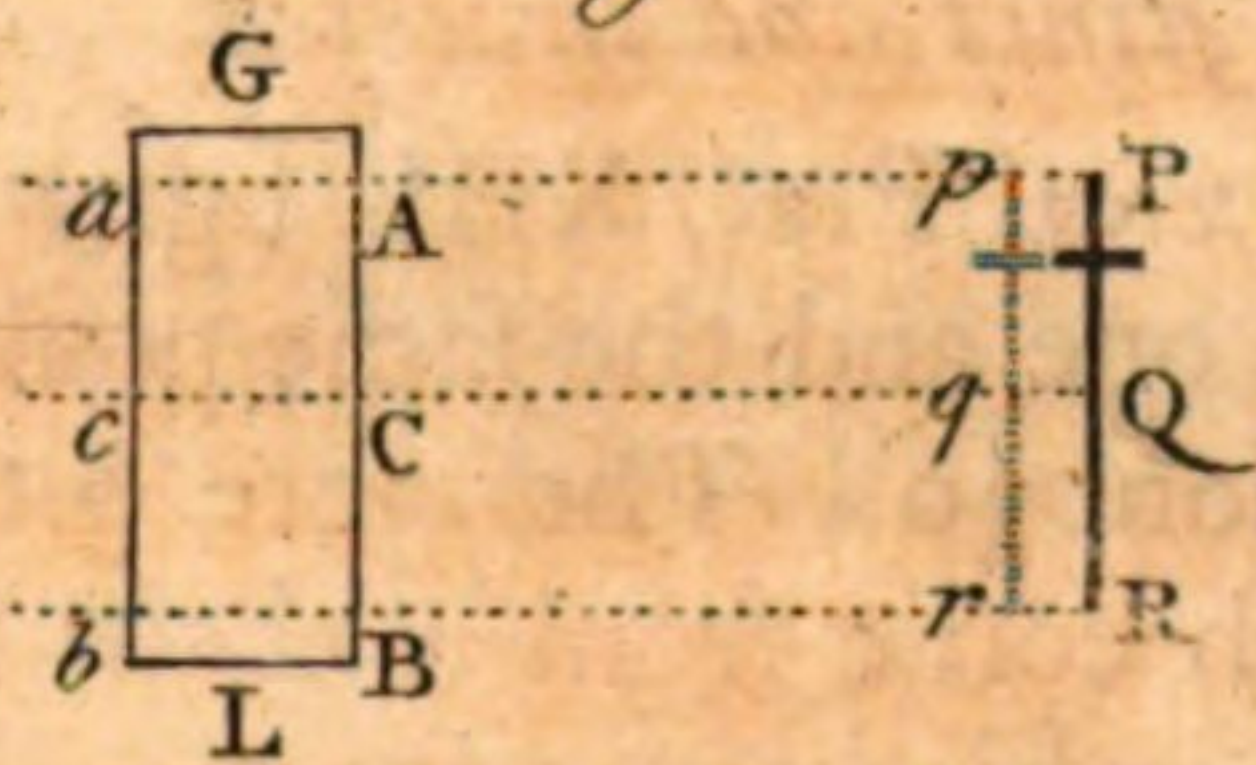


Fig. 8.

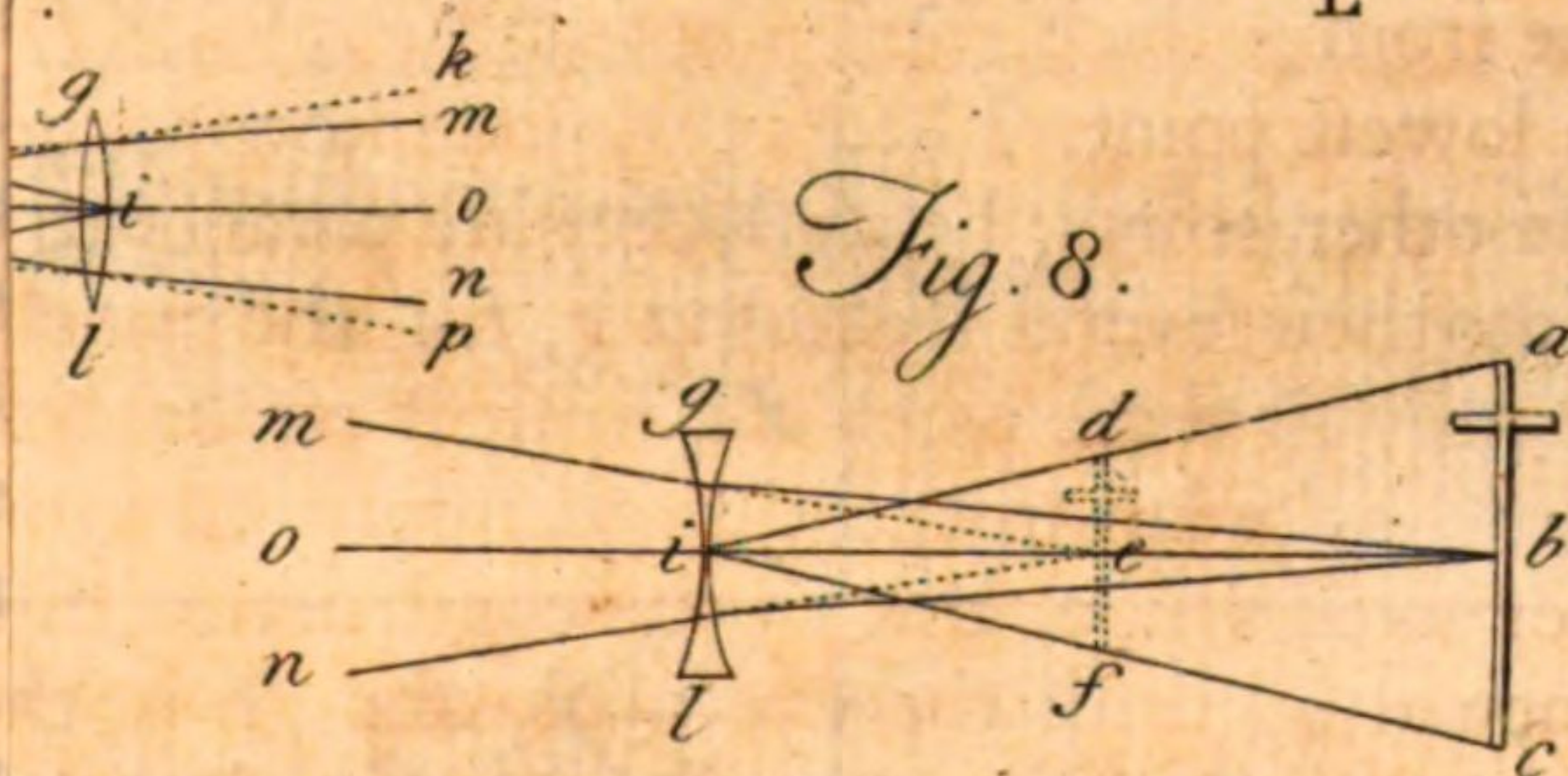


Fig. 10.

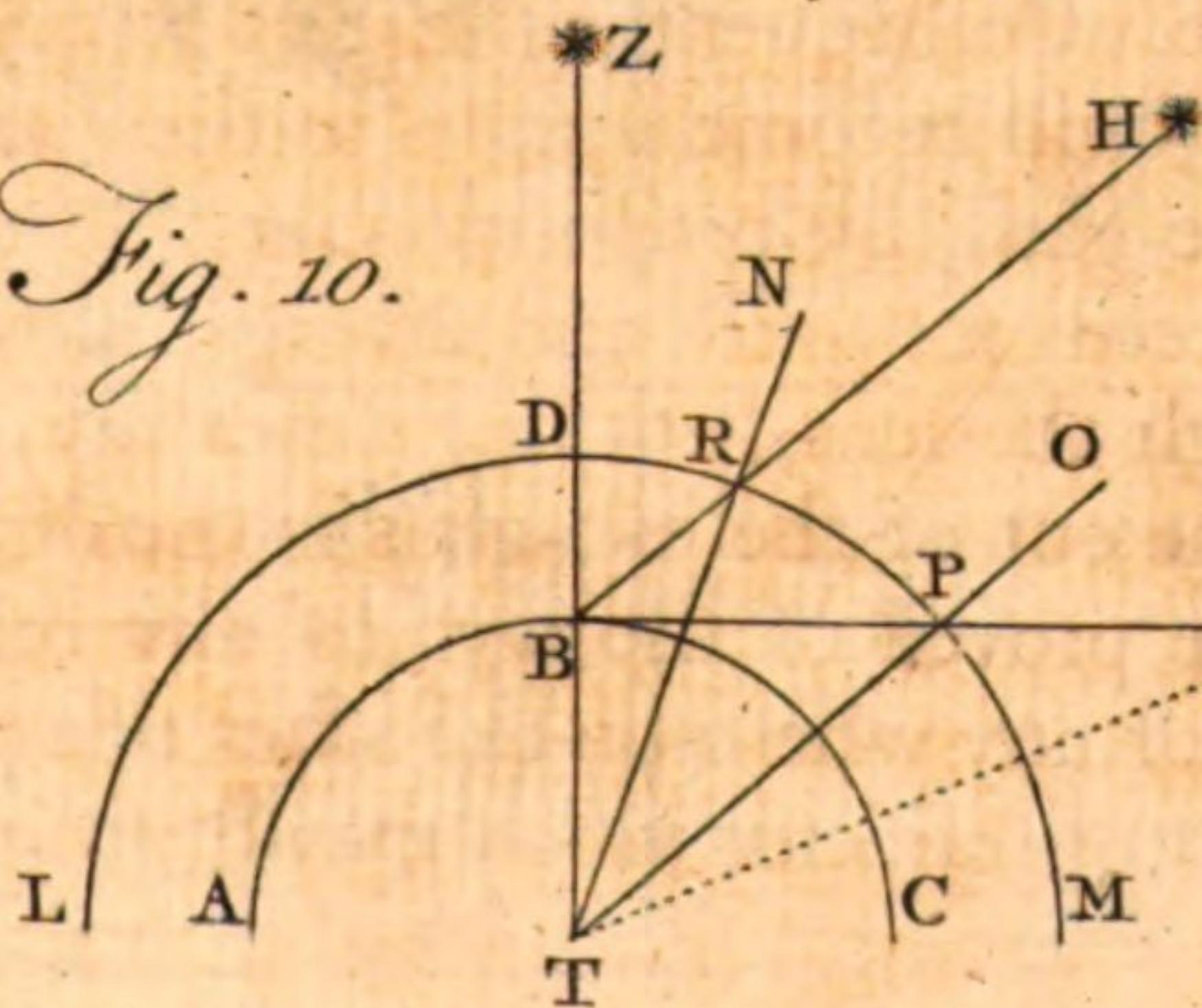


Fig. 11.



on the other hand, if the object is placed at a distance from the eye, the image is inverted and diminished. This is the case when the object is placed at a distance greater than the focal length of the eye.

The eye is a biconvex lens, and the image is formed on the retina. The distance from the eye to the object is called the object distance, and the distance from the eye to the image is called the image distance.

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of each beam are parallel to one another. It is plane from what has been said that the object thus seen will appear inverted; for the distinct picture is the object of refracted vision, and the distinct picture is inverted, by proposition 85.

173. *The apparent diameter of an object seen through an astronomical telescope, is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as the distance of the distinct picture from the object-glass to its distance from the eye-glass.*

In the telescope *mc*, Plat. XIII. fig. 1, the distance of the picture from the object-glass is *mf*, and its distance from the eye-glass is *fc*. The diameter of any remote object *zy* will appear longer through the telescope when the eye is at *o*, than it would appear to the naked eye placed at *m*, which is the station of the object-glass. This proposition determines how much longer the diameter of the object will appear; it is magnified in the proportion of *mf* to *fc*; or the diameter seems as much longer with the telescope than it would without it, as *mf* is longer than *fc*. Thus suppose that *mf* is to *fc* as 40 to 1, then the apparent diameter of the object *zy* will be magnified in the proportion of 40 to 1, or the object will appear 40 times longer and likewise 40 times broader through the telescope than it would appear to the naked eye at *m*.

If the distinct picture was received upon a paper at *efg*, the apparent diameter of the object seen by the naked eye at *m* the station of the object-glass is equal to the apparent diameter of the picture seen from the same station, as was proved under proposition 153. Now the real diameter of the picture is given, by proposition 88, because its distance *mf* from the lens is given: consequently the apparent diameter of the picture will be inversely as the eyes distance from it; by proposition 126, or 160. So that if the eye is any where at a less distance from the picture than *mf*, its apparent diameter will be greater in proportion as that distance is less. Suppose then that the paper was taken away and that there was no eye-glass, and let the eye be at *c* the station of the eye-glass; its distance from the place of the distinct picture is then *fc*. Therefore the picture to the eye in that station will appear bigger than it did at the station *m* in the proportion of the distances *fc* and *mf* inverted, or in the proportion of *mf* to *fc*. But the apparent diameter of the picture seen from the station *m* is equal to the apparent diameter of the object seen by the naked eye. Therefore the picture seen from the station *c* appears bigger than the object in the proportion of *mf* to *fc*. This would be the case, if there was no eye-glass and the eye was placed in that station where the eye-glass should

should be. But if the eye-glass was slipped between the eye and the picture, so that the eye should touch the eye-glass; this would make no alteration in the apparent diameter of the picture seen through it, by proposition 147. Therefore if the eye is close to the eye-glass of the telescope, the object seen through it will be magnified in the proportion of mf to fc . But since the picture is in the eye-glass's principal focus, by the construction, it appears of the same size whether the eye is close to this lens or at any distance from it, by proposition 154. Therefore, wherever the eye is, the apparent diameter of the object seen with the telescope is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as mf to fc , or as the distance of the distinct picture from the object-glass to its distance from the eye-glass. What is the most convenient situation for the eye to be placed in, shall be shewn hereafter.

174. *An astronomical telescope will not magnify an object unless the principal focal distance of the object-glass is greater than the principal focal distance of the eye-glass.*

Because the objects, at which we look through telescopes, are remote ones, so that, by proposition 30, the rays of each beam, that come from them, are nearly parallel to one another; it follows, by proposition 60, that the distinct picture gfe , Plat. XIII. fig. 1, will be in the principal focus of the object-glass; and this picture, by the construction of the telescope is likewise in the principal focus of the eye-glass. Therefore the distance of the distinct picture from the object-glass is the principal focal distance of the object-glass; and the distance of it from the eye-glass is the principal focal distance of the eye-glass. But, the apparent diameter of an object seen through a telescope is to the apparent diameter of the same object seen with the naked eye, as the distance of the picture from the object-glass to its distance from the eye-glass, by proposition 173, that is, by what has just been proved, as the principal focal distance of the object-glass to the principal focal distance of the eye-glass.

From hence it follows, that, if mf the principal focal distance of the object-glass is greater than fc the principal focal distance of the eye-glass, the telescope will magnify the object. But, if mf was equal to fc , the object will appear no larger with the telescope than it does to the naked eye. Or lastly, if mf was less than fc , the object would be apparently diminished. Now the less convex a lens is the greater is its principal focal distance, by proposition 53; and consequently since the principal focal distance of the object-glass must be greater than that of the eye-

eye-glass for a telescope to magnify an object, the object-glass must be less convex or must be a segment of a larger sphere than the eye-glass.

175. *An object may be equally magnified by two telescopes of very different lengths.*

If the principal focal distance of the object-glass is 20 feet, and that of the eye-glass is $\frac{1}{2}$ foot; one is to the other as 40 to 1; and such a telescope 20 $\frac{1}{2}$ feet long will magnify an object 40 times; so that its apparent diameter, when seen through the telescope, will be 40 times longer, than when seen with the naked eye, by proposition 173, 174. Another telescope 20 $\frac{1}{2}$ inches long would magnify in the same proportion, if the principal focal distance of its object-glass was 20 inches and that of its eye-glass $\frac{1}{2}$ inch: for in this telescope as well as in the former the principal focal distance of the object-glass is to the principal focal distance of the eye-glass as 40 to 1. And universally, since a telescope magnifies an object in the proportion of the focal distance of the object-glass to the focal distance of the eye-glass; all telescopes, in which this proportion is the same, will magnify equally, whatever may be the lengths of the telescopes. The reason why short telescopes that magnify an object very much are not so useful as long ones that magnify no more shall be shewn in its proper place.

176. *If an astronomical telescope is inverted, objects, that are seen through it, will appear to be diminished.*

The principal focal distance of the object-glass is greater than the principal focal distance of the eye-glass, when the telescope is used in its proper position, by proposition 174. But when it is inverted the eye-glass is turned towards the object, and therefore becomes the object-glass; and the object-glass is next the eye, and therefore becomes the eye-glass. So that in this position of the telescope the principal focal distance of the object-glass is less than the principal focal distance of the eye-glass; and consequently, by proposition 173, 174. the apparent diameter of the object will be diminished.

177. *The visible area, or space that may be seen at one view through a telescope, is proportional to the area of the eye-glass.*

Since the distance of the picture from the object-glass is equal to the lens's principal focal distance; the area of the picture is a given quantity, by proposition 96; but how much of that picture we can see at a time depends upon the breadth of the eye-glass. If a picture of any determinate

terminate size drawn upon canvas was to hang up against a wall, and we were to stand at a certain distance from it and to look at it through a hole in a piece of board, it is evident that we should see more of the picture as the area of the hole was greater, and less of it as the area of the hole was less; so that the quantity of the picture, which might be seen at one view, would be proportional to the area of the hole. This is the case of the distinct picture in a telescope: the size of it is determinate; it is at a certain distance from the eye-glass, for it is always in the eye-glass's principal focus; and the eye-glass is the hole through which we look at it. Therefore the quantity of the picture that we can see at one view, or the visible area, will be greater as this hole is greater, or less as this hole is less; that is, it will be proportional to the area of the eye-glass. Thus if *gfe* Plat. XIII. fig. 1. is the picture of the moon, the area of the eye-glass may be large enough to see the whole moon at once, or it may be so small that we can only be able to see a part of it at a time. Or if there is at *f* the distinct picture of any planet that we are looking at through the telescope, and *ef*, *gf* is the distinct picture of the blue sky round the planet; how much of this blue sky we shall see depends upon the area of the hole or eye-glass, through which we look.

There is a great inconvenience in using a telescope where the eye-glass is small, which is, that it will be difficult to find an object. If the area of the eye-glass is large the telescope will take in a large compass at one view; so that we may see a planet that we are looking for, though the telescope is not directed exactly towards it. For though it is necessary that it should be so directed in order to see the planet in the very middle of the visible space; yet without such exactness, where the space taken in at one view is large, the planet may be seen somewhere in it, though it is not in the middle. But if the area of the eye-glass, and, in consequence of that, the visible area is very small, unless the telescope is directed exactly towards the planet, we shall not be able to see it.

178. *The brightness of an object seen through a telescope depends upon the area of the object-glass: but whether the area of the object-glass is great or small the space that may be seen at one view will be the same, if the area of the eye-glass is given.*

The brightness of an object seen through a telescope depends upon the brightness of the distinct picture, and this, by proposition 101, is proportional to the area of the lens, through which the rays pass to form that picture. But this lens is the object-glass. Therefore an ob-
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ject seen through a telescope will appear the brighter for using an object-glass of a large area, and the fainter for using one of a small area.

But, by proposition 100, the size of the distinct picture is the same whether the area of the object-glass is great or small, and consequently, by proposition 177, if we look at it through a given hole, or through an eye-glass of a given area, the quantity that we can see at once, or the space that we see at one view, will not be altered by any alteration in the area of the object-glass.

179. *The distance of the eye from the eye-glass in looking through a telescope should be equal to the eye-glass's principal focal distance.*

Since the distinct picture gfe is in the principal focus of the glass dcb , wherever the eye is placed on the other side of the glass, it will see that picture equally magnified, by proposition 154. Therefore it is not for the sake of magnifying the object more, that the eye is to be placed at o , so that co the distance of it from the eye-glass may be equal to this glass's principal focal distance. The advantage gained by this position of the eye is, that we may see more of the object at one view, or that the visible area will be greater, when the eye is placed there, than when it is placed any where else. This will be best understood by considering the two glasses nl and db fixed, as they commonly are, in a tube whose sides are parallel to mc , which mc is the common axis of the two lenses continued, and is called the axis of the telescope. Now no rays can make any surface visible, but such as come from that surface, by propositions 110. 113. But no rays, that are parallel to the inner surfaces of the sides of the tube, come from those surfaces. Therefore no rays, that are parallel to the axis of the telescope, can make the sides of the tube visible. At o , which is the principal focus of the eye-glass bd , no rays meet but such as before refraction were parallel to the axis of the lens db , or to mc , which as it is the axis of the telescope, is the axis of the lens db continued, by proposition 60; consequently no rays that are collected at o will make the sides of the tube visible. And upon this account if the eye is placed at o , since it receives no other rays but such as are collected at o , the sides of the tube will not be seen through the eye-glass. But if the eye was placed any where else, either nearer to the eye-glass or farther from it than o its principal focus; some rays would enter the eye, which were not parallel to one another before refraction: so that in any other position of the eye, the sides of the tube will be seen through the eye-glass as well as the object that we are looking at. Now the area taken in at one view is proportional to the area of the eye-glass,

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glass, by proposition 177. Therefore a greater area of the object will be taken in at one view, when nothing else is seen through the eye-glass: but if we can see the sides of the tube, some part of the area of the eye-glass will be taken up by those sides of the tube; and the object will be seen only through the remaining part. Therefore more of the object will be taken in at one view, if the eye is at o , where the sides of the tube cannot be seen, than if it was placed in any other station.

The case would be the same though the glasses were used without a tube. Then indeed the eye placed any where besides at o could not see the sides of a tube, but it would see the open space nd , lb ; and when any part of the eye-glass is taken up with seeing that open space, the remaining area, through which the picture is to be seen, will be so much the less; and consequently so much the less of the picture will be seen at one view.

180. *A telescope consisting of four convex lenses is a double astronomical telescope.*

The two lenses nml , dcb , Plat. XIII. fig. 2. if they are placed at the distance mc equal to the sum of their principal focuses, make one astronomical telescope; and the two lenses ptq , rus placed at the distance tu , equal to the sum of their principal focuses is another, by proposition 171. If these two telescopes are fixed at the distance ct from each other, so as to be both of them used together, they make one telescope consisting of four convex lenses. In this compound telescope the lens, that is towards the object, is called the object-glass, and all the others are called eye-glasses. Of these eye-glasses, dcb , which is next the object-glass is called the first, ptq is the second, and rus , which is next the eye, is the third.

181. *An object appears distinct and erect through a telescope consisting of four convex lenses.*

Such a telescope is in effect two astronomical telescopes, by proposition 180. Therefore as the object would appear distinct through each of them, by proposition 172, one will not change the distinctness produced by the other; and consequently it will appear distinct through both of them, when they are used together. But since, by proposition 172, each of them used alone would invert the object, it follows that when they are both used together, the first will invert it and the second by inverting it again will shew it upright. Therefore through such a compound telescope the object appears erect.

It will be no great trouble to prove this proposition by tracing the rays through all the four lenses. The rays of each beam are parallel to one another when they fall upon the object-glass *nml*, by proposition 30, because the object is very remote: therefore a distinct inverted picture will be formed at *gfe* the principal focus of the object-glass, by propositions 60. 84. This picture, by the construction is in the principal focus of *dcb* the first eye-glass. Therefore the rays of each beam which diverge from the several points in the distinct picture *gfe*, will be made parallel to one another by passing through the first eye-glass, by proposition 61. Consequently the rays of each beam, when they fall upon the second eye-glass *ptq*, are parallel to one another, just as they were when they fell upon the object-glass *nml*. Therefore a distinct inverted picture will be formed again at *kib*, the principal focus of this second eye-glass. And this picture, by the construction, is in the principal focus of the third eye-glass *rus*. Therefore the rays of each beam, which diverge from the several points in this picture will be parallel to one another, when they have passed through the lens *rus* and come to the eye at *o*. Consequently the object seen by these rays will be seen as distinctly through the telescope as the naked eye can see any object where the rays of each beam are parallel.

From what has been said we may observe the reason why the object appears erect. The first picture *gfe* is inverted and so is the second *kib*. But the first is inverted in respect of the object, and the second is inverted in respect of the first. Therefore the second picture, which is the last image, or object of refracted vision, is in the same situation with the object itself.

182. *A telescope consisting of four lenses commonly magnifies an object in the proportion of the object-glass's principal focal distance to the first eye-glass's principal focal distance.*

The telescope *mu*, Plat. XIII. fig. 2. consists of two astronomical telescopes, by proposition 180. The first of them *mc* magnifies the object in the proportion of *mf* to *fc*, or in the proportion of the principal focal distance of the object-glass to the principal focal distance of the first eye-glass, by proposition 173, 174. But the second telescope *tu* is commonly made up of two convex lenses of equal convexities. Therefore this second telescope does not alter the apparent magnitude of the object, by proposition 174. From whence it follows, that, when both telescopes are used together, the object is magnified only by the first of them: so that the apparent diameter of the object seen through
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the telescope is to its apparent diameter seen by the naked eye, as mf the principal focal distance of the object-glass nml to fc the principal focal distance of the first eye-glass dcb .

183. *Galileo's telescope consists of a convex object-glass and a concave eye-glass; and the distance between them is the difference of their principal focuses.*

If yf , Plat. XIII. fig. 3. is the principal focal distance of the convex lens zyx , and cf is the principal focal distance of the concave lens bca ; then $yf - cf = yc$ is the difference of their principal focal distances. And if these two lenses are placed at this distance yc from each other, they make Galileo's telescope.

184. *An object seen through Galileo's telescope appears erect and distinct.*

The rays that come from each point in a very remote object are parallel to one another, by proposition 30: and if these rays pass through the convex lens zyx , Plat. XIII: fig. 3. they would converge to as many points at the lens's principal focus, by proposition 60, 83, and at gfe would paint the picture of the object, if the concave eye-glass was out of the way. Suppose therefore the eye was at c the station of the eye-glass, then the object seen through the object-glass alone would appear erect and confused, by proposition 149. And if the concave eye-glass was to be slipped between the eye and the object-glass, this would not change the apparent situation of the object, by proposition 163, so that as the object would appear erect through the convex lens alone, it will likewise appear erect when seen through both the lenses. But the concave eye-glass will make the object distinct, which would be confused without it. For since the several beams of rays, when they fall upon the concave glass are converging to several focuses g, f, e , &c. and all these focuses are at the distance cf from the lens, or, by the construction of the telescope, at the distance of the eye-glass's principal focus, it follows, from proposition 62, that the rays of each beam, when they are refracted, will become parallel, and the eye will see the object as distinctly through the telescope, as the naked eye sees an object when it is far enough off for the rays of each beam of rays, that come from it, to be parallel to one another.

185. *Galileo's telescope magnifies an object in the proportion of the object-glass's principal focal distance to the eye-glass's principal focal distance.*

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The apparent diameter of an object seen through the telescope yc , Plat. XIII. fig. 3. is as much longer than its apparent diameter seen with the naked eye, as yf the principal focal distance of the object-glass is greater than cf the principal focal distance of the eye-glass.

If the concave lens bca was taken away and the distinct picture gfe was received upon a paper, the apparent diameter of the object either through the lens zyx by the eye close to it at y , or by the naked eye at the same place is equal to the apparent diameter of this picture seen from y at the distance yf , by propositions 147, 153. And since the apparent diameter of the object seen through the lens zyx is inversely as the eyes distance from the distinct picture, by proposition 153, if the eye was to withdraw from y towards c the station of the eye-glass, the object seen through the lens zyx would keep encreasing as the distance yf decreased, so that when the eye was arrived at c the object would seem as much bigger than it did when the eye was at y as cf is less than yf , or as yf is greater than cf . But when the eye was at y its apparent diameter seen through the lens was equal to its apparent diameter seen with the naked eye. Therefore if the eye is at c the station of the eye-glass the object will be magnified by the lens zyx in the proportion of yf to cf , or in the proportion of the principal focal distance of the object-glass to the principal focal distance of the eye-glass. Now suppose, whilst the eye continues at c , the concave lens bca to be placed just before it so as to touch it; this lens, by proposition 147, in such a situation will make no alteration in the apparent magnitude of the object. Therefore the object seen through the telescope, or through both the lenses, will continue to be magnified in the same proportion as before, if the eye is close to the eye-glass: the only difference will be that the object so magnified is seen confusedly without the eye-glass, by proposition 149, but is seen distinctly with it, by proposition 184. And having thus shewn that the proposition is true, if the eye is close to the eye-glass, we may shew it to be equally true in any other station of the eye. For since the rays of each beam, when they come out of the eye-glass are parallel to one another, by proposition 184, the apparent magnitude of the object will not be altered by the departure of the eye from c , for a reason assigned in proposition 154; since, as the rays of each beam are parallel, the eye by departing from the eye-glass neither gets farther from the last image nor nearer to the distinct picture.

A telescope of this sort cannot be so contrived as not to magnify the objects, that are seen through it. For if the telescope does not magnify, the principal focal distances of the convex object-glass and the concave eye-

eye-glass must be equal. Now the distance of the two glasses from each other must be the difference between their principal focal distances, by proposition 189. Consequently where the principal focal distances are equal, that is, where the difference between them is nothing, the distance between the two glasses must be nothing, or the two glasses must touch each other; and then, instead of making a telescope, they will rather make a meniscus having its convexity on one side equal to its concavity on the other.

186. *The visible area in Galileo's telescope depends upon the breadth of the pupill.*

The rays of any beam, when they fall upon the eye-glass, are not far from their focus: therefore they do not spread over the whole surface of the lens; but extend only over a small part of it. Thus the beam, which is converging to the extreme focus g , Plat. XIII. fig. 3, covers but a small part of the lens at b the edge of it; and the beam, which is converging to the other extreme focus e , covers but a small part at a the other edge of the lens. Now if the pupill, when it is close to the lens, is wide enough to reach from b to a , the rays from the extreme parts of the object will enter it, and consequently those parts may be seen through the telescope. But if the pupill was narrower, if for instance it was no wider than do , the rays of the extreme beams would not enter it, so that the extreme parts of the object could not be seen. Therefore the visible area or space that may be taken in at one view will be greater as the breadth of the pupill is greater, or less as that breadth is less.

187. *The eye is in the most convenient position for looking through Galileo's telescope, when it is close to the eye-glass.*

The apparent magnitude of the object seen through the telescope would be the same, wherever the eye was placed, by what was shewn under proposition 185. But when the eye is close to the concave eye-glass, more of the object is seen at one view, than could be seen in any other station of the eye. The rays of each beam are indeed parallel to one another: the rays of the beam bk , for instance, are parallel amongst themselves; and so likewise are those of the beam al . But then the beams or rays of different beams are not parallel to one another: the beam bk is not parallel to the beam al . For since the beam bk before refraction was tending to g and the beam al to e , so that one of them was nearly parallel to the other; it follows that after refraction they will diverge from one another, by proposition 54. Therefore the farther these beams are from the eye-glass the farther they will be asunder, by proposition

10 : consequently though, when the eye is close to the glass, these extreme beams may be near enough to each other to enter the pupil, yet, when the eye is placed at any distance from the glass, they will be so far asunder that they cannot enter it : therefore the extreme parts of the object, which might be seen, if the eye was close to the eye-glass, will not be seen, if the eye is removed to any distance from it.

188. *A single microscope is only a very convex lens.*

A lens of a great convexity is a segment of a small sphere ; and therefore will have a short principal focus, by proposition 53. Suppose therefore such a lens to have its principal focus only $\frac{1}{10}$ inch : if any small object is placed in the principal focus or at the distance of $\frac{1}{10}$ inch of the lens, the eye on the other side will see it distinctly through the lens, and as big as the naked eye could see it, if the eye was at the station of the lens or within $\frac{1}{10}$ inch of the object, by propositions 147, 151, 154. But the naked eye does not see an object distinctly unless it is 5 inches from it. Therefore the apparent diameter of the object seen through the lens is as great as it would be, if the eye was 50 times nearer than it ever is, when we have distinct vision, and consequently is 50 times as great as we are ever used to see it.

189. *A double microscope consists of two convex lenses, of which the object-glass is more convex than the eye-glass : and the distance between them is equal to the distance of the distinct picture from the object-glass added to the principal focal distance of the eye-glass.*

The microscope, Plat. XIII. fig. 4, consists of a very convex object-glass *nml* and a less convex eye-glass *dcb*. The small object *zxy* is nearer to the object-glass than its principal focus, so that somewhere behind the glass there will be a distinct picture of it *gfe*, by proposition 84. The distance of this picture from the object-glass is *mf*, the distance of it from the eye-glass, or *cf*, must be equal to the eye-glass's principal focus. And the distance of the two glasses from each other must be *mc*, or *mf+fc*, which is the distance of the distinct picture from the object-glass added to the principal focal distance of the eye-glass.

From this construction it is evident that the distinct picture is always in the eye-glass's principal focus. And by this description of a compound microscope we may see what is the essential difference between such a microscope and an astronomical telescope. The object-glass of a compound microscope is more convex than the eye-glass : but, if an astronomical telescope magnifies, the object-glass of it is less convex than the eye-

eye-glass. How each of these instruments is fitted for their respective purposes by this construction will appear as we go along.

Some compound microscopes are made with three glasses; so that the rays, after they have been refracted by passing through the lens *nml*, are received upon another convex lens, and are refracted again before they form the distinct picture. In this case the rays are made to converge by the refraction of both these lenses, and they are both of them called object-glasses. The distinct picture so formed is in the principal focus of the third glass, which being next the eye is in this as in other microscopes called the eye-glass.

190. *An object seen through a double microscope appears distinct and inverted.*

The demonstration of this proposition is the same with that of proposition 172: and the letters of reference are made the same, in Plat. XIII. fig. 1, and 4, that the reader, if he pleases, may the more readily apply it.

191. *The apparent diameter of an object seen through a double microscope is to the apparent diameter of the same object seen by the naked eye in the station of the object-glass, as the distance of the distinct picture from the object-glass to its distance from the eye-glass.*

This proposition is demonstrated on Plat. XIII. fig. 4, in the same manner as proposition 173, is on fig. 1.

Now an astronomical telescope will not magnify an object unless the principal focal distance of the object-glass is greater than the principal focal distance of the eye-glass, or unless the object-glass is less convex than the eye-glass. For when the object is very remote, as objects are which we look at through a telescope, the distinct picture is in the principal focus of the object-glass; and consequently it cannot be farther from the object-glass than it is from the eye-glass unless the principal focal distance of the object-glass is greater than the principal focal distance of the eye-glass. But when the object is very near the object-glass, as objects are which we look at through a microscope, the distinct picture will be more remote from the glass than its principal focus, by proposition 61: and consequently the distinct picture *gfe*, Plat. XIII. fig. 4, may be more remote from the object-glass *nl* than it is from the eye-glass *df*, though the object-glass is more convex or has a shorter principal focus than the eye-glass.

Here we may observe, first; that an astronomical telescope will not answer the purposes of a microscope or cannot be used for magnifying
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near objects. For if the object zxy , Plat. XIII. fig. 1. was near the lens nml , the picture gfe would be so far from it as not to be within the compass of the instrument or between the two lenses. Secondly, a double microscope cannot be used for magnifying distant objects: because the object-glass has a shorter principal focus than the eye-glass, and consequently is like an astronomical telescope inverted, and instead of magnifying a remote object will diminish it, by proposition 176. Thirdly, an astronomical telescope inverted will not answer the purpose of a microscope in magnifying very near objects: for though if bd , Plat. XIII. fig. 1. is towards the object and nl is towards the eye, the object-glass will then be more convex than the eye-glass, as in a double microscope; yet db the eye-glass of an astronomical telescope is not convex enough to serve for the object-glass of a microscope: so that when the object is very near db , the picture gfe will be very far from this glass db , and consequently, unless the telescope is made much longer than it usually is, this picture will either not be between the lenses, or however will not be, as it ought, in the principal focus of the lens nl , which is next to the eye.

192. *An object seen through a microscope appears magnified in the proportion of the limit of distinct vision to the distance of the object-glass from it.*

The magnifying power of a microscope, as determined in proposition 191, is in reference to the eye at the station of the object-glass m , Plat. XIII. fig. 4. Thus if mf is to cf as 10 to 1, the apparent diameter of the object seen through the microscope will be 10 times greater than the naked eye would see it at m . But if mx the distance of the object-glass from the object is less than the distance at which the naked eye can see an object distinctly, or less than the limit of distinct vision, the apparent diameter of the object seen by the naked eye at m will be as much greater than we are ever used to see it, as mx is less than that limit, by proposition 126. Thus for instance, the limit of distinct vision being supposed 5 inches, we are never used to look at an object nearer than 5 inches, because if we were to hold it nearer to our eye, we should see it confusedly. Therefore if mx is only 1 inch, the object would appear to the naked eye at m 5 times longer than we usually see it. Therefore the object seen through the microscope appears larger than usual upon two accounts. First it is 10 times larger than the naked eye at m would see it, because fm is 10 times fc . And secondly, the naked eye at m would see it 5 times larger than it has ever been used to

see it, because mx is 5 times less than the limit of distinct vision, or 5 times less than the distance at which the naked eye has been used to look at it. But if the apparent diameter is 10 times greater than usual upon one account, and 5 times greater upon the other account, it follows, that upon both accounts together it will be 50 times greater. For the microscope magnifies in the proportion of 10 to 1, and of 5 to 1, that is in the proportion of 50 to 1. From whence we may deduce the following general proposition for determining how much larger an object will appear through a microscope than the naked eye has been used to see it.

193. *A microscope magnifies the apparent diameter of an object in a proportion compounded of the distance of the distinct picture from the object-glass to its distance from the eye-glass, and of the limit of distinct vision to the distance of the object-glass from the object.*

It magnifies in the first proportion, by proposition 191, and in the second, by proposition 192.

194. *When the same eye-glass is made use of, the magnifying power of a microscope is encreased, by encreasing the convexity of the object-glass.*

If the microscope cm , Plat. XIII. fig. 4. magnifies an object in the proportion of 50 to 1 with the object-glass m , it would magnify the same object in a greater proportion, if this object-glass was changed for a more convex one.

The more convex an object-glass is, the nearer will be the distinct picture, if the object was to continue at the same distance, by proposition 95. But by bringing the glass nearer to the object the picture will depart from it, by proposition 98. And thus, whatever is the convexity of the glass, by altering its distance from the object the picture is always kept, where it ought to be, at gfe the principal focus of the eye-glass. Now when the place of the distinct picture, and consequently its distance from each of the glasses, is given; the microscope magnifies in the proportion of the limit of distinct vision to the distance of the object-glass from the object, by proposition 192. Therefore the nearer this glass is to the object the more the microscope will magnify: and since a more convex object-glass must be brought nearer to it than a less convex one, the more convex one will magnify the most.

If mf is to cf as 10 to 1, and mx is 1 inch or $\frac{1}{5}$ of the limit of distinct vision, we found, by proposition 192, that the objects apparent diame-

diameter seen through the microscope will be 50 times greater than usual. But suppose a more convex object-glass was made use of; so that to keep the picture at gfe , the lens must be brought within $\frac{1}{2}$ inch of the object. Then mf is still to cf as 10 to 1. But mx , being now only $\frac{1}{2}$ inch or $\frac{1}{10}$ 5 inches or $\frac{1}{18}$ the limit of distinct vision, the microscope will magnify the object again in the proportion of 10 to 1, by proposition 192. Since therefore the object is magnified in the proportion of 10 to 1 and of 10 to 1, upon both accounts together it will be magnified by this more convex lens in the proportion of 100 to 1, or twice as much as it was by the less convex one. And universally, since in the same microscope the distinct picture is always kept in the same place, whatever object-glass we make use of, the magnifying power of it will always be inversely as the distance of the object-glass from the object.

We need say nothing here about the visible area through a double microscope, or about the proper position for the eye; since the similitude between this instrument and an astronomical telescope plainly shews that these particulars are to be determined by the same rules in both.

195. *The aperture of the object-glass in a microscope should be very small.*

For as the object-glass is very near to the object, the outermost rays of every beam will diverge very much, and would hinder the distinctness of the picture, by proposition 99, unless they were excluded by making the aperture small. But the smallness of the aperture will make the object appear faint, by proposition 178. Therefore to make amends for this, many ways are contrived to illuminate the object as much as possible. For this purpose it is placed in the sun-shine, or the light is collected by a convex lens, or a concave mirror, and is thrown upon it.

196. *Short-sighted persons see best through a telescope or microscope if it is shortened a little.*

When an astronomical telescope or double microscope is of the usual length the distinct picture is in the eye-glass's principal focus, and the rays of each beam come out parallel, by propositions 172. 190. But if either instrument is shortened a little, the eye-glass is brought nearer to the picture, and then, as the picture is nearer to it than the principal focus, the rays of each beam, that diverge from thence, will continue to diverge after they have passed through it, by proposition 67. So likewise in Galileo's telescope the rays of each beam, when they fall upon the concave eye-glass converge to its principal focus, and therefore when they

have passed through it they become parallel, by proposition 184. But by shortening the telescope the eye-glass is moved farther from the place of the distinct picture, for which reason the rays of each beam converge to points more remote than its principal focus; and therefore, after refraction they will diverge, by proposition 65. Since therefore shortening any of these instruments a little will make the refracted rays diverge, and since, by proposition 168, short-sighted persons see better by diverging rays than by parallel ones, it follows that such persons will see more distinctly through any of these instruments, when they are shortened, than when they are of the usual length.

197. *Old persons see best through a telescope or microscope, if it is lengthened a little.*

When a telescope or double microscope is of the usual length the rays of each beam are parallel when they come to the eye, by propositions 172, 190, 184. But if an astronomical telescope or double microscope is lengthened by drawing the eye-glass farther from the object-glass, the distinct picture will be more remote from the eye-glass than its principal focus, and consequently the rays of the several beams, which diverge from the picture will converge, after they have passed through the glass, by proposition 63. And if Galileo's telescope is lengthened, the rays, which fall converging on the eye-glass, will converge to points nearer than the glass's principal focus, and will therefore continue to converge after refraction, by proposition 69. But old persons see more distinctly, when the rays converge, than when they are parallel, by proposition 169. Therefore such persons will see the better through any of these instruments, if they are lengthened a little.

CHAP. XII.

Of the cause and laws of the reflection of light.

198. *The reflection of light from transparent bodies, is either partial or total. The partial reflection happens either at the first or second surface; the total one at the second surface only.*

WHEN a beam of light falls upon a thick piece of polished glass; all the rays will not pass through it; but some of them will be reflected from the first surface of the glass, where the beam enters. For if a candle is placed before the glass, when the beams that come from the

the candle fall upon it, though the greatest number of rays pass into it, yet a faint image of the candle may be seen by reflection from the first surface, which shews that some few of the rays are reflected there, whilst the rest enter the glass. Secondly, when as much of the beam as entered the glass is come to the second surface, all the rays, that come thither, will not pass out of it into the air, the greatest part may pass out, but some will be reflected from thence. For a second faint image of the candle may be seen there. These are the two partial reflections; and these happen, whatever is the obliquity of the rays, when they fall upon the first or second surface. The total reflection at the second surface does not happen, unless the rays fall very obliquely upon it. If the angle of incidence at the second surface is under 40 degrees, greatest part of the beam will pass out of the glass into the air, and only some few rays will be reflected, as we said before. But if the angle of incidence or obliquity is greater than 40 or 41 degrees, none of the rays will come out of the glass, but the whole beam, that is, as much of the beam as came to the second surface, will be reflected.

By shewing the cause of these reflections from transparent substances we shall see to what cause the reflection of light from other substances, such as polished metals, is owing.

199. *The rays of light are not reflected by striking upon the solid parts of bodies.*

When rays of light fall upon a thick piece of polished glass, some of them are reflected from the first and some from the second surface, by proposition 198, and as many or more rays are reflected at the second surface, where they are to pass out of the glass into the air, as at the first surface, where they are to pass out of air into glass. Now this could scarce be the case, if these reflections were occasioned by the striking of the rays upon the solid parts of bodies: for upon this supposition, since glass is denser, or in a given space contains more solid parts, than air, a greater quantity of rays ought to be reflected, when they are passing out of air into glass, than when they are passing out of glass into air.

When a beam of light is to pass out of glass into air, if its obliquity is less than 40 degrees, greatest part of it will pass out, but if the obliquity is greater than 41 degrees, it will all be reflected, by proposition 198. But it is scarce probable that the beam at one obliquity should meet with nothing but pores or interstices in the air, and so pass freely into it, and at a little greater obliquity should meet with nothing but solid parts and so be totally reflected. Or if the reader imagines, that what is so unlikely should however be possible; we may observe farther that if there was
water

water behind the glass instead of air, the rays would not be reflected from the water at the second surface, though their obliquity was greater than 41 degrees. But since water is denser than air, and the rays which would have been reflected back into the glass, if there had been air behind it, will pass freely out of it, if there is water behind it, we cannot imagine that the reflection of them from the air could be occasioned by their striking upon its solid parts: for if air reflects them by this means, then certainly water, which is more dense than air, could not fail of reflecting them.

Nor can any one suppose that the total reflection from the second surface, though it is not occasioned by the rays striking against the solid parts of the air, may yet be owing to their striking against the solid parts in the second surface of the glass itself. This supposition is attended with the same difficulty as the other. For it is scarce possible that the rays at one degree of obliquity should meet with nothing but pores in that surface, and at a greater obliquity should meet with nothing but solid parts. Nay indeed this supposition is attended with a greater difficulty. For if the rays, falling at an obliquity greater than 41 degrees, meet with nothing but solid parts in the second surface of the glass, and are reflected by striking against those parts; then this reflection would happen, whether there was air or water behind the glass; whereas at the obliquity of 43 degrees, they will be reflected, if there is air behind it, but would be transmitted if there was water behind it.

200, *Bodies reflect and refract light by one and the same power differently exercised in different circumstances.*

When rays of light are to pass out of one medium into another of a different density, they are refracted; and this refraction will be so much the greater as the mediums differ more from one another in density, by proposition 37. Rays are more refracted, when they pass out of air into a diamond, than when they pass out of air into glass, and more still than when they pass out of water into glass. And in these or any other instances, where the refraction is the strongest, the reflection is likewise the strongest. There is a stronger reflexion, when light is passing out of air into a diamond, than when it is passing out of air into crystal, or into common glass: and if a diamond, or a piece of crystal or of common glass is immersed in water, as they differ less in density from water than from air, the refraction is the weaker, and the reflexion from their surface becomes the fainter.

We are therefore to explain the reflection of light upon the same principles, that we made use of in explaining its refraction. Bodies act upon
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the rays of light two ways, by proposition 5 of mechanics, they attract all those rays that are very near them, and repel those that are a little farther from them. Any ray, if it is at a certain distance from the surface of a piece of glass, is attracted by the glass, but is repelled, if it is at a distance somewhat greater. The space contiguous to the surface of the glass, where the rays are attracted, we may call the attracting surface; and the space beyond this or farther from the surface of the glass, where they are repelled, we may call the repelling surface. Farther we must observe of the rays themselves that they have fits of easy reflection and easy transmission. That they have such fits shall be proved hereafter. What these fits are owing to is not easy to determine; perhaps the rays are put into them at their first emission from luminous bodies, and continue in them during all their progress. And it is not unlikely that those rays, which are in a fit of easy transmission, move the fastest.

Now if GGMM, Plat. XIII. fig. 5. is the repelling surface of a piece of glass and any ray Ra falls upon it; this ray when it enters the surface will be retarded by the repulsion, and consequently will be refracted from a perpendicular, so that instead of going straight forwards in the line Rn , it will be bent into bo , by proposition 38: for though the repelling surface may not be a rarer medium than the air out of which the ray passes; yet the effect is the same; the ray will be retarded as much by passing into the repelling surface as it would be by passing out of a denser medium into a rarer. Now this repulsion encreases as the ray approaches the glass, till it gets into the surface of attraction. And consequently, as the ray moves in the repelling surface, it is constantly retarded, by which means it will be constantly refracted from a perpendicular. The ray therefore will not go straight forwards in the direction bo , but will be bent, as it passes along, into the directions ct , dv , and at last into fw , so as to be parallel to MM the limit of the repelling surface. But the ray cannot proceed straight in this direction fw , for being repelled by the glass it will be inflected from it, into the line gp ; and whilst it is thus moving from the glass, and is still in the surface where it is repelled from the glass, the repulsion will accelerate the ray, and consequently, by proposition 37, it will be constantly refracted towards a perpendicular, and will be bent, as it goes along, into the several directions bp , iq , kl , and will emerge from the surface at l in the line lS . Thus a constant retardation refracted the ray from a perpendicular, till it became parallel to the limit of repulsion at f , it was there inflected from the surface of the glass, and then a constant acceleration refracted it towards a perpendicular,
till

till it was reflected or came out of the repelling surface on the same side where it entered it at first. And in the course of its motion, it described the curve *abcdefghikl*. Thus the reflexion is occasioned at the first surface, or when the rays are about to enter the glass. But then this reflexion will be only a partial one: those rays only will be reflected, which are moving the slowest, the moment of the rest which have a greater velocity, will be sufficient to carry them through the repelling surface; and when they have passed that, they will then be attracted by the glass, and will be transmitted to its second surface.

The partial reflection at the second surface is produced in the same manner; only as it was produced, by the repelling surface, when the rays were entering the glass, it will be produced by the attracting surface when they are coming out of it. Let *GG* therefore now be considered as the second surface of the glass, *MM* the limit between the attracting and repelling surface, and consequently the space between *GG* and *MM* is the attracting surface. As the ray *Ra* comes out of the glass, it will continue to be attracted towards it, so long as it continues within the space *GGMM*: this attraction will retard the ray and bend it from a perpendicular, till at last the direction of the ray becomes parallel at *f* to *MM* the limit of attraction. At this place the attraction will inflect the ray, and will make it begin to move towards the glass; so that afterwards, when the ray is moving towards the glass, the same attraction will accelerate it and bend it towards a perpendicular, till it enters the glass again at *l*, and then passing back again through the substance of the glass it will come out of it at its first surface, or will be reflected. This likewise will be a partial reflection: those rays, which are in a fit of easy transmission, or which are moving with the greatest velocity, will make their way through the attracting surface, and those only will be reflected, which were moving with the least velocity.

The total reflection at the second surface, which happens when the rays fall very obliquely is planely nothing but a refraction. Let *fō*, Plat. VIII. fig. 5. be a ray of light passing out of the glass *Z* into the air *X* through the plane surface *ab*, this ray will be so refracted from a perpendicular, by proposition 38, as to describe the line *or*, when it comes into the air: and *rc* the sine of the refracted angle is to *fd* the sine of incidence as 3 to 2. Now if the ray *fō* falls more obliquely, or if the sine of incidence *fd* encreases, the refracted sine *rc* will encrease with it, by proposition 40. Suppose therefore the obliquity to be so great, or by encreasing the angle *fod* make the sine of incidence *fd* so great, that *oa* the radius may be to *fd* as 3 to 2, then the radius *oa* will be the refracted sine,
that

that is, coa will be the refracted angle, or the ray will be refracted along the surface of the glass. Here therefore the reflection will begin: for if the obliquity of the incident ray is ever so little encreased, the refracted angle will be greater than coa or greater than a right one; and then the refracted ray will not describe *or* or any other line on that side of oa , for any such line would make a less angle than a right one with co ; nor will it describe oa , for then the refracted angle would be a right one; but it will describe some line on the other side oa within the glass; that is, it will return into the substance of the glass and so passing back again through it, will come out at the same surface, where it first entered, or will be reflected. And universally, when a ray is to pass out of a denser medium into a rarer, if the obliquity of the ray is so great that the sine of incidence is to the refracted sine, as the sine of incidence to the radius, the refracted sine will be the radius, and the total reflection will begin at that obliquity. Therefore the more the sine of incidence differs from the refracted sine, so much less the sine of incidence may be, and yet the refracted sine will be equal to the radius. Now the sine of incidence differs most from the refracted sine, when the mediums differ most from each other in density. Consequently the total reflection will begin at a less obliquity, when rays are passing out of glass into air, than when they are passing out of water into air: because glass and air differ more in density, than water and air.

Now here we must observe, that, when a beam of light is totally reflected at the second surface of the glass, the retardation, which occasions this total reflection is not made all at once. For when the rays come so near to the surface as to make the attraction towards the glass greater than that towards the air, the retardation, and consequently the refraction begins. Thus if MM , Plat. XIII. fig. 5. is the second surface of the glass, supposing the distance between MM and any imaginary plane GG in the glass to be equal to the semidiameter of the sphere of attraction of any single particle of the glass; then till the ray Ra comes to a it is equally attracted all ways; but as soon as it is at b , or nearer to MM than GG it will be more attracted by the particles, that are behind it than by those that are before it; for there will be fewer particles attracting it between b and MM than there are attracting it the contrary way. Consequently, since the rays are gradually retarded till they are reflected, they will describe such a curve as $abcdef$, and after reflection being accelerated in the same manner as they return in the glass's substance they will describe a similar curve $fgbikl$.

201. *In all cases of reflection, the complement of reflection is equal to the complement of incidence, and the angle of reflection is equal to the angle of incidence.*

In all cases of reflection the rays describe such a curve as *abcdefghikl*, as has been proved under proposition 200: and since they describe one half of this curve *abcdef* by being retarded, and the other half *ghikl* by being similarly accelerated, one half will be similar to the other: and consequently, since they are similarly placed, one half will make the same angle with any plane surface *GG* that the other half makes with it. Therefore the angle *RaG*, which is the complement of incidence, will be equal to the angle *SlG*, which is the complement of reflection. But since the complements are equal, the angles themselves are likewise equal, the angle of incidence equal to the angle of reflection.

In the reflection of light whether it is partial or total, the bending of the rays is made within a very small compass: so that the curve *abcdefghikl* will be so small that we may neglect it, and may look upon the point of incidence *a* as coinciding with the point of reflection *l*: and upon this account we may consider the reflection of light in the same manner, as in mechanics, we have considered the reflection of elastic bodies, where the body is always reflected from the point of incidence.

202. *All reflection is reciprocal*

If any reflected ray is turned back again to the reflecting surface, it will be reflected into the line of incidence. If the ray *AC*, Plat.VII. fig. 3. after it has been reflected in the line *CB*, is turned back again in the direction *BC*, it will be reflected into the line *CA*, which it had before described at its incidence. For the angle of reflection is always equal to the angle of incidence, by proposition 201. Therefore if, when the angle of incidence is *ACO*, the angle *OCB* will be the angle of reflection, because it is equal to it; for the same reason, when *BCO* is the angle of incidence, *OCA* will be the angle of reflection.

CHAP. XIII.

Of the reflection of light from plane mirrors.

203. *The cathetus of incidence is the axis of any beam, or is a perpendicular drawn from any radiant to the plane of a mirror, on which rays fall that come from the radiant.*

IF A Plat.VII. fig. 7. is a radiant and rays diverging from it fall on the mirror *Ce*, the line *AC*, or *Aa* which is *AC* continued behind the mirror, is the cathetus of incidence. Or if *be* is the whole length of the

the mirror so that no line can be drawn from the radiant A perpendicular to the mirror itself, yet still the line AC is the cathetus of incidence, though the mirror reaches no farther than *b*: for AC is perpendicular to the plane of the mirror, since *bc*, though no part of the mirror, is in the same plane with it.

This explanation of the term in plane mirrors will sufficiently shew the reader what is meant by it, when we come to apply it to such mirrors as are not plane.

204. *Rays that are parallel, when they fall upon a plane mirror, will be parallel, when they are reflected from it.*

If ED, Plat. XIII. fig. 6, is a plane reflecting surface, or is a line drawn along the surface of a common looking-glass, and any rays, FA, LC, GB, which are parallel to one another, are reflected from thence, the reflected rays, AH, CM, BI, will likewise be parallel. FA and LC are parallel at their incidence, by the supposition, therefore they are equally inclined to the plane surface, or to the right line ED drawn upon it; that is, the angle FAE, which one ray makes with it, is equal to the angle LCE which the other makes with it. For, since FA and LC are parallel, the right line ED, which crosses them, makes the external angle FAE equal to LCE the internal opposite angle on the same side. Euc. b. I. prop. 29. But FAE and LCE are the complements of incidence of the two rays FA and LC. And since the complements of incidence are equal, the complements of reflection will likewise be equal, by proposition 201, or the angles HAD, MCD, which the reflected rays AH and CM make with the plane ED, will likewise be equal. But MCD is the external angle, and HAD is the internal opposite one on the same side. Therefore, since these are equal, AH and CM, or the reflected rays are parallel to one another. Euc. b. I. prop. 28. In the same manner we might prove, that if GB is parallel to FA, when they fall upon the plane ED, BI will likewise be parallel to AH, when they are reflected. And that all rays, which are parallel before reflection, will likewise be parallel after it.

205. *If diverging rays are reflected from a plane mirror, the distance of the imaginary radiant behind the mirror, is equal to the distance of the real one before it; and the divergency of the rays is not altered by the reflection.*

Let A, Plat. VII. fig. 7, be the radiant, from which the rays Ab, Ad, Ae diverge, when they fall upon the surface of a looking-glass or the surface of smooth water, they will be reflected from thence in the

directions bf , dg , eb , so as to make the complements of reflection equal to the complements of incidence, by proposition 201. In this case ab , the distance of the imaginary radiant a , from which the reflected rays appear to diverge, will be equal to Ab , the distance of the real radiant. And the angle bAd or the divergency of any two rays at their incidence will be equal to the angle bad or fag , which is the divergency of the same rays after reflection.

The angle AbC is equal to the angle fbe , by proposition 201; for one is the complement of incidence of the ray Ab and the other is the complement of reflection of the same ray. But Cba is likewise equal to fbe , because they are opposite angles made at the vertex, where the two right lines Ce and af cross. Euc. b. I. prop. 15. Therefore since AbC and abC are each of them equal to fbe , they are likewise equal to each other. In the same manner we may prove that AdC is equal to adC . But since AbC is equal to abC , it follows that Abe is likewise equal to abe ; for Abe is the complement of AbC and abe is the complement of abC to two right angles; and where the angles themselves are equal, their complements will be so too. Thus we have shewn that in the two triangles Abd and abd the angles at b and at d in each of them are equal; and the side bd is common to both. Consequently the triangle Abd is in all respects equal to the triangle abd . Euc. b. I. prop. 26. The side Ab in one is equal to the side ab in the other, and the side Ad is likewise equal to the side ad ; that is, the distance of the real radiant A either from the point b or from the point d is equal to the distance of the imaginary radiant a from the same point. And the angle bAd contained between the incident rays Ab and ad is likewise equal to the angle bad contained between the reflected rays bf and dg ; that is, the divergency of the incident rays is the same with the divergency of the reflected ones.

206. *If converging rays are reflected from a plane mirrour, the distance of the imaginary focus behind the mirrour is equal to the distance of the real focus before it; and the convergency of the rays is not altered by the reflection.*

If fb , gd , be , Plat. VII. fig. 7, rays converging to the focus a , fall upon the mirrour be , and are reflected from it, before they come to a , then a is the imaginary focus; and the point A , where the rays meet after reflection is the real one. Now ab the distance of the imaginary focus from the point b , or ad its distance from the point d , will be equal respectively to Ab and Ad or to the distances of the real focus from those points; and the angle bAd or fag , which is the convergency of the incident

cident rays, will be equal to the angle bAd , which is the convergency of the same rays when they are reflected.

This is proved exactly in the same manner as proposition 205.

CHAP. XIV.

Of the reflection of parallel rays from convex and concave mirrors.

207. *When parallel rays fall upon a concave spherical mirror, they will be made to converge by being reflected from it; and the distance of the focus will be half a semidiameter of the mirrors concavity.*

IF a concave mirror MBN, Plat. XIII. fig. 7, is part of a sphere whose semidiameter is CB; any rays CB, LD, that are parallel, when they fall upon the reflecting surface, will converge after reflection to a focus F, and BF the distance of this focus from the mirror will be equal to half CB.

Any ray CB, which passes through C the center of the concavity, is perpendicular to the surface of the mirror, and consequently will be reflected back again in BC the line of incidence, by proposition 201, and proposition 88, of mechanics. This ray, as the line it describes is not altered by the reflection, may be considered as the axis of the beam. But any collateral ray, LD, will fall obliquely upon the mirror; for, at the point of incidence D, DC drawn from that point to the center is the perpendicular. The angle of incidence here is LDC; and the ray will be reflected in the line DF, so as to make the angle of reflection CDF equal to LDC, by proposition 201. Now all the perpendiculars drawn from the surface to the center C, as BC, DC, HC, converge to the axis BC. Therefore any collateral ray LD, which was parallel to the axis when they fell on one side of the perpendicular, will converge by being reflected in the line DF on the other side. BF the distance of the focus from the surface is half BC. The angle of incidence of the ray LD is LDC. LD and CB are parallel rays, by the supposition, and consequently CD, which crosses them, makes the alternate angles LDC and DCF equal. Euc. b. I. prop 29. Thus DCF is equal to the angle of incidence; because it is the alternate angle. And CDF is equal to the angle of incidence; because it is the angle of reflection. Therefore the two angles at the base of the triangle CFD are equal to one another, because each of them is equal to the angle of incidence; and consequently the sides FC and FD, which

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subtend those angles, are equal. Euc. b. I. prop. 6. But when D and B are very near together FB is very nearly equal to FD. Therefore FC and FB are very nearly equal, or the point F, to which the ray DF converges, divides the semidiameter CB in halves, so that BF is half the semidiameter.

208. *When parallel rays fall upon a convex spherical mirror, they will be made to diverge by being reflected from it; and the distance of the virtual or negative focus will be half a semidiameter of the convexity.*

If the concave mirror MBN, Plat. XIII. fig. 7. was to unbend itself, the semidiameter BC would lengthen and the focus F, which is always at the distance of half a semidiameter, would depart from the mirror. When the surface was quite unbent, or was become plane, the semidiameter BC would be infinite, and the reflected rays would converge to an infinitely distant focus, or would be parallel, by proposition 30. 204. But if MBN was to unbend itself still farther, or to become less concave than when it was plane; its concavity would become less than nothing, or it must bend itself the contrary way and so become convex, or the concave mirror MBN would then be changed into the convex one *mbn*, fig. 8. So that a convex mirror is only a negative concave one. When the concavity is affirmative, the semidiameter BC, and the focus F, are on the same side with the incident rays. When the concavity is nothing, or when the mirror is plane, the semidiameter is infinite, and the focus at an infinite distance. But when the concavity is negative, or when the mirror is convex the semidiameter will be greater than infinite, and the focus will be at more than an infinite distance; and consequently they will be negative, by proposition 49, that is, they will change sides, and will be on the contrary side of the mirror to the incident rays. Since therefore the focus is affirmative, when the mirror is concave, but negative when it is convex; the convergency, which is affirmative, when the mirror is concave, will become negative when it is convex, or the reflected rays will diverge. Thus if *mbn* is the mirror, and *xb*, *ld*, two parallel rays that fall upon it; *xb*, because it is perpendicular to the surface, will be reflected back again in the line *bx*; but *ld* which is oblique to the mirror, will be reflected in the line *dz* on the other side of the perpendicular *cy*, and will diverge from the virtual or negative focus *f*. Now when the semidiameter is affirmative the focus F is before the mirror and divides the semidiameter BC into two equal parts; and consequently, when the semidiameter is negative or lyes behind the mirror, the
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negative focus will be just as far in proportion from the mirrour as the affirmative one, and will likewise divide the semidiameter bc into two equal parts; so that fb the distance of this virtual or negative focus behind the mirrour will be half the semidiameter bc .

209. *When rays, that were parallel at their incidence, are reflected from a concave mirrour, they do not all of them converge exactly to one and the same focus; and when they are reflected from a convex mirrour, they do not all of them diverge exactly from one and the same virtual focus.*

LD, and GH, are two parallel rays falling upon the concave mirrour MN. LD, which is nearest to the axis, is reflected in the line DF to the focus F: but GH will not be reflected in the line HF to the same focus. For if it was reflected thither, then, by what has been proved in proposition 207, CFH must be an isosceles triangle, or FC must be equal to FH. But FC was there proved to be equal to FD. And consequently, if the ray GH was reflected to F, the same line FC must be equal both to FH and to FD. But it cannot be equal to each of them, because they are not equal to one another; for FH is longer than FD. Therefore the ray GH is not reflected to the same point F with the ray LD. If it be asked therefore whether the focus, to which GH is reflected, is nearer to the mirrour than F or farther from it; we answer it will be nearer. For as the distance of F from C is equal to FD, so the distance of the focus to which GH is reflected will be equal to FH; and since FH is longer than FD, the focus, to which GH is reflected, will be at a greater distance from the center than CF, and consequently will be at a less distance from the mirrour than FB.

That the rays which fall far from the axis CB, will be reflected to a less distant focus than those which fall near to it, may be otherwise proved. For the line BC, in which the axis itself is reflected, is the same that the incident ray described; because the axis is perpendicular to the surface. And consequently, since any ray LD, which falls near to the axis, is nearer perpendicular, than any other ray GH, which falls far from it, the line, in which LD is reflected, will differ less from DL the line of incidence than the line in which GH is reflected will differ from HG. But the less the line of reflection differs from the line of incidence so much less will the reflected ray deviate from being parallel to its axis; for all the lines of incidence are parallel to it, by the supposition. But the less a converging ray deviates from being parallel to its axis the less it converges, and the less it converges so much more remote

mote is the focus, by proposition 29. Therefore those rays, which fall nearest the axis, will have a remoter focus than those which fall at a greater distance from it. Now the focus of those rays, which fall nearest to the axis, is at the distance of half a semidiameter of the concavity, by proposition 207, and consequently the focus of those, which fall farther from the axis, is at a distance something less than this.

But since the proposition is true, when the concavity of the mirror, the convergency of the reflected rays, and the focus are all of them affirmative, it will likewise be true for the same reasons, when they are all of them negative. Therefore when the mirror is convex, when the reflected rays diverge and the focus is negative, all the rays after reflection will not diverge from exactly the same point; but those, which fall nearest to the axis, will have their virtual focus at a greater distance from the mirror, than those have, which fall farther from it.

The reader however may observe, that in rays which do not fall very remote from the axis, the difference of the focal distance is so small that we may neglect it, and may consider them after reflection as all converging to the same point, if the mirror is concave, or as all diverging from the same point, if it is convex.

CHAP. XV.

Of the reflection of diverging rays from a concave mirror, and of converging rays from a convex one.

210. *The principal focal distance of a concave mirror is the distance, at which rays, that were parallel at their incidence, are united after reflection. And the principal focal distance of a convex mirror is the distance, from which rays, that were parallel at their incidence, diverge after reflection.*

THE principal focus of either a concave or a convex mirror is at the distance of half a semidiameter of the sphere, of which the mirror is a segment, by proposition 207, 208.

From the similitude, which the reader must have already observed, between the effects of a concave mirror and a convex lens, and between the effects of a convex mirror and a concave lens, it follows that, as in proposition 60, there will be three cases of diverging rays, when they fall upon a concave mirror, and three cases likewise of converging rays, when they fall upon a convex one. For, when the mirror is

is concave and the rays are diverging, the radiant may be either in the principal focus, or more remote than the principal focus, or nearer than the principal focus. And, when the mirror is convex and the rays are converging, the imaginary focus may be either in the principal focus, or more remote than the principal focus, or nearer than the principal focus. And indeed the reader will find the same similitude between the reflection and the refraction of light in most instances.

211. *If the radiant is in the principal focus of a concave mirror, the reflected rays will be parallel. (See prop. 61.)*

After parallel rays have been reflected by the concave mirror MBN, they converge to F, Plat. XIII. fig. 7. which is the principal focus of the mirror. Therefore if these rays were turned back in the lines of reflection, they would diverge as from a radiant in the principal focus. And when rays are so turned back, they will be parallel after reflection, by proposition 202. consequently, when a radiant is in the principal focus, the rays, which diverge from thence, will be parallel after reflection.

212. *If the imaginary focus of converging rays is in the principal focus of a convex mirror, the reflected rays will be parallel. (See prop. 62.)*

If the incident rays zd , xb , Plat. XIII. fig. 8, converge to f the principal focus of the convex mirror mbn ; that is, if the imaginary focus of these rays is at f , the reflected rays dl , bx will be parallel. For parallel rays ld , xb , after they have been reflected by the mirror diverge from f , in the lines dz , bx . Therefore these rays, if they were turned back again in the lines of reflection zd , xb , or any other rays converging in the same manner to f , will be reflected in the lines dl , bx , and will be parallel to one another after reflection, by proposition 202.

213. *If the radiant is more remote from a concave mirror, than the mirror's principal focus; the rays, that diverge from it, will converge after they are reflected. (See prop. 63.)*

The reflection from the mirror destroys just as much divergency as rays have, which come from the mirror's principal focus, and the reflected rays then become parallel, by proposition 211. But if the radiant is at E, Plat. XIII. fig. 9. so that the distance BE is greater than the mirror's principal focal distance, the rays EB, ED, that come from it, will have less divergency, than they would have if the radiant was in the principal focus, by proposition 28. Consequently the reflection will destroy more than all their divergency, or will make their divergency

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negative, or will make them converge: so that they will be reflected in the lines BF, DF and will meet in a focus at F.

214. *If the radiant is more remote from a concave mirrour than the principal focus; as the distance of the radiant from the mirrours surface to its distance from the center of the concavity, so is the distance of the focus from the surface to its distance from the center.*

The rays which diverge from E, Plat XIII. fig. 9. will be reflected to some focus F; where EB is the distance of the radiant from the surface, and EC its distance from the center: EB is a perpendicular ray and consequently is reflected in the line BC: FB is the distance of the focus from the surface, and FC its distance from the center. Now in this case EB is always to EC, as FB to FC. CD drawn from the center to the surface is a perpendicular, ED is the incident ray, DF is the reflected ray, EDC is the angle of incidence, and CDF is the angle of reflection. But, by proposition 201, EDC and CDF are equal angles. Consequently the line DC divides the angle EDF into two equal parts, so that one half of EDF is the angle of incidence, and the other half is the angle of reflection. But if in the triangle EDF the angle at E is thus divided into two equal parts, the line DC, which so divides it, will divide the side FE into two parts EC and FC, which are in the same proportion with ED and FD the other two sides of the same triangle, Euc. b.VI. prop. 3; that is, ED is to EC, as FD to FC. But if B and D are very near together, EB will be nearly equal to ED, and FC will be nearly equal to FD. Therefore EB is to EC, as FB is to FC.

215. *If the imaginary focus is more remote from a convex mirrour than the mirrours principal focus; the rays that converge to that imaginary focus at their incidence, will diverge after they are reflected.* (See prop. 65.)

If the rays yd , xb , Plat. XIII. fig. 10. converge to e a point more remote than the mirrours principal focus, they will be reflected in the lines bx , dx , so as to diverge from f . If these rays had converged to the mirrours principal focus, the reflection would have just destroyed all their convergency, and would have made them parallel, by proposition 212. Therefore if their convergency is less, as it is, by proposition 29, when they converge to a point more remote than the principal focus, the reflection will destroy more than all their convergency, that is, it will make their convergency negative or will make them diverge.

216. *If the imaginary focus is more remote from a convex mirrour than the principal focus; as the distance of the imaginary focus from the mirrours surface is to its distance from the center of the convexity, so is the distance of the imaginary radiant from the surface after reflection to its distance from the center.*

If the rays yd , xb , Plat. XIII. fig. 10. converge to the imaginary focus e , and are reflected in the lines dx , bx so as to diverge after reflection from the imaginary radiant f ; then eb is the distance of the imaginary focus from the surface, and ec is its distance from the center; fb is the distance of the imaginary radiant from the surface, and fc is its distance from the center. And eb is always to ec as fb to fc . This proposition is the same with proposition 214, only every thing is taken on the contrary side, or fig. 10, is the negative of fig. 9. For, first E is a radiant before the mirrour, from whence the incident rays diverge, and e is an imaginary focus behind the mirrour, to which the incident rays converge. Secondly, MBN is a concave mirrour, and mbn is a convex one. Thirdly, the principal focus of a concave mirrour is before it, and that of a convex mirrour is behind it. Fourthly, the reflected rays BF , and DF , converge, and the reflected rays bx , dx , diverge. Therefore fifthly, the focus F is before one of the mirrours, and the imaginary radiant f is behind the other. Since then whether diverging rays fall upon a concave mirrour, or converging rays fall upon a convex one, every thing is the same only taken on the contrary sides, provided the radiant in one case and the imaginary focus in the other are more remote than the principal focus; this proposition is the same with proposition 214, only it is in negative terms. But making the terms negative does not alter the proportion. Therefore since as EB to EC , so FB to FC , it follows that as eb to ec , so fb to fc .

217. *If a radiant is more remote from a concave mirrour than its principal focus, as the radiant approaches to the mirrour, the focus departs from it; and as the radiant departs, the focus approaches.*
(See prop. 71.)

As the radiant E , Plat. XIII. fig. 9. moves towards the mirrour, the focus F moves from it. Now as the radiant E approaches to the mirrour EB decreases, and so likewise does EC . But then they decrease equally, because just as much as is taken from the length of EB by E 's approach, just so much is taken from the length of EC . And consequently if these two lines decrease equally, the longer of the two EB will decrease slower in proportion than the shorter EC . But if in the

approach of the radiant to the mirrour, EB decreases slower in proportion than EC, then the proportion of EB to EC is changed, so that EB, by having decreased slower than EC, will bear a greater proportion to EC, than it did before. But EB is always to EC as FB to FC. Consequently, when in the approach of the radiant the proportion of EB to EC is encreased, the proportion of FB to FC must encrease likewise. And since the proportion of FB to FC cannot encrease unless FB encreases, it follows, that whilst the radiant approaches, FB or the distance of the focus from the mirrour will encrease, that is, the focus will depart.

In like manner, when the radiant departs from the mirrour, EB encreases, and EC encreases equally at the same time. But EB is longer than EC, therefore when equal parts are added to both of them, EB so encreased will be encreased less in proportion than EC, or whilst they encrease equally EB encreases slower in proportion than EC: and consequently in the equal encrease of these two quantities, the proportion of EB to EC decreases. And since as EB to EC so FB to FC, when the proportion of EB to EC decreases, the proportion of FB to FC must decrease at the same time. But the proportion of FB to FC cannot decrease unless FB decreases. Therefore when EB encreases FB must decrease, or as the radiant departs from the mirrour, FB, which is the distance of the focus, decreases, that is, the focus approaches to it.

218. *If the radiant is in the center of the mirrours concavity, the focus will be in the same place.*

If the radiant is in the center, all the rays diverging from it will be perpendicular to the mirrours surface, since they describe lines drawn from the center to the surface. But all perpendicular rays are reflected in the lines of their incidence, by proposition 202. Therefore the rays, which diverged from the center before reflection, will converge to it afterwards, or the focus will be in the center of the concavity, if the radiant is there.

This proposition may be proved in another manner. For when the radiant is at the center it is more remote than the principal focus of the mirrour, by proposition 210. Therefore as EB, Plat. XIII. fig. 9. to EC so FB to FC, by proposition 214. But when the radiant E is at the center C then EC is nothing; and consequently because EC, or the second term in the proportion, is nothing, FC, or the fourth term, which is the distance of the focus from the center, will likewise be nothing; that is, the focus will be at the center.

219. *If the radiant is on one side of the center of a mirrours concavity, the focus will be on the other side.*

Whilst the radiant E, Plat. XIII. fig. 9. approaches to the mirrour the focus F departs from it, by proposition 217. Now both the radiant by approaching to the mirrour and the focus by departing from it approach to the center C; and when one of them arrives at the center, the other will be there too, by proposition 218. And if the radiant still approaches, the focus will still depart as before, so that if the radiant was on one side of the center at E, the focus would be on the other side at F; or if the radiant was at F the focus would then be at E, by proposition 202.

220. *If the radiant is nearer to a concave mirrour than its principal focus; the rays, which diverge from thence, will continue to diverge, but less than they did at their incidence. (See prop. 67.)*

As the radiant approaches to the mirrour the focus departs from it, by proposition 217. And when the radiant has got into the principal focus, the reflected rays will be parallel, by proposition 211, and the focus will be at an infinite distance, by proposition 31. Consequently if the radiant E, Plat. XIII. fig. 11, gets nearer to the mirrour, the focus still departing from the mirrour will be at a distance greater than infinite, or its distance will be negative. Now whilst the radiant was more remote than the principal focus of the mirrour, and the distance of the focus, to which the rays were reflected, was finite, the rays converged after reflection and this focus was before the mirrour, as at F, fig. 10. Therefore when the radiant is nearer than the principal focus, and the distance of the focus, to which the rays are reflected, becomes greater than infinite, this distance will be negative, so that the focus will be behind the mirrour, and the reflected rays will have a negative convergency or will diverge, by proposition 49. Thus if the radiant E, fig. 11, is nearer to the mirrour than its principal focus, the rays EB, ED, which diverge from it, will be reflected in the lines BE, DS, so as to diverge after reflection from a negative focus or imaginary radiant F behind the mirrour. However these reflected rays will diverge less than the incident ones. For the reflection will destroy just as much divergency as rays have that come from the principal focus, by proposition 211. Therefore, though these rays have a greater divergency because the radiant is nearer, and consequently the reflection cannot destroy it all and make the rays parallel, yet it will destroy a part of it, and will make them diverge less than they did at their incidence.

221. *If the radiant is nearer to a concave mirrour than the principal focus; as the distance of the real radiant from the surface of the mirrour to its distance from the center of the concavity, so is the distance of the imaginary radiant from the surface after reflection to its distance from the center.*

If E, Plat. XIII. fig. 11, is the radiant and F is the negative focus or imaginary radiant; then EB is to EC as FB to FC. This is the proportion when E, as in fig. 9, is more remote than the principal focus, by proposition 114, and, from what has been said under proposition 220, it appears that there is no difference between that case and this, except only that the focus which is affirmative in one case is negative in the other. Therefore this proportion is likewise true, when the radiant is nearer than the principal focus.

222. *If rays converge at their incidence upon a convex mirrour to an imaginary focus which is nearer than the principal focus; the reflected rays will continue to converge, but less than they did at their incidence. (See prop. 69.)*

If *fb* and *sd*, Plat. XIII. fig. 12, converge to an imaginary focus *e*, which is nearer to the mirrour than its principal focus; they will be reflected in the lines *bf*, *df*, so as to converge still, only less than they did before. If the imaginary focus had been in the principal focus, the reflection would have destroyed all the convergency of the rays, by proposition 112. Therefore if the imaginary focus is nearer, and the rays, by proposition 29, converge more, the reflection will destroy only a part of their convergency; and consequently the reflected rays will continue to converge, but less than the incident ones.

223. *If rays converge, at their incidence, to an imaginary focus, which is nearer to a convex mirrour than its principal focus; as the distance of the imaginary focus from the surface of the mirrour is to its distance from the center of the convexity, so after reflection is the distance of the real focus from the surface to its distance from the center.*

The imaginary focus to which the incident rays *fb* and *sd*, Plat. XIII. fig. 12, tend is *e*, *c* is the center of the convexity, and *f* is the real focus, where the reflected rays meet. And *eb* is to *ec* as *fb* to *fc*. This proposition is the same as proposition 221, only every thing is taken on the contrary side, or fig. 12 is the negative of fig. 11. For first, in fig. 11, E is a radiant nearer than the principal focus, and fig. 12, *e* is an imaginary-
gi-

ginary focus on the contrary side of the mirrour. Secondly MBN is a concave mirrour, and *mbn* is a convex one. Thirdly, the principal focus of the concave mirrour is before it, and that of the convex one is behind it. Fourthly, the reflected rays BE and DS diverge, and the reflected rays *bf* and *df* converge. Therefore fifthly, the imaginary radiant F is behind one mirrour, and the real focus *f* is before the other. Since then, whether diverging rays fall upon a concave mirrour, or converging rays upon a convex one, every thing is the same, only taken on contrary sides, provided the radiant in one case, and the imaginary focus in the other is nearer than the mirrours principal focus, the proportions will be the same in both cases: for if the proposition is true when the terms are affirmative, it will not be made less true by changing them from affirmative to negative. Therefore since EB is to EC as FB to FC, *eb* is likewise to *ec* as *fb* to *fc*.

224. *If the radiant is nearer to a concave mirrour than its principal focus, as the real radiant on one side approaches to the mirrour, the imaginary radiant on the other side will approach to it, and as the real radiant departs the imaginary one will depart.* (See prop. 72.)

If the radiant E, Plat. XIII. fig. 11, is nearer to the mirrour MBN than its principal focus, then as E approaches to the mirrour, F the imaginary radiant will likewise approach to it. EB is to EC as FB to FC, by proposition 221. Now if E approaches to the mirrour, EB must decrease, and the proportion of EB to EC must decrease, consequently the proportion of FB to FC must decrease or FB must decrease, that is, F must approach to the mirrour. If the reader is not aware how the proportion which FB bears to FC will decrease by shortening FB, since as F moves towards B, it likewise moves towards C and consequently FC shortens at the same time; in order to understand this he need only recollect what we have often observed, that these two lines shorten equally, as F moves towards B, and consequently the shorter of the two or FB decreases faster in proportion than the longer of the two FC.

In like manner, if the radiant E departs from the mirrour the imaginary radiant F will depart too. For since we have seen that, whilst EB decreases FB will likewise decrease, it follows that whilst EB encreases by the radiants motion from the mirrour, FB will encrease or the imaginary radiant will move from the mirrour. This motion of the imaginary radiant will continue no longer than till the real radiant E has got to the mirrours principal focus: for then the distance of F is infinite, by proposition 30; and if the radiant departs farther this distance will be greater

greater than infinite, and the radiant will change sides, or will be before the mirrour, by what is proved under proposition 49; that is, when E is more remote than the principal focus of the mirrour, F will be a negative radiant, or a focus, before the mirrour, as in propositions 213, 214.

225. *If the real radiant is close to the surface of a concave mirrour, the imaginary radiant will likewise be close to it.*

The distances of the real radiant E, Plat. XIII. fig. 11, and of the imaginary radiant F is thus determined, by proposition 221, as EB to EC, so is FB to FC. Now when the real radiant E is close to the mirrour at B, then EB is nothing. But when the first term or EB is nothing, the third term or FB will be nothing, and consequently F will be close to B or the imaginary radiant will be close to the mirrour.

We may now recapitulate what has been said in the foregoing propositions concerning the motion of E and F or of the radiant and focus in respect of a concave mirrour. When E, Plat. XIII. fig. 9, is at an infinite distance, so that the incident rays are parallel, F will be before the mirrour in its principal focus, by proposition 207. As E approaches to the mirrour F will depart from it, and unless E is at an infinite distance F will be farther from the mirrour than its principal focus, by propositions 213, 214. Whilst E keeps approaching, F will keep departing, till they both meet at C in the center of the concavity, by propositions 217, 218. When E is between the center and the surface, F will be on the opposite side of the center, by proposition 219. And in the nearer approach of E to the mirrour F will still keep departing, till E comes to the principal focus, and then the distance of F is infinite, or the reflected rays are parallel, by propositions 217, 211. If the radiant E approaches nearer to the mirrour than the principal focus, as in fig. 11, then the distance of F is greater than infinite, or F will be negative and on the contrary side of the mirrour or behind it, by propositions 220, 221. After this if E approaches to the mirrour F will approach to it likewise, by proposition 224, till E is close to the mirrour and then F will meet it again, by proposition 225.

CHAP. XVI.

Of the reflection of converging rays from a concave mirror, and of diverging rays from a convex one.

226. *If rays converge, when they fall upon a concave mirror, they will converge more after reflection. (See prop. 74.)*

IF the rays SD, EB, Plat. XIII. fig. 11, converge to the imaginary focus F, but, before they arrive there, are reflected by the concave mirror MBN, they will converge after reflection more than they did before, and will meet at E. For, by propositions 220, 221, if the rays EB, ED, were to diverge from E, they would be reflected in the lines BE, DS, and would diverge less than they did at their incidence, just as if they had come straight from F. Therefore if these rays BE and DS are turned back again in the same lines EB, SD, so as to converge to F, they will be reflected in the lines of their first incidence BE and DE, so as to be made to converge more and to meet at the point E, by proposition 202. In like manner any other rays which diverged before reflection, and diverge likewise after it, will diverge less, by proposition 220. But all converging rays may be considered as such rays turned back again: and since all such rays, when turned back again will be reflected in the lines of their incidence, by proposition 202; it follows that all converging rays, when they have been reflected, will be made to converge more.

227. *When converging rays are reflected from a concave mirror, as the distance of the imaginary focus from the surface of the mirror to its distance from the center of the concavity, so is the distance of the real focus from the surface to its distance from the center.*

If the rays EB, SD, converge to the imaginary focus F before reflection, they will converge to the real focus E after reflection, by propositions 221, 202. And as FB is to FC, so is EB to EC. This follows from proposition 221. For we have there proved that as EB to EC, so FB to FC, and consequently as FB to FC, so is EB to EC.

228. *If rays diverge, when they fall upon a convex mirror, they will diverge more after reflection. (See prop. 76.)*

If there is any radiant at *f*, Plat. XIII. fig. 12, from which the rays *fb*, *fd* diverge; when they are reflected from the mirror *mbn*, they will go on in the lines *bf*, *ds*, so as to diverge more, and will have the same

direction in respect of each other, as if they had come strait forwards from a radiant e . If the rays fb , sd , which converge to e , were reflected from the mirrour, they would converge more after reflection, and would meet at the focus f , by propositions 222, 223. And if these rays were to be turned back again from f in the lines fb , fd , so as to diverge from f , they would be reflected into the lines of their former directions bf and ds , by proposition 202. And consequently would be made to diverge more. In like manner all rays, which converge at their incidence, and converge likewise after reflection, will converge less after reflection than they did before it, by proposition 222. And all rays, which diverge at their incidence, may be considered as such rays turned back again. But since all such rays, when thus turned back again, will be reflected into the lines of their incidence, and will therefore be made to diverge more; it follows that all rays, which diverge when they fall upon a convex mirrour, will diverge more after they are reflected from it.

229. *When diverging rays are reflected from a convex mirrour, as the distance of the real radiant from the surface of the mirrour to its distance from the center, so is the distance of the imaginary radiant from the surface after reflection to its distance from the center.*

If rays diverge before reflection from the focus f , Plat. XIII. fig. 12. they will diverge after reflection from the imaginary focus e , by propositions 222. 202. And as fb to fc so is eb to ec . For we have proved, in proposition 223, that eb is to ec as fb to fc ; from whence it follows, that fb is to fc as eb to ec .

230. *As the real radiant moves either towards or from a convex mirrour, the imaginary radiant will move the same way. (See prop. 78.)*

If the real radiant f approaches to the mirrour mbn , Plat. XIII. fig. 12, the imaginary radiant e will likewise approach to it: or if f departs from the mirrour, e will likewise depart. For when f approaches to the mirrour fb decreases faster in proportion than fc ; or the proportion of fb to fc decreases, consequently, since as fb to fc so eb to ec , by proposition 228, the proportion of eb to ec must decrease at the same time. But the proportion of eb to ec cannot decrease unless eb decreases. Therefore when the real radiant f approaches to the mirrour, or when fb decreases, eb will likewise decrease, or the imaginary radiant e will approach to it. In like manner if the real radiant f departs from the mirrour, or if fb encreases, it will encrease faster in proportion than fc , and consequently the proportion of fb to fc will encrease; for which reason

Fig. 1.

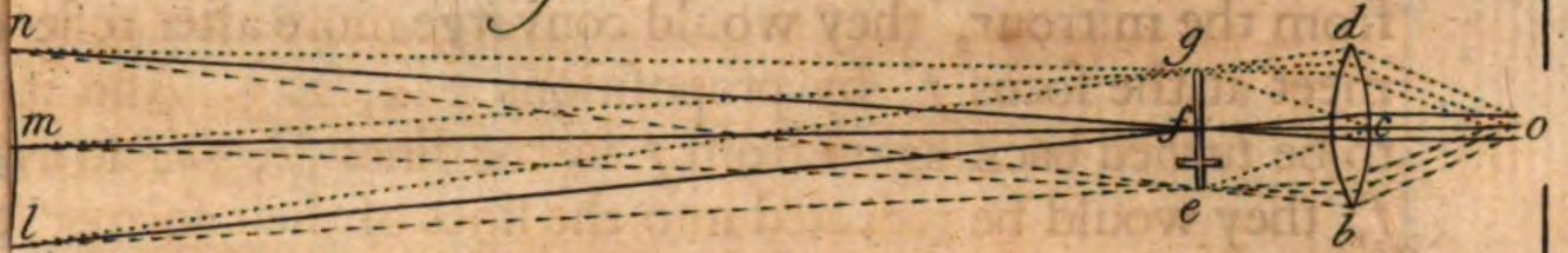


Fig. 2.

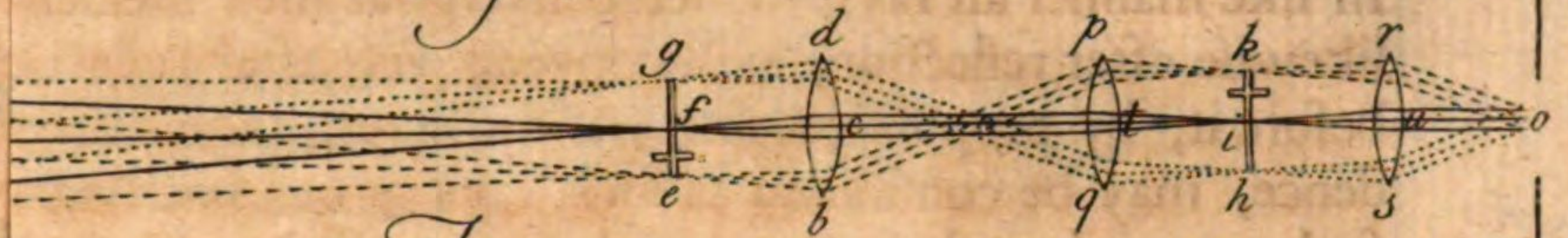


Fig. 3.

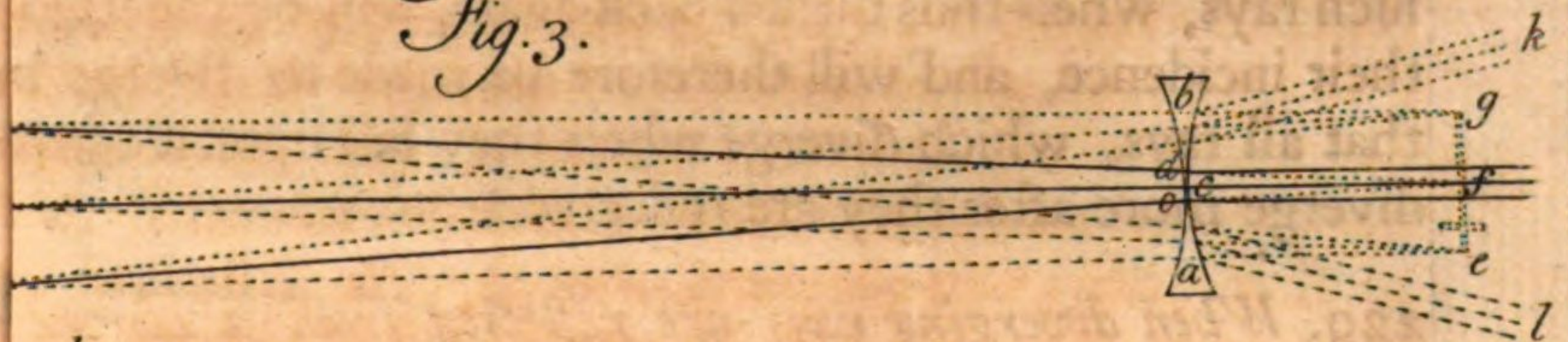


Fig. 4.

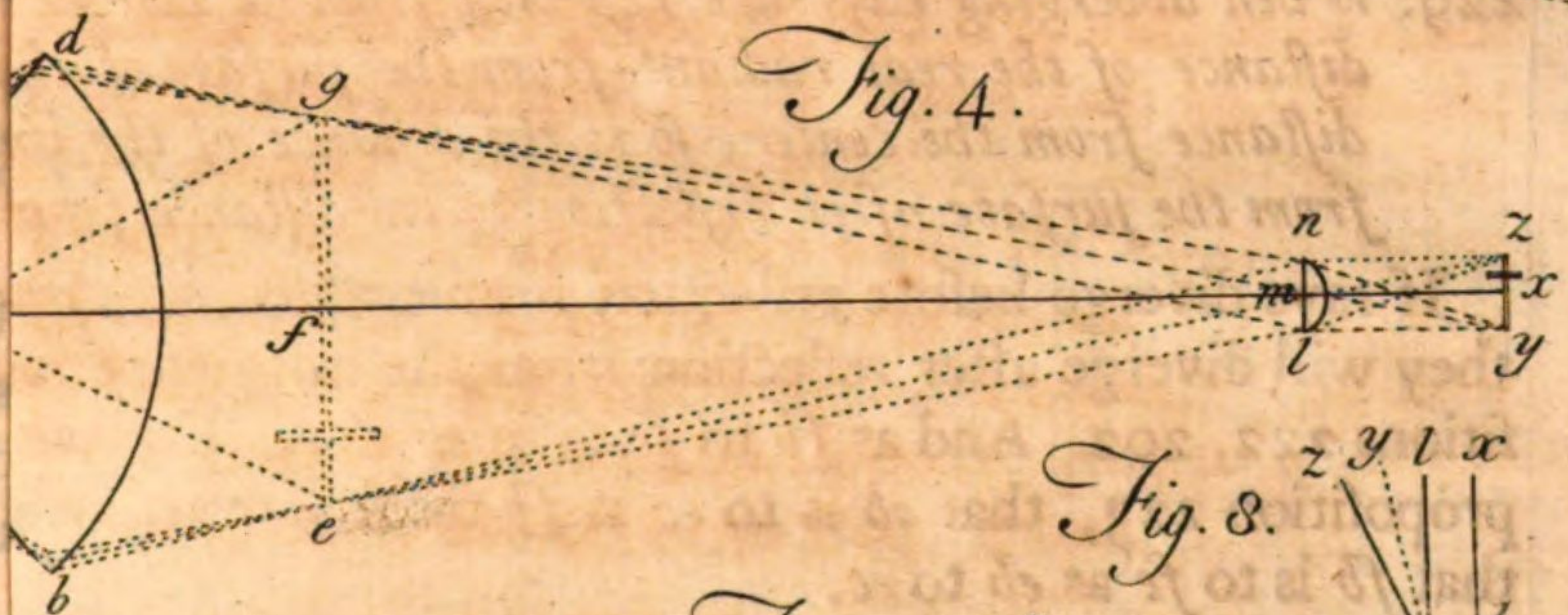


Fig. 8.

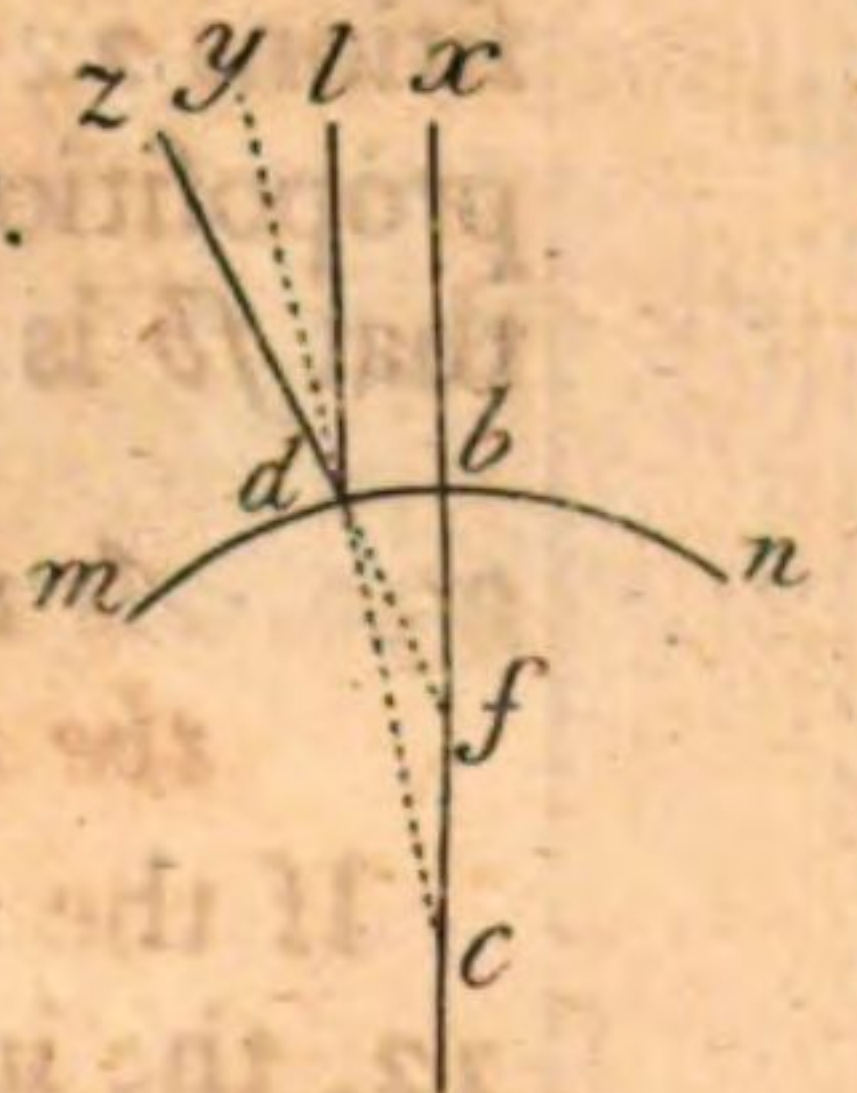


Fig. 7.

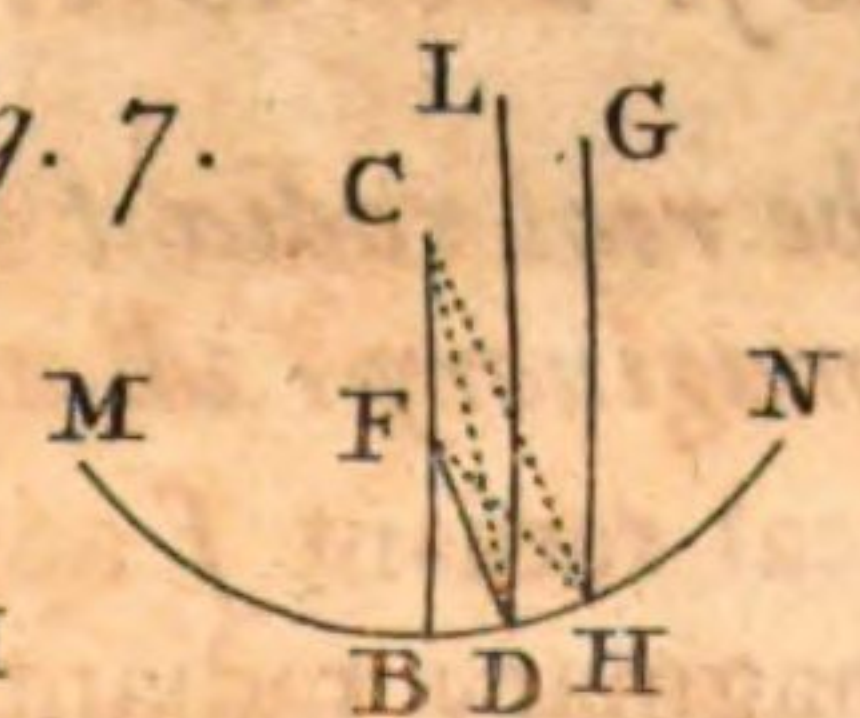


Fig. 6.

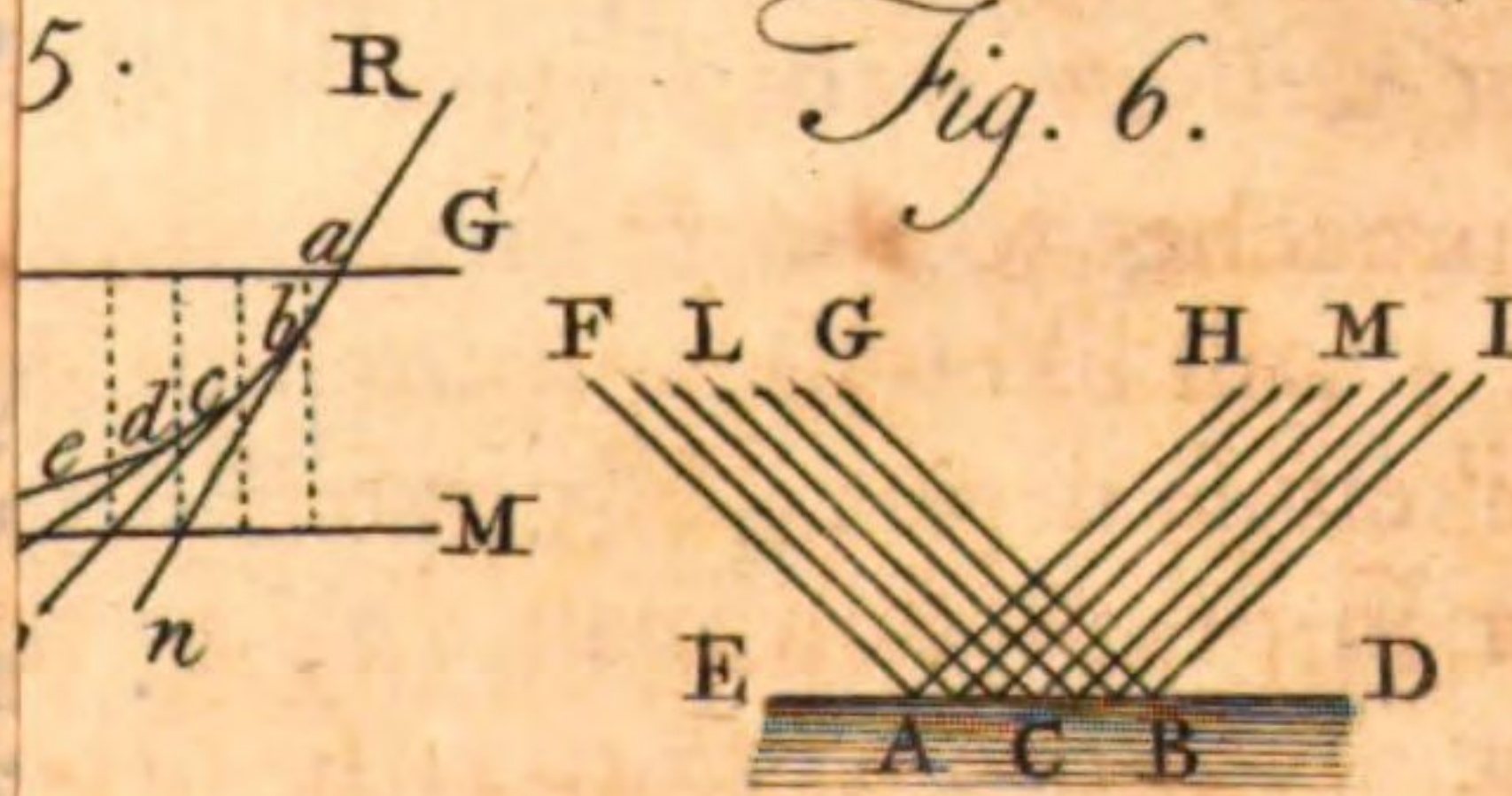


Fig. 11.

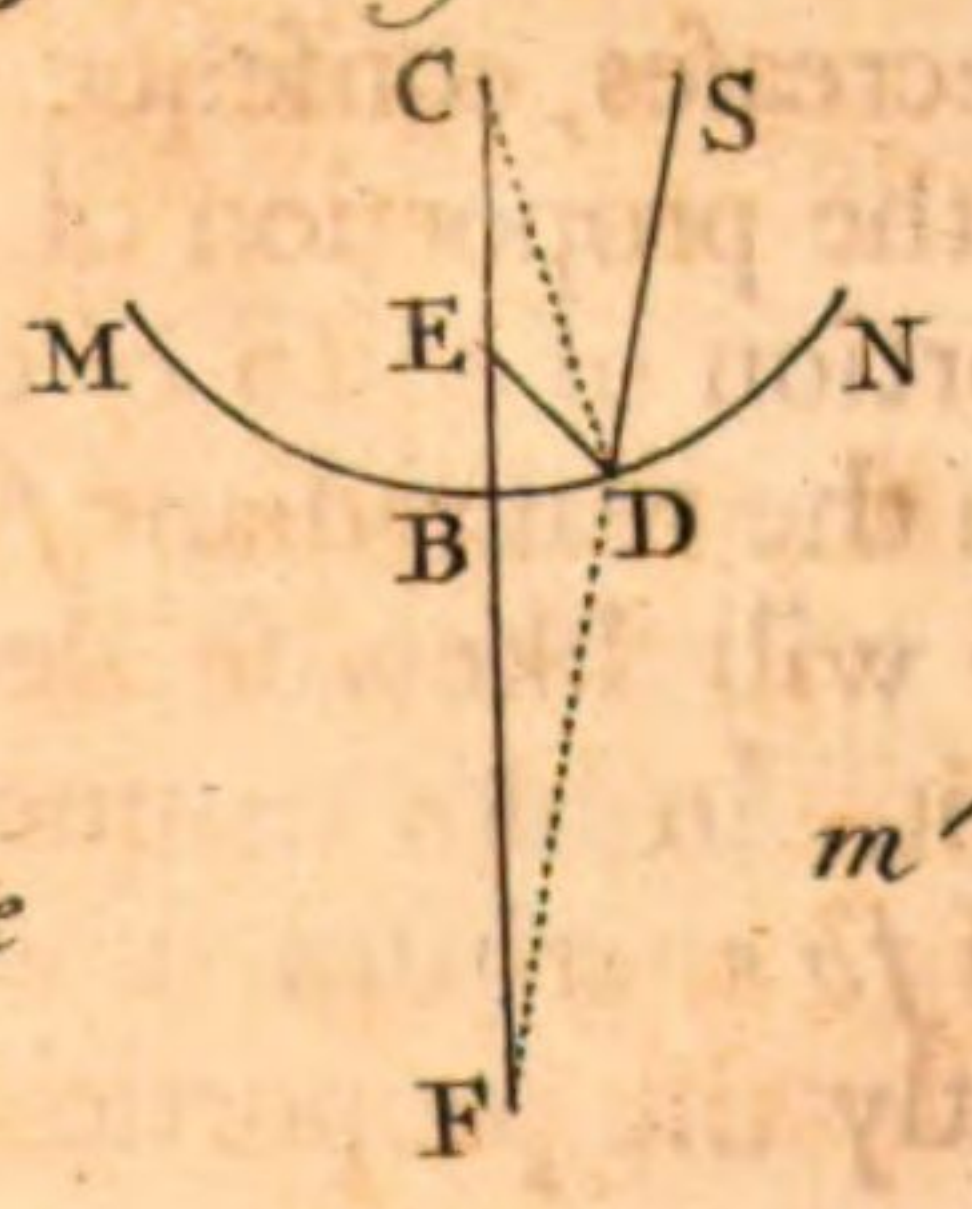


Fig. 12.



Fig. 10.



the proportion of eb to ec will encrease, that is, eb the distance of the imaginary radiant e will encrease, or the imaginary radiant will depart at the same time.

231. *If the real radiant is close to a convex mirrour, the imaginary radiant will likewise be close to it. (See prop. 79.)*

As fb to fc , Plat. XIII. fig. 12. so is eb to ec , by proposition 228. But when the real radiant f is close to the mirrour, fb or its distance from the surface is nothing. And when the first term of this proportion is nothing, the third term ec , or the distance of the imaginary radiant from the surface, will be nothing, that is, the imaginary radiant will be close to the surface.

CHAP. XVII.

Of reflected vision in plane mirrours.

232. *The passage of reflection is the incident ray added to the reflected ray.*

WE may sufficiently explaine this definition in an instance of reflected vision from a plane mirrour ab , Plat. XIV. fig. 1. though the term defined is used in all other instances of reflected vision. Let de be an object, and suppose the rays df , dr , which diverge from d the highest radiant in it, to be reflected in the lines fc , rb ; here df is the incident ray and fc is the reflected one, and df added to fc is the passage of reflection; or dr is the incident ray and rb is the reflected one, and dr added to rb is the passage of reflection.

233. *When any small object or any point of an object is seen by reflected light, it appears in the direction of that line, which the rays describe after their last reflection.*

This is proved in the same manner as proposition 137. For the picture of an object seen by reflection will be in the same part of the retina, that it would have been in, if we had looked with the naked eye at an object situated in the place from whence the reflected rays diverge; since the reflection gives the same direction to the rays as if they had originally come from that place: so that in reflected as well as in refracted vision the last image consists of imaginary radiants, from which the rays diverge when they come to the eye.

234. *In all mirrours, whether plane or spherical, the place of the imaginary radiant, when it can be determined, is the intersection of the cathetus of incidence and of any reflected ray.*

If d , Plat. XIV. fig. 1, is a radiant in any object de , and any ray df , which comes from this radiant, is reflected in the line fc , then draw the cathetus of incidence di perpendicular to ab the surface of the plane mirroure, and the plane of the imaginary radiant from whence fc appears to diverge is thus determined; continue the cathetus of incidence di behind the mirroure, and if the reflected ray cf is continued in the same manner till it crosses the cathetus at l , this point of intersection l will be the place of the imaginary radiant. All the rays, which diverge from one and the same radiant d before reflection, diverge from one and the same imaginary radiant after reflection. Now amongst the innumerable rays, which come from d in all directions there will be one that describes di the cathetus of incidence. This ray will be reflected in the line of its incidence id , because it falls perpendicularly upon the mirroure, by proposition 88 of mechanics. Consequently, since all the rays diverge after reflection from one and the same imaginary radiant, this radiant must be somewhere in the line di continued, for otherwise the reflected ray id could not diverge from it. The ray fc likewise diverges from the same radiant; and consequently this radiant must be somewhere in the line cf continued. But it is impossible that the imaginary radiant should be both in the line cf and in the line di , unless it is at some point which is in both these lines. And no point can be in both these lines, but l , which is the point of their intersection. Therefore l is the place of the imaginary radiant.

This is universal, whenever the ray, which describes the cathetus of incidence, is reflected either from a plane or a spherical mirroure. For the imaginary radiant will always be somewhere in this ray; that is, it will be somewhere in the cathetus of incidence; and it will always be somewhere in any other reflected ray. Therefore it will always be in the common point of intersection of the cathetus of incidence and of any reflected ray. Thus in the concave mirrours gil , fig. 2, and 3, and in the convex one gil , fig. 4, if a is a radiant in the object abc , the place of the last imaginary radiant, when the rays, which come from a , are reflected by the mirroure, may be determined by this rule. The ray ai is reflected in some line ic ; draw the cathetus of incidence kad , which in spherical mirrours always passes through k the center of the sphere of which the mirroure is a segment; then continue the reflected

ray

ray ic till it crosses the cathetus; and the point of intersection d is the place of the last imaginary radiant.

The place of the last radiant is determined in the same manner, though it may happen, either from the position of the object or from the shortness of the mirror, that the ray, which describes the cathetus of incidence, does not fall upon the mirror's surface, and consequently is not reflected. Thus if the plane mirror is not so long as ab , fig. 1. if, for instance, the length of it is only fb , so that the ray di does not fall upon it, the last imaginary radiant will still be at l , or at the point where the cathetus di continued intersects any reflected ray cf or br continued. This would be the place of the last radiant, if the mirror was long enough to reflect the ray which describes the cathetus, as has been proved already. But the length of the mirror does not alter the place of the imaginary radiant, by proposition 205. Therefore this will be the place of the last radiant, or the reflected rays fc , rb , will diverge from the same point; though the length of the mirror is only fb , so that the ray di , which describes the cathetus of incidence is not reflected.

235. *In plane mirrors, the distance of the last image from the mirror is equal to the distance of the object from it: and the distance of any point in the last image from the eye is equal to the passage of reflection.*

The distances of each imaginary radiant l , m , Plat. XIV. fig. 1, behind the mirror are respectively equal to the distance of the corresponding real radiants d , e , before it, by proposition 205. Therefore the distance of the last image lm , which is made up of imaginary radiants, is equal to the distance of the object de , which is made up of real ones.

The distance of l or the highest point in the image from the eye at c is lfc , and lfc is equal to dfc the passage of reflection. For since lf is equal to df , by proposition 205, $lf + fc = lfc$ must be equal to $df + fc = dfc$. So likewise mnc , which is the distance of the lowest point in the image from the eye at c is equal to enc , which is the passage of reflection. For $mn = en$, by proposition 205: and consequently, $mn + nc = en + nc$.

236. *In plane mirrors the image is equal and similar to the object.*

The image lm , Plat. XIV. fig. 1, is just as high as the object de . For if d is the highest point of the object, then the highest point of the image is in the cathetus of incidence dil , and if e is the lowest point of the

the object, the lowest point of the image is in the cathetus exm , by proposition 334. But dil and exm are parallel to one another, because, by proposition 203, they are both of them perpendicular to the same plane surface ab . Euc. b. XI. prop. 6. Now the height of the object is de the distance between those two lines at the object, and the height of the image is lm the distance between the same lines continued to the image. But these lines are parallel to one another, and consequently the distance between them is the same at both places. Therefore the height of the last image lm is equal to the height of the object de . In the same manner we might shew that the breadth of the image is equal to the breadth of the object. And consequently one is equal to the other in all respects. But surfaces that are every way equal to one another must be similar. Therefore the image and object are similar surfaces. What we mean by the surface of the object, when we say that the surface of the image is similar to it, may be seen in proposition 90, where we explained the same thing in reference to the picture of an object in a dark room.

237. *If a plane mirrour and the object that is seen in it are both of them perpendicular to the horizon, the object seen in the mirrour appears erect.*

If the object de , Plat. XIV. fig. 1, and the mirrour ab are perpendicular to the horizon, the image lm , or the object of reflected vision, is in the same position as the object itself. The highest point d of the object appears in the cathetus of incidence di drawn from thence, and the lowest point e appears in the cathetus ex , by proposition 234. But these cathetuses being both perpendicular to the surface of the mirrour are parallel to each other; Euc. b. XI. prop. 6, and consequently they do not cross each other any where between the object de and the image lm . Therefore that cathetus, which is highest at the object, will be highest at the image, and that, which is lowest at one, will be lowest at the other: and consequently the image, which is the object of reflected vision, will be in the same position with the object itself.

238. *When the object is parallel to a plane mirrour, the length of that part of the mirrour, upon which the image appears, is to the length of the object, as any reflected ray is to the passage of reflection.*

If the object de is parallel to the mirrour ab , and the image lm is seen by the eye at c , then fn the length of that part of the mirrour, which is taken up by the image, subtends the optic angle lcm , or
the

the angle under which the image appears. For since all the visible length of the image is included within that angle lcm , by proposition 125, no more of the mirror can be taken up by that visible length than what is likewise included within the same angle. Now the length of the image lm is equal to the length of the object de , by proposition 236. And fn the length of that part of the mirror, which is taken up by the image, is to lm the length of the image as any reflected ray fc is to dfc the passage of that rays reflection, or as the distance of the eye from the mirror, which is the reflected ray fc to lfc the distance of the eye from the image, which is equal to the passage of reflection, by proposition 235. For the apparent length of fn that part upon the mirrors surface, where the image is seen, is equal to the apparent length of the image; because they both subtend the same angle; by proposition 125. And from hence it follows, by proposition 128, that their real lengths are to each other as their respective distances from the eye. But the eyes distance from that part of the mirror, where the image appears is equal to the reflected ray fc , and the eyes distance from the image is lfc equal to the passage of the reflection. Therefore fn the length of that part of the mirror where the image appears is to lm the length of the image, or to $de=lm$ the length of the object, as fc a reflected ray is to $lfc=dfc$ the passage of reflection.

Or if the eye was at b , then lbm is the optic angle, and ro is the length of that part in the mirror where the image appears: and in this position of the eye ro is to lm or to de as rb any reflected ray is to $lrh=drh$ the passage of reflection.

In the same manner we might shew that the breadth of that part in the mirror, which is taken up by the image, is to the breadth of the object, or that any diameter of that part is to the corresponding diameter of the object, as a reflected ray is to the passage of that rays reflection.

239. *Though a common looking-glass should be so short that, at a certain distance of the eye and of a given object from it, we cannot see the whole length of the object in the glass by reflection; yet, by bringing the eye nearer to the glass or by removing the object farther from it, we may see the whole length of the object.*

The length of glass, which is requisite to see the whole length of the object, bears the same proportion to the length of the object, that a reflected ray bears to the passage of reflection, by proposition 238. From hence it follows that the less the reflected ray is in proportion to the passage of reflection, so much shorter a glass may be in proportion to any ob-

object, the entire length of which is seen by reflection in the glass. Now as the eye c , Plat. XIV. fig. 1. approaches to the glass, the length of a reflected ray cf decreases: at the same time indeed the passage of reflection dfc , or $lfc = dfc$, decreases too. But cf is shorter in itself than lfc , and by the approach of the eye to the glass these two unequal lines are equally diminished; and consequently this equal decrease bears a greater proportion to cf than to lfc ; that is, though in the approach of the eye to the glass an equal part is taken away both from cf and from lfc , yet this equal part bears a greater proportion to cf than it does to lfc ; upon which account by taking an equal part from cf and from lfc , the reflected ray cf will be more diminished in proportion than the passage of reflection lfc . Therefore in the approach of the eye to the glass the proportion of the reflected ray to the passage of reflection decreases, since the former is more diminished in proportion than the latter; and consequently the length of glass, in which we see the object, is shorter in proportion to the objects length: so that if a glass was too short to shew us the whole length of the object, at a certain distance of the eye, yet it may be long enough for this purpose, if the eye comes nearer to it.

The same purpose may be answered, by removing the object farther from the glass, whilst the eye continues at the same distance. For this removal of the object will diminish the proportion of the reflected ray to the passage of reflection. If the eye continues at c , the reflected ray is cf , but as the object departs from the glass df , and consequently dfc , or $lfc = dfc$, which is the passage of reflection, encreases. But the longer lfc is, so much less proportion the given line cf will bear to it. Therefore as the object departs from the glass the proportion of the reflected ray to the passage of reflection decreases.

Since the reader has been shewn that, as the eye approaches to the glass or as the object departs from it, the length of glass, upon which a given object is seen, decreases; he will easily infer that the contrary cause would produce a contrary effect, or that, as the eye departs from the glass or the object approaches to it, the length of glass, upon which a given object is seen, will encrease. For in the departure of the eye or in the approach of the object, the proportion of the reflected ray to the passage of reflection encreases.

240. *That part of a plane mirrours surface, upon which an object appears by reflection, is always similar to the surface of the object.*

The length of the object is to the length of that part in the mirrours surface where the object appears, and the breadth of the object is to the
breadth

breadth of this surface, as a reflected ray to the passage of reflection, by proposition 238. Therefore the length of the object is to the length of this surface as the breadth of the object to the breadth of this surface; since each is to the other as a reflected ray to the passage of reflection. But two surfaces are similar, when their lengths and breadths are respectively proportional to each other. Therefore that part of a mirrors surface, on which any object is seen, is similar to the surface of the object itself.

241. *The area of that part in a plane mirrors surface, upon which an object appears, is to the area of the objects surface, as the square of a reflected ray to the square of the passage of reflection.*

That part of a mirrors surface, where any object appears is similar to the objects surface, by proposition 240. But the areas of similar surfaces are to one another as the squares of their corresponding diameters. Euc. b. VI. prop. 20. corol. 1. Therefore the area, which the image covers on the mirrors surface, and the area of the objects surface, are to each other as the squares of their respective lengths. Now the length of that part in the mirror, which the image covers, is to the length of the object, as a reflected ray to the passage of reflection, by proposition 238. Therefore their areas are to each other as the square of a reflected ray to the square of the passage of reflection.

242. *If a person sees himself entirely in a plane looking-glass, which is parallel to him; the glass must be half as long as he is, and half as broad as he is; or its surface must be equal to a quarter of his surface.*

The length of a mirror must be to the length of a person, who sees himself entirely in it, as a reflected ray to the passage of reflection, by proposition 238. Now when a person is looking at himself, the incident ray is his distance from the glass, and the reflected ray is equal to it, for it is the glass's distance from him. The passage of reflection, is the incident ray added to the reflected ray, and consequently is equal to twice his distance from the glass. Therefore the reflected ray is to the passage of reflection as 1 to 2. From whence it follows that the length of the glass, in which he can see himself from head to foot, must be to his length as 1 to 2; that is, the length of the glass must be half his length.

In a plane mirror, that is parallel to a person who is looking at himself, he will not be able to see himself from head to foot, if the mirror is at all shorter than this, whether he is near to the mirror or far from

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it. If he was looking at any other object, then indeed the nearer he comes to the mirror, the more of the object he might see, by proposition 239: because by coming near to the mirror his eye approaches to it. But when he himself is the object that he is looking at, as he comes nearer to the mirror his eye comes nearer to it, but then the object comes just as much nearer at the same time. And consequently though in a short mirror he might see more of himself from the approach of the eye, yet he will see just as much less of himself from the equal approach of the object, by proposition 239. Therefore upon both accounts together he will see just as much and no more of himself, when he is near to the mirror, than he did, when he was farther from it; but if it was too short to see himself entirely in it at any one distance, it will likewise be too short for that purpose at any other less distance. In like manner if he was looking at any other object, the farther the object is from the mirror the more he might see of it. But when he himself is the object that he is looking at, though by going farther from the mirror, he might upon this account see more of himself; yet as he cannot go farther off, but his eye will depart equally from the mirror, consequently upon this second account he will see just as much less of himself. Therefore upon both accounts together he will see neither more nor less of himself for encreasing his distance: but if the mirror was too short to see himself from head to foot at any one distance, it will likewise be too short for that purpose at any other greater distance.

In the same manner that we have proved a mirror must be half a mans length for him to see his whole length in it, we might likewise prove that it must be half his breadth for him to see his whole breadth.

But the surface of the mirror can only be a quarter of his surface, if it is but half as long and but half as broad as he is. This last part of the proposition may be demonstrated otherwise. The area of a mirror, which is covered by the image of any object, is to the area of that objects surface, as the square of a reflected ray to the square of the passage of reflection, by proposition 241. But when a man is looking at himself, the reflected ray is to the passage of reflection as 1 to 2, by what has been already shewn, and consequently the square of the reflected ray is to the square of the passage of reflection as 1 to 4. Therefore the area of a mirror, which will be just covered by a mans image, when he is looking at himself, is to the area of his surface, as 1 to 4; that is, the surface of the mirror in which he can just see his whole length and his whole breadth is $\frac{1}{4}$ of his surface.

243. *In a plane mirrour what is on the right-hand side in the object, is on the left-hand side in the image, and what is on the left-hand side in the object will be on the right-hand side in the image.*

This will be best explained and proved by the instance of a person looking at himself in a plane mirrour. His face is then towards the face of his image in the glass. Now if any other person stood with his face towards ours, it is evident that his right-hand would be towards our left, and his left-hand towards our right. And for the same reason it will be so in the case of our own image seen in the glass. If we hold a pen in our right-hand, the image with its face towards us will hold the pen in that hand which is on the same side: and if any person with his face towards us was to hold a pen in the hand which is on the same side with our right-hand, we should say that he held it left-handed.

In reading a book we begin the lines from the left-hand, and read towards the right. But if we turn the book towards a looking glass we must then read the lines seen by reflection from the left-hand of our image in the glass towards the right-hand of that image. This we shall find is reading the contrary way from what we usually do, for the lines will appear inverted and we must read them from our own right-hand towards our left.

Something like this may be seen in some prints that are copies from paintings; for all the figures in such prints are frequently left-handed. If the figures engraved upon the copper-plate stand the same way that they do in the original painting and are all of them right-handed; then in the print taken off from that plate they will stand the contrary way and will be left-handed. For in taking off the print from the engraving, any figure upon the plate, as for instance, the figure of a man with a spear in his right-hand has its face towards the figure upon the paper; consequently the right-hand of the man upon the plate will be towards the left-hand of the same man in the print: therefore as one of them holds the spear in his right-hand the other will hold it in his left.

244. *If a plane mirrour is parallel to the horizon such objects, as are perpendicular to the horizon, when they are seen it, appear inverted.*

This is the case of trees, or buildings, or men, standing on the bank of a river, when they are seen in the water, which is a plane mirrour. The bottom of any tree or other object is nearest to the mirrour and the top of it is farthest from the mirrour. Now, by proposition 205, each imaginary radiant is at the same distance behind the mirrour, that the real radiant is at before it. Therefore in the image the bottom of

the tree will be nearest to the surface of the water and the top will be farthest from it. But in this position of the image the top of the tree seen in the water will be nearer to the center of the earth than the bottom of it, or the tree will appear inverted.

If a man holds a looking-glass above his eye with the reflecting surface turned downwards and parallel to the horizon, he will see himself inverted in the glass. In this position his head is nearer to the glass than his feet, and consequently, by proposition 205, in his image on the other side the head will likewise be nearer to the glass than the feet. Therefore the feet of his image will be farther from the center of the earth than its head is, or the image will appear with its feet upwards.

245. *If a plane mirrour is inclined to the horizon so as to make with it half a right angle, any object, which is parallel to the horizon, will appear erect in the mirrour; and on the contrary any object, which is erect, will appear parallel to the horizon.*

If AB, Plat. XIV. fig. 1, is a plane looking-glass with its reflecting surface downwards, and CD is an object parallel to the horizon, then the mirrour AB, which makes half a right-angle with the horizon, will likewise make half a right-angle with the object, or ABC will be an angle of 45 degrees. Now at whatever distance any radiant C in the object is from the mirrour, the distance of the corresponding radiant *c* in the image will be the same, by proposition 205; or CE will be equal to *cE*. For the same reason any other radiant D in the object will be just as far distant before the mirrour, as the imaginary radiant *d* is behind it; or DF will be equal to *dF*. And since every radiant in the image may in this manner be proved to be at the same distance behind the mirrour that the real radiant is at before it; it follows, that the image *cd* will make half a right angle *cBA* with the mirrour on one side as the object CD makes half a right angle CBA on the other. Consequently $cBC = cBA + CBA$ is a right angle, or the image *cd* is perpendicular to the object CD; and therefore, being perpendicular likewise to the horizon to which the object is parallel, it will appear erect.

If *cd* is the object and the mirrour AB has its reflecting surface upwards; then the object is erect; the mirrour is inclined to the horizon at half a right angle; and the image will be CD parallel to the horizon. If CB is an horizontal plane, the object *cd*, because it is erect, by the supposition, will make a right angle *cBC* with the plane. But ABC the mirrours inclination is one half of this angle, therefore the other half AB*c* is likewise half a right angle; that is, the object makes

makes half a right angle with the mirrour before it, and consequently, from what has been said in proving the other part of the proposition, the image will make half a right angle with the mirrour behind it. Therefore the object cd and its image CD will make two half right angles or one whole right angle with each other. But the object cd makes likewise a right angle with the horizontal plane CB . Therefore the image CD will be in this plane or will be parallel to the horizon.

246. *If an object is placed between two plane mirrors inclined to one another at any angle, more images than one of that object may be seen.*

If two plane mirrors CB and CA , Plat. XIV. fig. 6. are inclined to one another so as to contain the angle BCA , and the object F is placed between them; then let FD be drawn from the object perpendicular to the mirror CA , and in FD , which is the cathetus of incidence, let KD be taken equal to KF , and the image of this object will appear at D , by proposition 235. In like manner, if FG is drawn perpendicular to the mirror CB , and G is just as far behind the mirror as F is before it, the image of the same object will appear at G , by proposition 235. Thus an image of the object will appear in each mirror.

There will be a small difference between these two images. For as the object F has one side towards the mirror CA , and the contrary side towards the mirror CB ; one of these images will represent one side of the object, and the other image will represent the other side of it. If F is the face of the spectator himself, and he looks strait forwards the right side of his face will appear in the glass CA and the left side in the glass CB . If he stands with his face towards CA , then D will be the representation of his face and G of his back. Indeed if both sides of the object are alike, as suppose F to be a candle, then though this difference is still the same as in any other object, yet the difference will not be perceivable, because both images will be alike.

But besides the images D and G there will be many more seen in each of the mirrors. Some of the rays, which are reflected from the mirror CA and diverge from the image D , come to the eye of the spectator placed any where between the mirrors, and make that image visible. But all the reflected rays will not come to the eye; for since they diverge from D in all directions, some of them will fall upon the opposite mirror CB . So that the image D may be considered as an object placed before the mirror CB : and consequently when the rays, that apparently diverge from D , are reflected by this mirror CB , if DHE is drawn

drawn perpendicular to CB, it will be the cathetus of incidence, and if EH is taken equal to DH, these reflected rays will represent the image of D at the point E, which is just as far behind the mirror CB as D is before it, by proposition 235. In like manner the rays which diverge from this second image E will some of them fall upon the mirror CA, and if a cathetus EM is drawn from E to this mirror CA, the image of E or a third image of the object will appear in this cathetus at the point M, which is as far behind the mirror as E is before it. For the same reason, since rays will diverge from M as from an object before the mirror CB, the image of M or a fourth image of the object F will appear in this mirror CB, in a cathetus drawn from M to a point as far behind the mirror as M is before it. And thus as long as the image represented in one mirror is before the other, so long an image of that image will be produced. Now all the images, that have been hitherto described, begin from the mirror CA, and D is the first of them. Therefore whatever side of the object is towards CA, all these images will represent that side.

But there will be another set of images beginning from the mirror CB, and of this set the image G is the first, and consequently all this set will represent the same side of the object that is represented at G, or the side, which is towards the mirror CB. For as the rays reflected from CB diverge from G in all directions some of them will fall upon the opposite mirror CA; and if the cathetus CL is drawn these rays, when reflected from CA, will produce an image of G at the point L, which is as far behind the mirror as G is before it. In like manner some of the reflected rays, which diverge in all directions from L, will be returned to the mirror CB, and being reflected from thence will produce an image of L. And thus again as long as the image represented in one mirror is before the other, so long an image of that image will be produced.

247. All the images, that appear in two plane mirrors inclined to each other, are in the circumference of a circle, the semidiameter of which is the distance of the object from the vertex of the angle contained between the mirrors.

The several images D, E, G, L, M, &c, Plat. XIV. fig. 6, are in the circumference of a circle, and CF the distance of F the object from C, the vertex of the angle made by the mirrors CA and CB, is the semidiameter of this circle.

CF the distance of the object from the vertex is equal to CD. For FK the distance of the object before the mirror is equal to DK the distance

stance of the image behind it, by prop. 235; CK is common to both the triangles CFK and CDK; and the angle CKF is equal to the angle CKD, because they are both right ones, since FD is the cathetus of incidence. But when two sides in one triangle are respectively equal to two sides in another triangle, and the angle contained between the equal sides in one is equal to the angle contained between the equal sides in the other, those two triangles are equal to each other in all respects. Euc. b. I. prop. 4. Therefore the side CF in the triangle CFK is equal to the side CD in the triangle CDK.

CE is likewise equal to CD. For DE is the cathetus, and consequently the angles CHD and CHE are equal, because they are both right ones; D is the place of one image before the mirror, and E is the place of the image of this image behind the mirror, and consequently DH is equal to EH; and lastly, in the two triangles CDH, CEH, the side CH is common. Therefore all the sides of one triangle are respectively equal to all the sides of the other, so that CE is equal to CD.

Since then CD is equal both to CF and likewise to CE, it follows, that CF and CE are equal to one another. And in the same manner we might prove that any line drawn from C to G or to L or to M or to any other image, that appears in the mirrors will be equal to CF. Therefore if one foot of the compasses is placed at C, and a circle is struck with the opening CF, this circle whose semidiameter is CF will pass through D, E, G, L, M, and every other image, that appears in the mirrors. And consequently all those images are placed somewhere in the circumference of a circle, whose semidiameter is the distance of the object F from C the vertical point.

248. *When two plane mirrors are inclined to one another, the images of each set end, when each cathetus of incidence ends between the two mirrors continued.*

In proposition 246, we observed that, if two plane mirrors, CA, CB, Plat. XIV. fig. 7, are inclined to one another and any object S is placed between them, there will be two sets of images seen in the mirrors: one set begins from the mirror CA, and a_1 will be the first image in it; the other begins from the mirror CB, and b_1 will be the first image. Every image in these two sets will be in the circumference of a circle, whose semidiameter is CS the distance of the object from the vertex C, by proposition 247. And the place of each succeeding image is determined by drawing a cathetus of the image to that mirror before which the preceeding image stands, by proposition 235, and by what was proved in

in proposition 246. Thus the image a_1 stands before the mirror CB, and if a cathetus is drawn from a_1 to the mirror CB and is continued to the circumference of the circle at a_2 , this will be the place of the next image. This image a_2 stands before the mirror CA: therefore if a cathetus is drawn from a_2 to the mirror CB and is continued to the circumference at a_3 , this will be the place of the third image in this set. This image a_3 stands before the mirror CB, and consequently to determine the place of the succeeding one, a cathetus must be drawn from a_3 to the mirror CB and must be continued to the circumference of the circle, in which all the images are placed. Now as the angle a_3CB , which a_3 makes with the mirror CB, is an obtuse one or is greater than a right one, no cathetus or perpendicular can be drawn from a_3 to the mirror itself. But such a cathetus may be drawn from a_3 to CL, which is the plane of this mirror continued from C to L. And the image seen by reflection will appear in the cathetus, whether it falls upon the mirror itself or upon the plane of the mirror continued, by propositions 234, 235. Therefore if a cathetus is drawn from a_3 to CL and is continued to a_4 , this will be the place of the fourth image. The image a_4 stands before the mirror CA, and to determine the place of the fifth image a cathetus must be drawn from a_4 to CR, which is CA continued, and when this cathetus is continued to the circumference a_5 , this point at the circumference will be the place of the fifth image. The image a_5 stands before the mirror CB, and consequently if from a_5 a cathetus is drawn to CL, and continued to a_6 , this will be the place of the sixth image. The cathetus a_5a_6 ends between CR and CL, which are the two mirrors continued: and the proposition affirms that the set of images which begins from CA, will end, where the cathetus of incidence drawn alternately from one mirror to the other ends between the two mirrors continued: and consequently a_6 is the last image of this set. That this will be the last image is evident from the following consideration. The reflecting surface of the mirror CB is towards the object S, and so likewise is the reflecting surface of the mirror CA. Consequently the image a_6 , which is placed between the mirrors continued to R and L, will be behind both of them, it is behind BCL, and behind ACR. But no rays can be reflected by either mirror, if they diverge from an image that is behind them. And unless there is such a reflection no new image will be produced. Therefore a_6 the image, which is placed between the two mirrors continued, will be the last of this set.

In like manner we might trace out the images of the other set, which begin from CB. The first of them is b_1 , and the last b_6 , where the cathetus

thetus ends between CR and CL the two mirrors continued. For the reason just assigned is universal. Whenever the cathetus ends between the two mirrors continued, the image is behind both of them, and consequently, as no farther reflection can be made from either of them, this image will be the last.

249. *When two plane mirrors are inclined to one another, the angular distance between the two first images of each set is equal to double the angle of inclination.*

The angle of inclination is BCA, Plat. XIV. fig. 7, or is the angle contained between the two plane mirrors AC and BC. The first image of one set is a_1 ; and the first of the other set is b_1 . The angular distance between these first images is the angle b_1Ca_1 . This angular distance is double the angle of inclination; or b_1Ca_1 is equal to twice BCA.

The angle of inclination BCA is equal to the angular distance of the object S from one mirror BC added to its angular distance from the other mirror AC, or $BCA = BCS + ACS$: this is self-evident. Now BCS is equal to BCb_1 , and ACS is equal to ACa_1 , by proposition 235. Therefore $BCb_1 + ACa_1 = BCS + ACS = BCA$. Consequently $BCb_1 + ACa_1 + BCA = 2BCA$. But the angular distance between b_1 and a_1 , or between the two first images of each set is $BCb_1 + ACa_1 + BCA$. Therefore this angular distance is equal to $2BCA$ or to double the angle of inclination.

250. *When two plane mirrors are inclined to one another, the angular distance between any two images of each set, that are produced by the same number of reflections, is greater than the angular distance between the two preceeding ones by double the angle of inclination.*

The image a_1 , Plat. XIV. fig. 7, is the first of one set, and the image b_1 is the first of the other: the angular distance between them is b_1Ca_1 . The image a_2 belongs to one set, and the image b_2 belongs to the other: each of these images is produced by the same number of reflections, for each of them is produced by two reflections: the angular distance between them is b_2Ca_2 . We are therefore to prove that b_2Ca_2 the angular distance between a_2 and b_2 is greater than b_1Ca_1 the angular distance between the two preceeding images a_1 and b_1 , by twice the angle of inclination BCA. The difference between a_1Cb_1 and a_2Cb_2 is $a_1Cb_2 + b_1Ca_2$, this difference is equal to $2BCA$; for it is equal to b_1Ca_1 , and b_1Ca_1 has been proved to be equal to $2BCA$, in proposition 249. That $a_1Cb_2 + b_1Ca_2 = b_1Ca_1$ may be thus shewn. First

$a_1Cb_2 = b_1CS$: for $b_2CA = b_1CA$, by proposition 235; consequently, if from these equal angles equal angles are taken, the remainders will be equal. But $ACa_1 = ACS$, by proposition 235. Therefore $b_2CA - ACa_1 = b_1CA - ACS$. But $b_2CA - ACa_1 = a_1Cb_2$ and $b_1CA - ACS = b_1CS$. Therefore $a_1Cb_2 = b_1CS$. Secondly $b_1Ca_2 = a_1CS$: for $a_2CB = a_1CB$; and $BCb_1 = BCS$, by proposition 235; consequently $a_2CB - BCb_1 = a_1CB - BCS$. But $a_2CB - BCb_1 = b_1Ca_2$, and $a_1CB - BCS = a_1CS$. Therefore $b_1Ca_2 = a_1CS$. Now by the first step, $a_1Cb_2 = b_1CS$, and by the second step, $b_1Ca_2 = a_1CS$. Therefore thirdly $a_1Cb_2 + b_1Ca_2 = b_1CS + a_1CS = b_1Ca_1 = 2BCA$. In the same manner we might shew that $a_3Cb_2 + b_3CA_2 = a_1Cb_2 + b_1Ca_2 = 2BCA$, or that the difference between the distances of the third images in each set and the distances of the second images in each set is equal to twice the angle of inclination. And the same may be shewn in the same manner of all the images in the whole circumference.

251. *When two plane mirrours are inclined to each other, as many images may be seen in both of them together as there are angles contained in a whole circle equal to the angle of inclination.*

If the angle of inclination BCA , Plat. XIV. fig. 7, is 30 degrees, there are 12 angles of 30 degrees contained in a whole circle, and consequently in both the mirrours BC and CA 12 images may be seen. The distance between the images of each set encreases by double the angle of inclination, as appears from proposition 250. Therefore in both sets taken together there will in the whole circumference of the circle be two images for every double angle of inclination, or one image for every angle of inclination.

If the angle of inclination was 90 degrees, by this rule there would be 4 images; if it was 40 degrees there would be 9 images; if 10 degrees there would be 36 images. The rule is indeed liable to some restrictions according as the angle or double the angle of inclination is or is not exactly commensurate to a circle, and according as the object is placed in the middle between the mirrour or nearer to one of them than it is to the other. But it will not be worth the while to trouble the reader with these restrictions. One of them however we may take notice of. If the double angle of inclination is exactly commensurate to a circle, the distance between the two last images of each set will be the circumference of a circle, and consequently these two images will be in the same place. Thus if the angle of inclination is 30 degrees the double angle of inclination is 60 degrees: the first two images a_1 and b_1 are 60 degrees asunder;

der; the two next a_2 and b_2 are $60+60=120$ degrees asunder; the two next a_3 and b_3 are $120+60=180$ degrees asunder; the two next a_4 and b_4 are $180+60=240$ degrees asunder, the two next a_5 and b_5 are $240+60=300$ degrees asunder; and the two last a_6 and b_6 are $300+60=360$ degrees asunder. Therefore a_6 and b_6 are at ab_6 the same point of the circle, and consequently coincide, so that though there are 12 images only 11 will be seen in either mirror.

252. *If two plane mirrors are parallel to one another and an object is placed between them, innumerable images of that object may be seen in each of them; and all the images will stand in a right line.*

This proposition may be deduced from propositions 251, 247. In two plane mirrors BC and AC, Plat. XIV. fig. 7, that are inclined to one another, as many images may be seen as there are angles of inclination contained in a whole circle, and all these images will stand in the circumference of a circle, whose semidiameter is CS the distance of the object S from the vertex C of the angle of inclination BCA. Now if the two mirrors were to continue at the same distance from one another at B and A, but were to be opened a little at C so as to make a smaller angle, or to be less inclined, or to be nearer parallel to one another; by this means the number of images will be encreased, by proposition 251; because the angle contained between the mirrors is diminished: and at the same time the circumference of the circle, in which the images are placed, will be enlarged, by proposition 247; because, when the mirrors continue at the same distance from one another at A and B, and are opened wider at the other end C, the vertex of the angle contained between them will be more remote than C, and consequently the object will be more remote from the vertex, or SC will be encreased; but SC is the semidiameter of the circle in which the images are placed, therefore as this semidiameter is encreased the circumference will be encreased. By opening the mirrors still farther at the end towards C, the number of images will keep encreasing, and so will the circumference of the circle, in which these images are placed. And we must farther observe that, when the mirrors are far asunder towards C, the images in that part of the circle, which is beyond C, will not appear: because, unless the mirrors are close together at that end, there will be no reflecting surface to make them visible. From hence then it follows, that the many images, which are seen, stand in a part of the circumference of a circle, and that the arcs, in which they stand, are parts of a greater circle in proportion as the mirrors are nearer parallel to one another. Now the greater the

circumference of any circle is, the less is the curvature of any given part of it. Consequently as the mirrors, by opening them towards C, become nearer parallel to one another, the curvature of the lines in which the images are placed, will be less. But when the mirrors are parallel, the angle contained between them is nothing; so that innumerable such angles are contained in a circle, therefore there will be innumerable images, by proposition 251. And since, when the mirrors are parallel, the vertex C is at an infinite distance. SC the distance of the object or the semidiameter, and consequently the circumference of the circle, in which the images are placed, will be infinite. But only a part of this circumference will be seen in each mirror; and any finite arc of an infinite circle has no curvature or is a right line. Therefore the images in each mirror will stand in a right line.

253. *In one single plane mirror, that is made of thick glass with quicksilver behind it, many images of any bright object may be seen.*

Suppose the object to be a candle. Some of the rays, which fall upon the glass, will be reflected from the first surface without entering it, by proposition 198. These rays produce the first image, which will not be a very bright one. But most of the rays will enter the glass, and when they are reflected from the quicksilver behind the glass, they will come out of the glass again through the first surface, and produce the second image, which will be the brightest of them all. But when the rays are thus reflected from the quicksilver, though most of them come out of the glass, yet those which are in a fit of easy reflection will not come out there, but will be returned back to the quicksilver: and when they are reflected from thence a second time, they will produce a third image. Neither will all the rays, that are thus reflected a second time from the quicksilver, come out at the first surface of the glass, and produce this third image; for such of them, as are in a fit of easy reflection, when they come to the first surface, will be returned again to the quicksilver and being reflected from thence a third time will produce a fourth image. As often as this happens the image will be repeated, so that, if the eye and the candle are oblique to the glass nine or ten images may be seen. But as there are fewer rays left to produce every image after the second, all after this grow fainter, till the last of them are so faint as to be scarcely visible.

CH A P. XVIII.

Of reflected vision in concave and convex mirrours.

254. *In spherical mirrours, whether concave or convex, when the place of the last image is determinate, the object and the last image are in the same situation if they are both on the same side of the center, but in contrary situations, if they are on opposite sides.*

LET *abc*, Plat. XIV. fig. 2, be an object placed nearer to the concave mirrour *gil* than its principal focus; and let *k* be the center of the concavity. Because the radiants *a*, *b*, *c*, &c. in the object are nearer, by the supposition, than the mirrours principal focus, the reflected rays will diverge, by proposition 220, and the distances of the corresponding imaginary radiants *d*, *e*, *f*, &c. may be determined, by proposition 221. Now as *k* is the center of the concavity, the real radiants in the object *abc* are on the same side of the center *k* with the imaginary radiants in the last image *d*, *e*, *f*. In this case the object and image will be in the same position. For the imaginary radiant, which corresponds to the real one *a*, is in the cathetus of incidence drawn from *a* to the place of the image; and so likewise the imaginary radiant, which corresponds to *c* is in the cathetus drawn from *c* to the place of the image, by proposition 234. But the cathetus of incidence, because it is perpendicular to the mirrours surface, always passes through the center; so that *kad* is the cathetus of one extreme radiant, and *kcf* is the cathetus of the other extreme radiant. Since therefore these cathetuses are all drawn from the center, they cross each other no where but at the center; and consequently if the object *abc* and the image *def* are on the same side of the center, that is, if the center *k* is not between them, the extreme cathetus *ad* cannot cross the other extreme cathetus *cf* in any place between the object and the image. Upon this account the two extreme cathetuses, as they have not crossed, will be in the same position in respect of each other both at the object and at the image; that, which was the highest at the object, will be the highest at the image, and that, which was the lowest at one, will be the lowest at the other. But every point of the object appears in its cathetus of incidence at the image, by proposition 234. Therefore the highest point in the object will appear the highest in the image, and the lowest point in the object will appear the lowest in the image, that is, the object and image will be in the same situation.

In like manner if the object *abc*, fig. 4, is placed before a convex mirrour *gil*, the rays, which diverge at their incidence, will diverge after

reflection, by proposition 228, and the place of the imaginary radiants *def*, which compose the last image is determined by proposition 229. And here each point of the object appears at its cathetus of incidence in the image, by proposition 234. Therefore if the object *abc* and the image *def* are both on the same side of *k* the center, the extreme cathetuses *ak* and *ck*, which meet at the center but cross each other no where else, will not cross between the place of the object and the place of the image; and consequently that cathetus, which is the highest or the lowest at the object, will be the highest or the lowest at the image. Therefore if *a* is the highest point of the object, it will appear at *d* the highest in the image; and *c*, if it is the lowest point at the object, will appear at *f* the lowest in the image, so that the image will be in the same situation with the object.

But if the object *abc*, fig. 3, is farther from the concave mirrour *gil* than its principal focus, the reflected rays will converge, by proposition 213, and the place of the focuses, to which they converge, is determined, by proposition 214. Let *fed* be the place of the several focuses, to which the rays, that come from the radiants *a*, *b*, *c*, &c. converge after reflection, then if a white paper is held at that place a distinct picture of the object will be painted upon the paper, by proposition 84. For the effect in this respect is planely the same whether the rays are collected into their respective focuses by refraction or by reflection. But if the eye is more remote from the mirrour than the place of the distinct picture, and the paper is taken away then the rays, that are collected into the several focuses *def*, as they pass on from thence, will diverge from these focuses as from so many radiants, and the distinct picture will become the last image, as in proposition 138. Now the radiants *a*, *b*, *c*, which compose the object, and the focuses *d*, *e*, *f*, which compose the distinct picture, are always on contrary sides of the center *k*, by proposition 219. Upon which account the picture or last image will be inverted in respect of the object. For here likewise each point of the object, as *a* or *c*, will appear in its cathetus of incidence, by proposition 234. The extreme cathetuses *gad* and *lcf* cross each other at *k*, which is the center of the concavity and, by the supposition, is between the object and the image. But, when the cathetuses have crossed each other between the object and the image, that, which is the highest at one, will be the lowest at the other: and consequently, if *a* is the highest point of the object, it will appear at *d* the lowest in the image, and for the same reason if *c* is the lowest point of the object, it will appear at *f* the highest in the image, so that the image in respect of the object will be inverted.

If

If def was the object, then abc would be the image, that is, if the real radiants were at d, e, f , the focuses would be at a, b, c , by proposition 219. Therefore in this case likewise the object and image would be on contrary sides of the center, and the image abc would be inverted in respect of the object def .

255. *In spherical mirrors, whether convex or concave, the diameter of the object is to the diameter of the last image as the distance of the object from the center to the distance of the image from the center.*

The length of the object ac , Plat. XIV. fig. 2, 3, 4, bears the same proportion to the length of the image df , that ak the distance of the object from the center of the spherical surface, bears to dk the distance of the image from the same center. If the eye is any where in the line ib or in that line continued, the length of the object and the length of the image considered as visible extensions are parallel to one another: for in this position of the eye bi is the optic axis. Therefore the visible length of both is proportional to a subtense of the optic angle perpendicular to bi , by proposition 129. But if their lengths considered as visible extensions are perpendicular to the same right line, they must be parallel to one another. The same thing would be equally true, if the eye was in the line lc or ga or any other line perpendicular to the mirrors surface. We may therefore say universally, that the visible length of the object is parallel to the visible length of the image. The angle akc is equal to the angle dkf . They must be equal in fig. 2, and 4, because they are one and the same angle; and in fig. 3 they are equal, because they are vertical. Euc. b. I. prop. 15. The angle kac is likewise equal to the angle kdf . For, in fig. 2, 4, one is the external angle and the other is the internal opposite one on the same side made by the right line kd crossing the two parallel lines ac and df ; and in fig. 3, they are the alternate angles. Euc. b. I. prop. 29. For the same reason the angle kca is equal to the angle kfd . Since therefore the three angles of the triangle ack are respectively equal to those of the triangle dkf ; the sides of these triangles are proportional, Euc. b. VI. prop. 4. ac the visible length of the object is to df the visible length of the image, as ak the distance of the object from the center to dk the distance of the image from the center.

256. *In spherical mirrors, whether concave or convex, the diameter of the object is to the diameter of the image, as the distance of the object from the surface of the mirror to the distance of the image from the same surface.*

When

When the rays, that are reflected from a concave or convex mirrour, diverge, as in fig. 2, 4, Plat. XIV, the distance of the real radiant from the surface of the mirrour is to its distance from the center, as the distance of the imaginary radiant from the surface to its distance from the center, by propositions 221, 228. And when the rays that are reflected from a concave mirrour, converge, as in fig. 3, the distance of the radiant from the surface is to its distance from the center, as the distance of the focus from the surface to its distance from the center, by proposition 214. Now the object *abc* consists of real radiants, and the last image *def* either of imaginary radiants or of focuses, as was observed under proposition 254. Therefore *bi* the distance of the object from the surface is to *bk* the distance of the object from the center, as *ei* the distance of the image from the surface is to *ek* the distance of the image from the center. And consequently the distance of the object from the surface is to the distance of the image from the surface, as the distance of the object from the center is to the distance of the image from the center; that is, since as *bi* to *bk*, so *ei* to *ek*, it will likewise be as antecedent to antecedent so consequent to consequent, or as *bi* to *ei* so *bk* to *ek*. Euc. b.V. prop. 16. But the diameter of the object is to the diameter of the image, as the distance of the object from the center to the distance of the image from the center, or *ac* is to *df* as *bk* to *ek*, by proposition 254. Therefore the diameter of the object is likewise to the diameter of the image as the distance of the object from the surface to the distance of the image from the surface, or *ac* is to *df*, as *bi* to *ei*. We may bring the parts of this demonstration nearer together in this manner. *bi* is to *bk*, as *ei* to *ek*, by propositions 221, 228, 214. Therefore *bi* is to *ei*, as *bk* to *ek*, Euc. b.V. prop. 16. But *ac* is to *df* as *bk* to *ek*, by proposition 254. Therefore *ac* is likewise to *df* as *bi* to *ei*.

257. *If the eye is close either to a concave or convex-mirrour, the apparent diameter of the object is equal to the apparent diameter of the last image.*

The apparent diameters of visible objects are equal, when their real diameters are as their distances from the eye, by proposition 128. But the real diameters of the object and image are as their respective distances from the surface of the mirrour, by proposition 256. Therefore if, the eye is contiguous to the mirrours surface, their real diameters will be to one another as their respective distances from the eye, and consequently their apparent diameters will be equal. Thus if the eye is at *i*, Plat. XIV. fig.

fig. 2, 3, 4, the apparent length of the object *ac* will be equal to the apparent length of the image *df*, or both of them will subtend the equal angles *aic* and *dif*.

258. *If the eye is placed in the center of a concave mirrour, it can see nothing in the mirrour but its own image.*

The apparent length of the object and image seen from the center of a spherical mirrour, whether concave or convex, ought to be equal, by proposition 128: because their real diameters are as their respective distances from the center, by proposition 254, and consequently as their respective distances from the eye, if the eye could be placed in that center, or could see the image of the object in the mirrour, when it is placed there. But *k* the center of any convex mirrour *gil*, Plat. XIV. fig. 4. is behind it, and consequently if the eye was placed there, it could see nothing at all by reflection from the mirrour: and though *k*, fig. 2. 3, the center of the concave mirrour *gil* is before it, yet the eye, when placed in that center, will see nothing in the mirrour but its own image. For in this situation of the eye, all the rays, which diverge from it, will converge to it after reflection, by proposition 218, and no other rays from any part of the mirrour can enter it but those which come from itself: upon which account it will see its own image spread over the whole surface of the mirrour.

Or otherwise. When the eye is in the center of the concavity, the distinct picture or last image of the eye will be there too, by proposition 218. Consequently the eye being at the place of its own distinct picture will see itself infinitely magnified, and spread over the whole surface of the mirrour. That any object, and consequently that the eye when it is its own object, will appear infinitely magnified to an eye placed at the distinct picture has already been proved, in proposition 153, where this picture is produced by refraction. But the same reasoning is equally applicable, whether the picture is produced by refraction or by reflection.

259. *If an object is nearer to a concave mirrour than its principal focus, but does not touch the mirrour; then the image appears behind the mirrour, is farther distant from the mirrour and greater than the object, and is erect and distinct.*

If any object *abc*, Plat. XIV. fig. 2, is nearer to the concave mirrour *gil* than the mirrour's principal focus; the rays, which diverge from each point in the object before the mirrour, will diverge after reflection from as many points behind it, but less than they did at their incidence, by

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proposition 220. The imaginary radiants d, e, f , or the image, will be farther from the mirrour than the real radiants a, b, c ; by proposition 29; because the reflected rays diverge less than the incident ones. And the place of the last image def is determined, by proposition 221. But if the image is farther from the mirrour than the object is, the image will be bigger than the object, by proposition 256. Since the object, by the supposition, is nearer than the principal focus, it must be nearer likewise than the center; consequently the object abc and the image def are both on the same side of the center k ; and upon this account the image will be erect or in the same position with the object, by proposition 254. Lastly the image def will be seen distinctly, because the rays from it diverge but not very much, just as those do that come from objects, which are at a moderate distance and which the naked eye can see distinctly.

260. *If an object touches a concave mirrour the image will touch it likewise and will be equal to the object.*

When the real radiants, or object abc , Plat. XIV. fig. 2, come close to the surface of a concave mirrour, the imaginary radiants, or last image def ; will likewise be close to it, by proposition 225. And when they are both of them close to the surface of the mirrour, their diameters will be equal, by proposition 256.

261. *If an object is placed in the principal focus of a concave mirrour, the image is at an infinite distance behind the mirrour, is bigger than the object, and is erect and distinct.*

When the object abc , Plat. XIV. fig. 2, which consists of real radiants, is in the principal focus of the concave mirrour gil , the last image def , which consists of imaginary radiants, is at an infinite distance, by proposition 211. The image will be larger than the object upon account of its remoteness, by proposition 256. It will be seen erect, because it is on the same side of the center with the object, by proposition 254. And since the reflected rays of each beam are parallel to one another, by proposition 211, it will be seen as distinctly as the naked eye sees any remote objects.

262. *If an object is farther from a concave mirrour than its principal focus, and the eye is nearer than the place of the distinct picture; the image will be confused, will be behind the mirrour, will appear erect and will be larger than the object.*

Let abc , Plat. XIV. fig. 3, be an object, more remote from the concave mirrour gil than the mirrours principal focus; then the rays of each

each beam, that diverge from the several radiants a, b, c , &c. in the object, will be collected into as many focuses d, e, f , &c. by proposition 213, and if a paper is placed at def , the inverted image of the object will be painted there, in the same manner as these focuses, if the rays were brought together by refraction, would produce such a picture, by propositions 84, 85. Now if the eye is any where between c and i , that is, if it is nearer to the mirror than the distinct picture the rays of every beam are converging, when they come from the mirror to the eye, and consequently the image, which appears by reflection in the mirror, will be confused. For since no rays of the same beam converge naturally, but either are parallel or else diverge, nature has not given us a power of changing the figure of the eye so as to see distinctly by such rays. Now in this position of the object abc , the rays ag and ai , diverge from the same point a when they fall upon the mirror, and ag , because it is perpendicular to the surface will be reflected in the line ga , and ai will be reflected in the line id , so that both of them will converge to d . But if the eye is nearer to the mirror than d , the reflected ray id will not cross its cathetus of incidence gd in any place before the eye; for if these rays dg and di are continued ever so far before the eye, instead of crossing one another they would be farther asunder: consequently, by proposition 234, the apparent place of this or any other point of the image seen in the mirror will be indeterminate: and upon this account we can apply neither proposition 254 to find out whether the image will be erect or inverted, nor proposition 256 to find out whether it will be smaller or larger than the object. However we may shew that it will appear behind the mirror at an indefinite distance. For if the distance of the image was finite, the rays that come from it would diverge; but the rays that come from it converge; therefore the apparent distance of it is indefinitely great, and consequently it will seem more remote than the mirror or will appear to be behind the mirror. We may likewise shew that the image seen in the mirror will be erect. For fed is the inverted picture of the object, and, if it was received upon a paper, would be distinct; but if the eye is nearer than the place of this picture, then, instead of being painted upon the paper, it will be painted upon the retina, and though it would be distinct upon the paper it will be confused upon the retina, by proposition 113, because the vision in this case is confused, as has been already proved. But though the picture is confused upon the retina, it will still be inverted. Therefore, by proposition 115, the vision will be erect; or the image in the mirror, which is seen by means of this picture upon the retina, will appear in the same position

with the object. Lastly we may shew that the image will be larger than the object. For the object *abc* consists of radiant points: but the rays of each beam, as *gd*, *id*, or *lf if*, converge after reflection; and consequently the rays of any reflected beam will appear to come from a circular spot instead of a point. And since such circular spots are bigger than points, the object of reflected vision, or the image, which consists of such spots, will be bigger than the object itself, which is only made up of as many radiant points.

263. *When an object is placed in the principal focus of a concave mirror, the apparent magnitude of the image seen in the mirror is not altered by the departure of the eye. When the object is nearer than the principal focus, the apparent magnitude of the image decreases as the eye departs. When the object is more remote than the principal focus, then as the eye departs the apparent magnitude of the image encreases; provided it does not depart beyond the place of the distinct picture.*

Here we must recollect what has been proved in propositions 152, 153, in the case of refracted vision: for it is equally true in reflected vision. If the eye departs from the last image, its apparent magnitude will decrease. If the eye approaches to the distinct picture, the apparent magnitude of the last image will encrease. First then, if the object is in the principal focus of the mirror, the reflected rays of each beam are parallel to one another, by proposition 211. Therefore the distance of the imaginary radiants, or last image, behind the mirror, and of the focuses, or distinct picture, before it, will be infinite, by propositions 30, 31. For this reason as the eye departs from the mirror, it neither gets farther from the last image, nor nearer to the distinct picture: and consequently the apparent magnitude of the last image will neither decrease nor encrease. Secondly, if the object is nearer than the principal focus of the mirror, the reflected rays of each beam diverge from an imaginary radiant at a determinate distance behind the mirror, so that the place of the last image which consists of these radiants is determinate, by propositions 220, 221. But because the reflected rays diverge, there will be no focuses, or no distinct picture before the mirror. Consequently as the eye departs from the mirror, it gets farther from the last image but no nearer to the distinct picture: and upon this account the apparent magnitude of the last image will decrease in the eyes departure, without encreasing at all. Thirdly, if the object is more remote than the principal focus, the reflected rays of each beam converge to focuses at a determinate

nate distance, by propositions 213, 214. But the last image is at an indefinite distance, as was observed in proposition 262. Therefore as the eye departs from the mirrour, its gets nearer to the distinct picture but no farther from the infinitely distant last image: and upon this account in the departure of the eye the apparent magnitude of the last image will keep encreasing, without decreasing at all. This will be the case, till the eye gets to the place of the distinct picture, and then the last image seen in the mirrour will be infinite, as was observed in a similar case of refracted vision, proposition 153.

264. *When the eye is in the principal focus of a concave mirrour, the apparent magnitude of the last image is not altered by the departure of the object from the mirrour. When the eye is nearer than the principal focus, as the object departs, the apparent magnitude of the last image will decrease. When the eye is more remote than the principal focus, as the object departs, the apparent magnitude of the last image will encrease, provided it does not depart so far as to bring the distinct picture between the eye and the mirrour.*

This is proved in the same manner with the similar cases of refracted vision through a convex lens, proposition 156.

265. *If an object is farther from a concave mirrour than its principal focus, and the eye is farther from the mirrour than the place of the distinct picture, the image appears before the mirrour, is inverted, and may be seen distinctly.*

When the object *abc*, Plat. XIV. fig. 3, is at any distance *bi*, which is greater than the principal focal distance of the mirrour *gil*, the rays, that come from each point in the object *abc*, will converge to as many points *def*, by proposition 213, 214; and upon a white paper held there would paint the inverted picture of the object, by propositions 83, 84. Now if the paper was taken away, the rays of each beam, after they had met at the several focuses, *d*, *e*, *f*, would go strait forwards and diverge from those focuses, by proposition 15. Therefore to the eye placed any where beyond *def*, the distinct picture will be the last image, by proposition 233. But the distinct picture is before the mirrour, and is inverted: consequently the image, in this position of the eye and object, will likewise be before the mirrour and will be inverted. The rays of each beam, after they have passed their focus, diverge from their focus as from a radiant: consequently if the eye is not too near the image, the rays, that come from it, will diverge, when they fall upon the eye, just

as

as rays do that come from the several radiants in an object, when we view that object at a moderate distance with the naked eye: so that as objects appear distinct to the naked eye at a moderate distance, this image before a concave mirrour will likewise appear distinct, if the eye is not too near it.

266. *When the image appears before a concave mirrour; if the object is nearer to the mirrour than the center of its concavity, the image will be larger than the object; if the object is more remote than the center, the image will be smaller than the object; and if the object is in the center, the image will be equal to the object.*

What position of the object and eye is requisite in order to make the image appear before the mirrour, has already been shewn in proposition 265. Now if the object *abc*, Plat. XIV. fig. 3, is between *k* the center of the concavity and *gil* the surface of the mirrour, so that the distance *ib* is more than half *ik*, it will then be nearer than the center but more remote than the principal focus, by proposition 210. But when the object *abc* consisting of real radiants is on one side of the center, the distinct picture or last image *def* consisting of focuses will be on the contrary side of the center, by proposition 219. Therefore the image will be farther from the mirrour than the object is, and consequently will be bigger than the object, by proposition 256. If the object was at *fed* more remote from the mirrour than *k* the center of its concavity, then the distinct picture or image would be at *abc* on the contrary side of the center, by proposition 219. Therefore the image in this case will be less than the object, because it is nearer than the object to the surface of the mirrour, by proposition 256. Lastly if the object is in the center of the concavity at *k*, the image will be there too, by proposition 218. Consequently the object and image will be equal to each other, by proposition 256, since they are both of them at the same distance from the mirrours surface.

In the motion of the object it will be easy to see both the motion of the image, and the alterations that are made in the magnitude of the image whilst it moves. If a person standing at some distance and looking at a concave mirrour *gil* stretches out his arm so that his hand is at *e*, he may see the image of his arm come out from the mirrour towards the place where he stands, and the image of his hand at *b* will be less than his hand. Whilst he moves his hand from *e* towards the mirrour, the image of his hand will come forwards from *b* to meet it: and this image, as it comes forwards, will keep encreasing in magnitude. When his hand gets to *k*,
the

the image of it will meet it at k , and will then be equal to it. If he moves his hand still farther towards the mirrour, the image of it will then be nearer to him than his hand itself is; for the hand and image will always be on the contrary sides of the center k ; so that if his hand was at b the image of it would be at e , and would then be bigger than his hand.

267. *When the image appears before a concave mirrour; if the eye moves either to the right hand or to the left, the image will appear to move the contrary way; but if the mirrour moves either to the right hand or to the left, the image will appear to move the same way.*

This proposition in reflected vision from a concave mirrour is the same and is proved in the same manner with proposition 162 in refracted vision through a convex lens.

268. *The inverted pictures of objects may be represented in a dark room by means of a concave mirrour.*

If *def*, Plat. XIV. fig. 3, is an object abroad and the rays which come from the several points in that object pass through a hole in a window-shutter into a dark room, and then fall upon a concave mirrour *gil* within the room, they will be collected into as many points or focuses *abc*; and a paper placed at *abc* to receive these focuses will have the inverted picture of the object *def* represented upon it. This has frequently been shewn already in the foregoing propositions.

269. *The rays of the sun, when they are reflected from a concave mirrour, will kindle fire.*

Because the sun is very remote, the rays of each beam are parallel to one another; and consequently after reflection from a concave mirrour they will converge to the principal focus of the mirrour, in the same manner as they are collected by refraction in the principal focus of a convex lens. See proposition 53. And the heat produced by this means will be sufficient to set fire to such bodies as are placed before the mirrour at the principal focal distance, which is half a semidiameter of the mirrours concavity.

The different degrees of heat at the focus of different concave mirrours is estimated in the same manner as we have already estimated the different degrees of heat at the focus of different convex lenses, in propositions 107, 108, 109.

The ancients made use of concave mirrours to rekindle the Vestal fires. Plutarch in the life of Numa says that the instruments used for this purpose

pose were *σκαφεῖα*, or dishes, which were made hollow by the side of a right-angled triangle, and that this triangle, for so I would understand him, converges from a circumference to a center. The dish so formed is, he says, placed opposite to the sun, and the combustible matter is placed in the center. Let us try what we can make out from this perplexed description. The quadrant or mixed triangle ACB, Plat. XIV. fig. 8, agrees very well with the description of the triangle, by which these burning-dishes were made hollow. It is a right-angled triangle, for ACB is a right angle; it is not indeed a plane triangle or does not consist of three right lines, and Plutarch does not describe it as a plane triangle, but says that it converged from a circumference to a center; and thus from a circumference, or rather part of a circumference ADB, the two sides AC and BC do converge to one and the same point C, which is the center of the side ADB. Now if the quadrant ACB was made of steel, and the edge ADB was sharp, by turning this edge round in a thick plate of brass or other metal, the brass would be scraped away and the plate would be made hollow. The concavity so made in the plate would be a concave spherical mirror: and such a mirror placed opposite to the sun would collect the rays into a focus and would answer the purpose, for which Plutarch says those dishes were used. The only difficulty, which remains, is, that Plutarch tells us the combustible matter was placed in the center; whereas the principal focus of a concave mirror is not in the center, or at the distance of a semidiameter of the concavity, but at the distance of half a semidiameter. But we may well suppose that Plutarch was mistaken in this particular: he may say that the combustible matter was placed in the center of the concavity, not because it was placed there, but because he imagined this point to be the mirror's principal focus. For the author of a treatise of catoptrics, which goes under Euclid's name, has fallen into the same mistake, and supposes the center of a mirror's concavity to be the burning point. Euc. catopt. prop. XXXI. And though we may be very certain that the author of this treatise was not Euclid the geometrician, yet, whoever he was, he understood mathematics in all probability as well as Plutarch. So that if he could make this mistake, Plutarch might make it too.

270. *When any object is placed before a convex mirror, the image of that object appears behind the mirror, is nearer to the mirror than the object, is less than the object, is seen distinctly, and is erect.*

If the object *abc*, Plat. XIV. fig. 4, is placed before the convex mirror *gil*, the rays, that diverge before reflection from each radiant

a,

a, b, c , &c. in the object will diverge after reflection from as many radiants d, e, f , &c. behind the mirror, by propositions 228, 229, and these imaginary radiants compose the image that is seen in the mirror, by proposition 233. The reflected rays diverge more than the incident ones, by proposition 228, and consequently, by proposition 28, the imaginary radiants or image def will be nearer to the mirror than the real radiants or object abc . But if the image is nearer to the surface of the mirror it will likewise be less than the object, by proposition 256. And since the reflected rays diverge from the several points in the image def , just as they would from any visible object placed there, this image will be seen as distinctly, as any object could be seen at the same distance from the eye. Lastly the image will be erect, by proposition 254, because it is on the same side of the center k with the object. That the image will always be on the same side of the center with the object may be easily shewn. For suppose the several radiants or object abc was at an infinite distance, so that the rays of each beam were parallel, then the imaginary radiants or image def would be in the principal focus of the mirror, that is, it would be in the middle between k the center and i the surface, by proposition 28. Therefore in the greatest distance of the object, the image will be on the same side of the center k that the object is on. But as the real radiants or object approaches to the mirror, the imaginary radiants or image will likewise approach to it, by proposition 230. Consequently at any lesser distance of the object the image will be nearer to the surface at i and farther from the center k . Therefore at any lesser distance of the object, it will still be on the same side of the center with the object.

171. *The image in a convex mirror is equal to the object, if the object is close to the mirror.*

If the object abc , or all the real radiants which compose that object, moves towards the surface of the mirror gil , then the image def , or all the imaginary radiants which compose that image, will move towards the surface at the same time, by proposition 230: and when the object is close to the mirror the image will likewise be close to it, by proposition 231. But when the object and image are both of them close to the mirror their diameters will be equal, by proposition 256.

272. *When either the eye or the object departs from a convex mirror, the apparent diameter of the image decreases.*

When the object abc , Plat. XIV. fig. 4, is represented at def in the convex mirror gil , if the object continues fixed in its place, the image

def will always be at the same distance from the mirrour, by proposition 229, and the real diameter of the image will be invariable, by proposition 256, Consequently the apparent diameter of the image will be inversely as the eyes distance, by proposition 126, so that, as the distance of the eye from the mirrour encreases, the apparent diameter of the image will decrease.

If the object *abc* depart from the mirrour, that is, if *ib* the distance of the object encreases, the proportion, which *ib* bears to *ie* the distance of the image, will encrease. Therefore by proposition 256, the proportion, which the diameter of the object bears to the diameter of the image, will likewise encrease, or the image will be less in respect of the object. But when the eye continues in the same place, the apparent diameter of the image will be directly as its real diameter, by proposition 127. Consequently since in the departure of the object the real diameter of the image decreases, its apparent diameter will likewise decrease.

C H A P. XIX.

Of reflecting telescopes and some other optical instruments.

273. *The portable camera obscura consists of a convex lens and a plane mirrour.*

ALL the optical part of a portable camera obscura is represented, Plat. XIV. fig. 9. it consists of CD a convex lens, and HK a plane mirrour inclined at half a right angle. The lens CD is fitted into a tube, that consists of two joynts, one of which slides into the other like the tube of a common perspective glass. One joynt of this tube is fixed in a hole at one end of a box, so that as the other joynt into which the lens is fitted slides in it, the lens may be moved nearer to or farther from EG the opposite end of the box. If we consider this box as a dark room, a distinct inverted picture of any object AB may, by means of the lens CD, be painted as EG, if the lens is drawn out to a proper distance, so as to make the focuses of each pencil of rays proceeding from the object fall at EG, by propositions 84, 85. But if the rays of each pencil, before they come to their several focuses at EG fall upon the plane mirrour KH, which makes an angle of 45 degrees with the bottom of the box and consequently with EG the end of the box; these rays will be reflected. The rays, which were converging to the focus E behind the mirrour, will converge after reflection to the focus M at an equal distance before it; those, which were converging to the focus G behind the mirrour,

roure, will after reflection converge to the focus N before it, by proposition 206. The same is true of every other pencil of rays: the reflection turns the rays out of the way, before they arrive at the imaginary focuses behind the mirroure; and makes them converge to as many real focuses at the same distances before it. Therefore whatever is the distance of all the imaginary focuses from E to G behind the mirroure, the same will be the distances of all the real focuses from M to N before it: and consequently since EG, by the construction, makes an angle of 45 degrees with the mirroure on one side, the picture MN, when reflected, will make with it an angle of 45 degrees on the other side. Therefore EG and MN will make an angle of 90 or twice 45 degrees with one another; so that as the picture would be at EG the end of the box, if the mirroure was out of the way, the reflection of the mirroure will throw it upon the top of the box. Now if the box, instead of being covered with a lid, is covered with a plate of plane glass that is left unpolished on one side, or with a sheet of oyled paper, the picture NM will be seen through the glass or the paper; and will be so bright and distinct that it may be traced out with a black-lead pencil upon the glass or the paper, if care is taken to screen it from the open day-light.

274. *An opera-glass consists of a common perspective and a plane mirroure.*

This instrument, which is represented, Plat. XIV. fig. 10, is so contrived, that you may look at a person in a public place, and no one can tell who it is you look at. One part of this instrument is *ibkl* a common perspective or short Galilean telescope consisting of a convex object-glass *ibb*, and a concave eye-glass *kal*, such as is described in proposition 183. The other part is a plane mirroure *fb*, which is fixed in a tube *fgib*, that screws on to the perspective at *ib*. This mirroure has its reflecting surface towards the object-glass *ibb*, and is placed so as to make half a right-angle with *abc* the axis of the perspective. In the tube *fgbi* there are three holes, one on the side at *o*, another in the opposite side at *t*, and a third in the end at *x*. The hole at *o* is principally to be considered: the use of the other two shall be mentioned presently. If there is any object PQR, which is parallel to *abc* the axis of the perspective, and rays come from it through the hole *o*, this object may be seen by reflection through the perspective. For since the mirroure *fb* makes an angle of 45 degrees with the axis of the perspective, and the object PQR is parallel to the same axis; by the plane of the mirroure continued will likewise make an angle of 45 degrees with the object. Therefore when the rays *Pd*, *Qc*, *Re* are reflected, they will appear to have come from an image *pqr*, which

will make an angle of 45 degrees with the mirrour behind it, as was shewn in a like case in proposition 245. Now since the object PQR makes an angle of 45 degrees with the mirrour and is before it, and since the image *pqr*, which is seen by reflection, makes an angle of 45 degrees with the mirrour and is behind it, the object and image make an angle of twice 45 or 90 degrees with one another. Therefore as the object PQR is parallel to *abc* the axis of the perspective, the image *pqr* will be at right angles with this axis, and consequently will be placed directly before the perspective so as to be seen through it. This makes it difficult to know what person we are looking at, because in common perspectives, the persons we are viewing must be directly before the perspective, and in this reflecting perspective, they must be on one side. The holes *x* and *t* will serve to make this still more difficult, and are likewise designed to answer another purpose. If an object is at S so near to us, that we could not see it distinctly through the perspective *ihkl*, then instead of turning the hole *o* towards it, we turn the hole *t* towards it and the mirrour *fb* has another reflecting surface on the side *v*, so that if we look through the hole *x*, we may see the object S by reflection in the mirrour, and the rays have no perspective to pass through before they come to the eye. The hole *x* has commonly a plane glass fixed in it, to make the whole instrument as like a common perspective as may be; and because the hole *o* is opposite to the hole *t*, and we may see an object through either hole according as we look through either the end *kal* or the end *x*; no one can easily tell on which hand of us the person stands, that we design to look at.

275. *The pictures, that are produced by a magic lantern, are to be explained upon the same principles with those, that are produced in a dark room.*

ABCD, Plat. XIV. fig. 11, is a tin lantern with a tube *nklm* fixed in the side of it. This tube consists of two joynts, one of which slips into the other: and by drawing this joynt out or pushing it in, the tube may be made longer or shorter. At *kl* in the end of the moveable joynt of the tube a convex lens is fixed, and an object painted with transparent colours upon a piece of thin glass is placed at *de* somewhere in the immoveable joynt of the tube; so that as the tube is lengthened or shortened, the lens will be either at a grearer or a less distance from this transparent object. In the side of the lantern there is a very convex lens *bhc*, which serves to cast a very strong light from the candle within the lantern upon the object *de*. Now when the rays, which shine through the ob-

object *de*, diverge from the several points as *d*, *e*, &c. in the object, and fall upon the lens *kl*, they will be made to converge to as many points *f*, *g*, &c. on the other side of the lens, and will paint an inverted picture of the object at *fg* upon a white wall, a sheet, or a screen of white paper, provided the object is farther from the lens than its principal focus, by propositions 83, 84, 85. To make this picture appear distinct and bright; it must have no other light fall upon it, but what comes through the lens *kl*, and for this reason the whole apparatus is to be placed in a dark room EFGH. The lens *kl* must be very convex so that the object *de* may be very near to it and yet not be nearer than its principal focus: for by this means, as the object is near to the lens, the picture *fg* will be at a great distance from it, by proposition 87, and consequently the picture will be much bigger than the object, by proposition 88. Since the picture is inverted in respect of the object, in order to make the picture appear with the right end upwards, it is necessary that the object *de* should be placed with the wrong end upwards.

From what has been said about the magic lantern, we may be able to understand the use of the solar microscope. If instead of the transparent picture *de*, any other transparent object was placed there, the distinct picture of such an object would be represented upon the white screen. Now small animals, and thin small parts of most substances, when they are strongly enlightened, are in some degree transparent. Therefore if any small animal is placed at *de* and a strong light shines upon it from the side where the lantern is placed in the figure, the rays that pass through it will paint the picture of this animal at *fg*. The picture will be larger than the animal, as has been shewn already, and by changing the lens *kl* for a more convex one the picture *fg* may be made as much larger than the object *de* as we please. For if the lens is more convex than *kl*, the object *de* must be brought nearer to it in order to keep the distinct picture upon the screen, by proposition 97. Now the diameter of the object *de* is to the diameter of the distinct picture *fg*, as the distance of the object from the lens to the distance of the distinct picture from the lens, by proposition 88. Therefore the less the distance of the object is in proportion to the distance of the picture, the less will the diameter of the object be in proportion to the diameter of the picture. Now as the object is nearer to the lens, its distance from the lens bears a less proportion to the distance of the picture, since the distance of the picture is a given quantity, because it is always represented upon the screen standing in the same place. Therefore the nearer the object is to the lens, the less proportion the diameter of the object will bear to the diameter of the picture. But the diameter of the object is a given
quan-

quantity; and consequently the less proportion its given diameter bears to the diameter of the picture, the greater must be the diameter of the picture. Therefore the nearer the object is to the lens, so much greater is the diameter of the picture, or the diameter of the picture, when its distance from the lens is given, is inversely as the distance of the object from the lens. Thus we have shewn how a transparent object at *de* may be magnified in any proportion, that we please, by using a lens of sufficient convexity, provided there is a very strong light shining through the object. The object-glasses of common microscopes will serve for this purpose. In order to procure a strong light the end *nm* of the tube *knlm* or of some other tube of the same sort with a microscopic lens at *kl*, instead of being fixed in a lantern, must be placed in the hole of a window-shutter of a dark room, so that the lens *kl* may be within the room; and then if the sun shines directly through the hole, any small object placed at *de* will be sufficiently enlightened, and a distinct picture of that object of any size that we chuse, may be represented upon a screen of white paper placed within the room at a proper distance. Though the sun is not directly before the window or is so high as not to shine directly through a hole in the shutter, yet a plane mirror may be fixed in such a position on the outside of the shutter as to throw the reflected rays directly through the hole upon the object, and these rays will enlighten it sufficiently.

276. *Reflecting telescopes of Sir Isaac Newtons invention consist of one concave mirror, one plane mirror and a convex lens.*

GH is the concave mirror, Plat. XV. fig. 1. KK is the plane mirror. LL is the convex lens. All these parts are fixed in their proper places in a tube ABCD. What their proper places are we will next consider. The concave mirror GH is placed towards the end BC of the tube ABCD, and is perpendicular to DC the side of the tube. The open end AC of the tube is turned towards the object PR. If the object PR is at so great a distance that any rays as E, *e*, that come from one and the same point in it, may be considered as parallel to one another, and that the rays F, *f*, that come from any other point are likewise parallel amongst themselves, by proposition 30; then these or any other beams, when they fall upon the concave mirror at *g*, *h*, and are reflected from it, will converge to *mn* at the principal focus of the mirror, and would represent an inverted picture of the object upon a paper held at that place, by proposition 268. But if the converging rays before they come to the place of the distinct picture, fall upon the plane mirror KK, which is inclined so as to make an angle of 45 degrees with DC the side of the tube,

PLATE XIV.

Fig. 2.

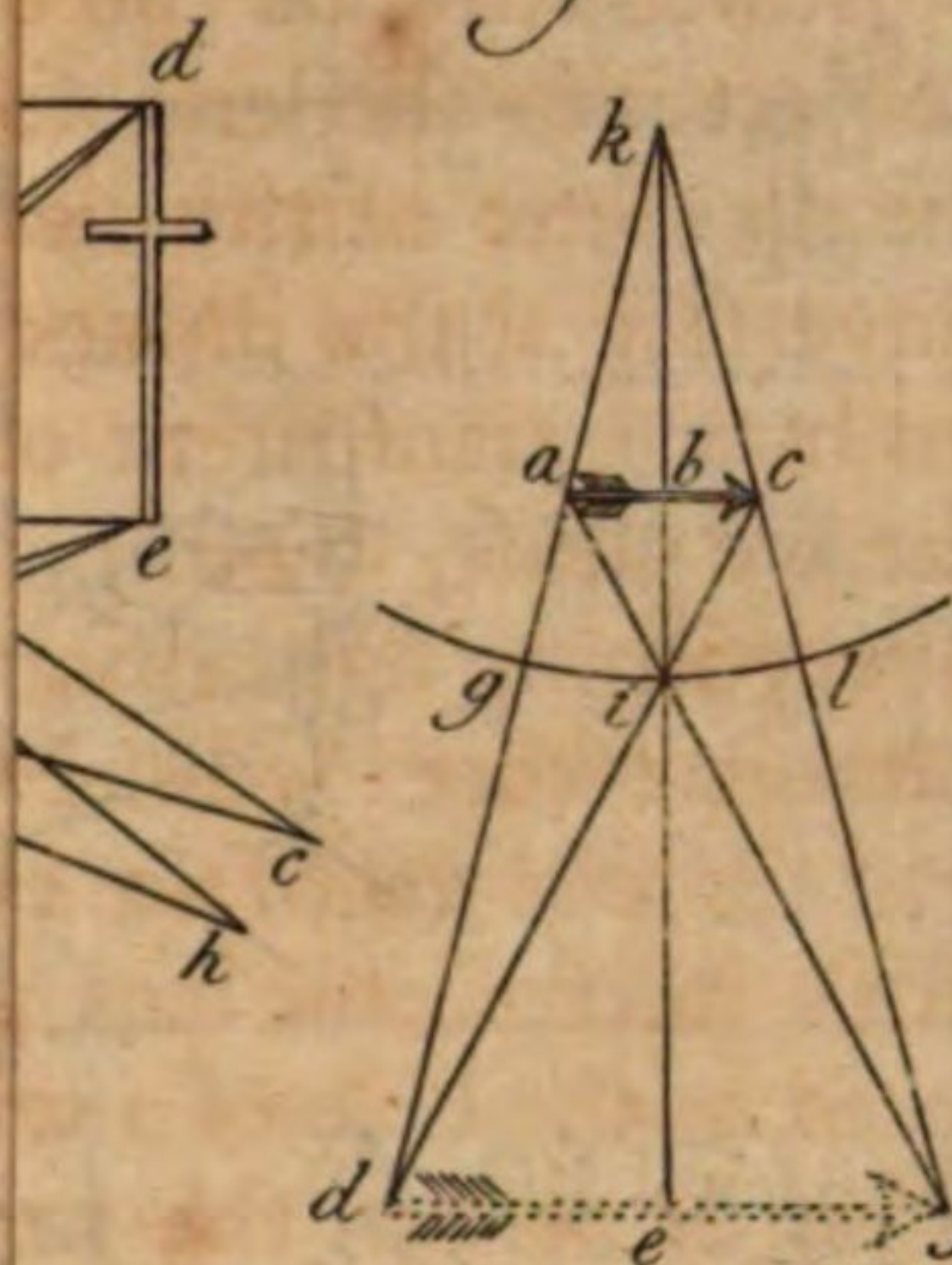


Fig. 3.

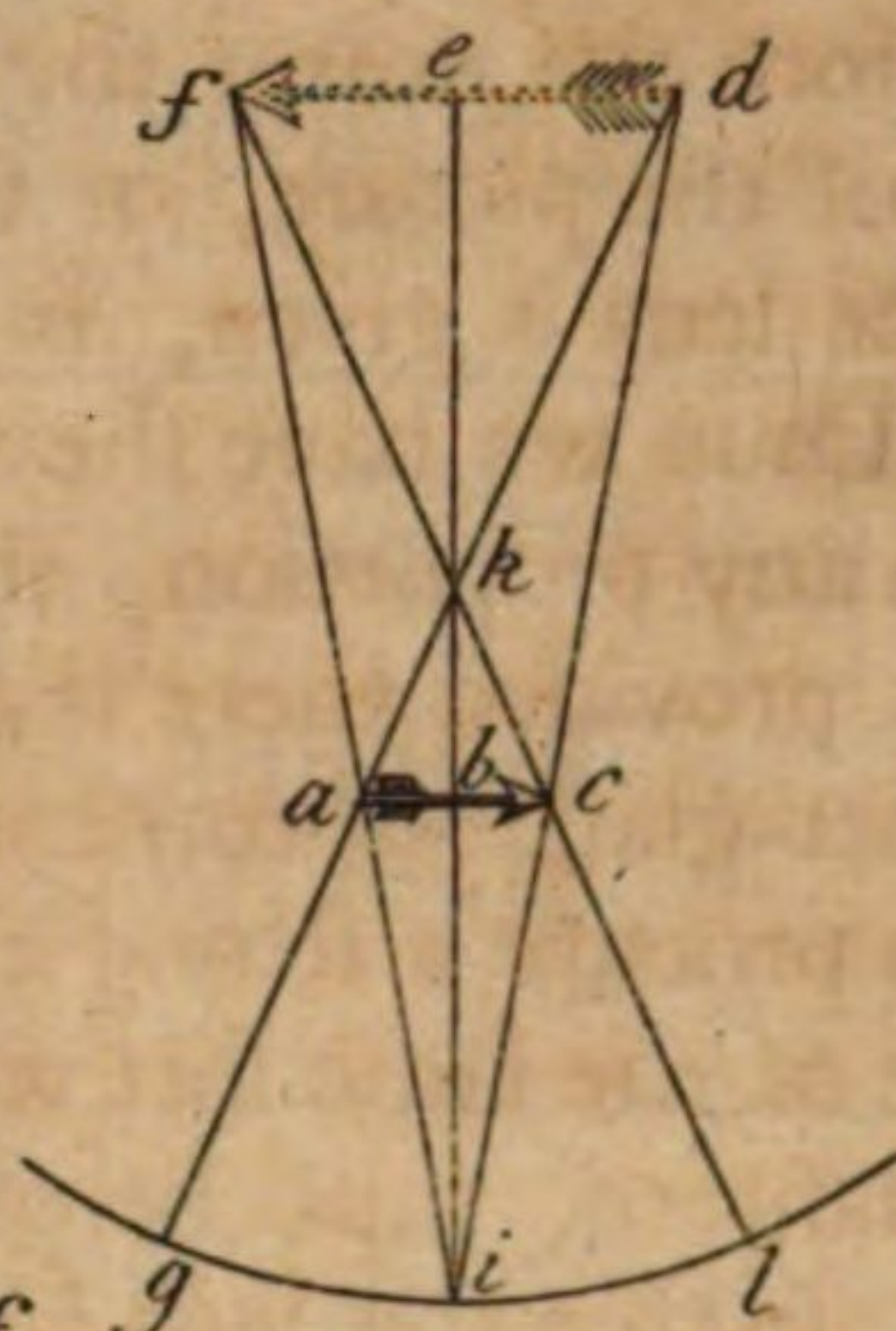


Fig. 4.



Fig. 7.

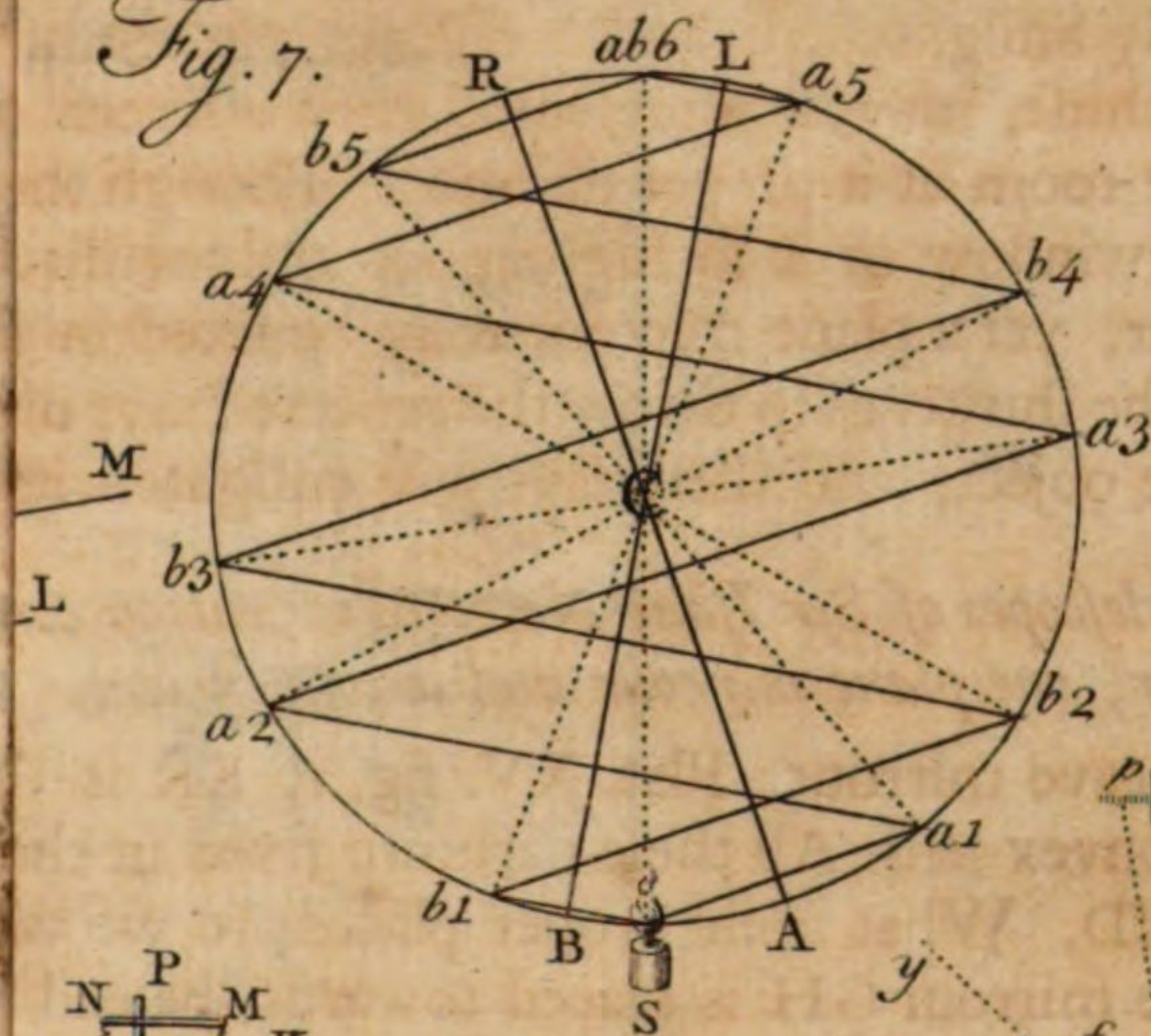


Fig. 8.

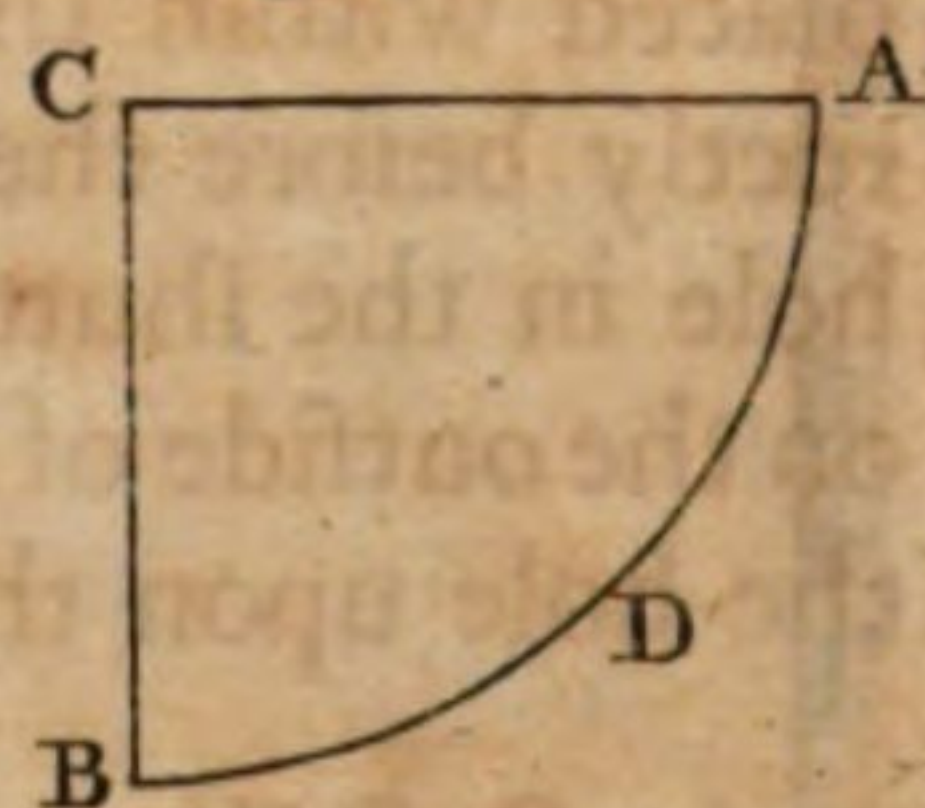


Fig. 10.

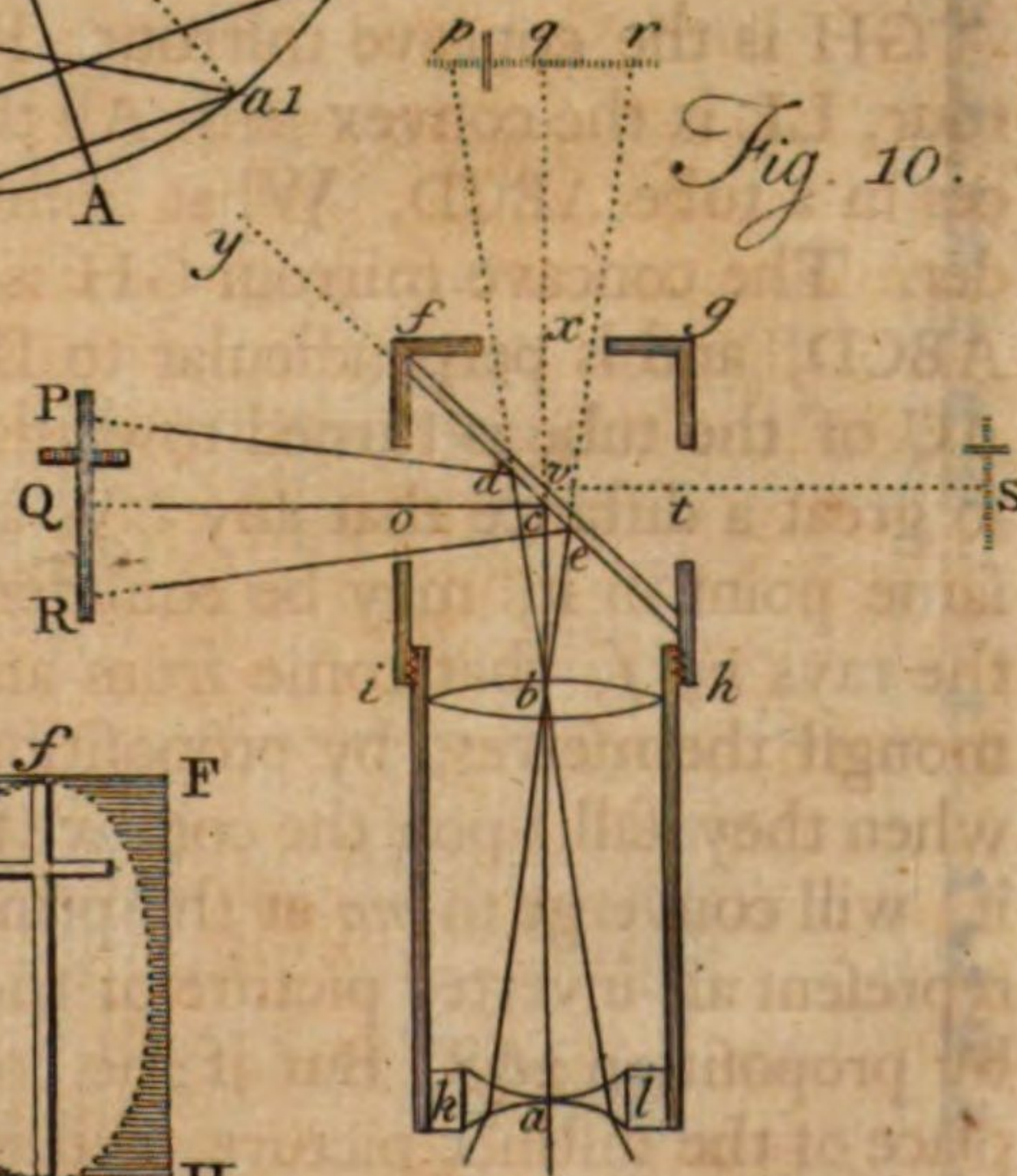
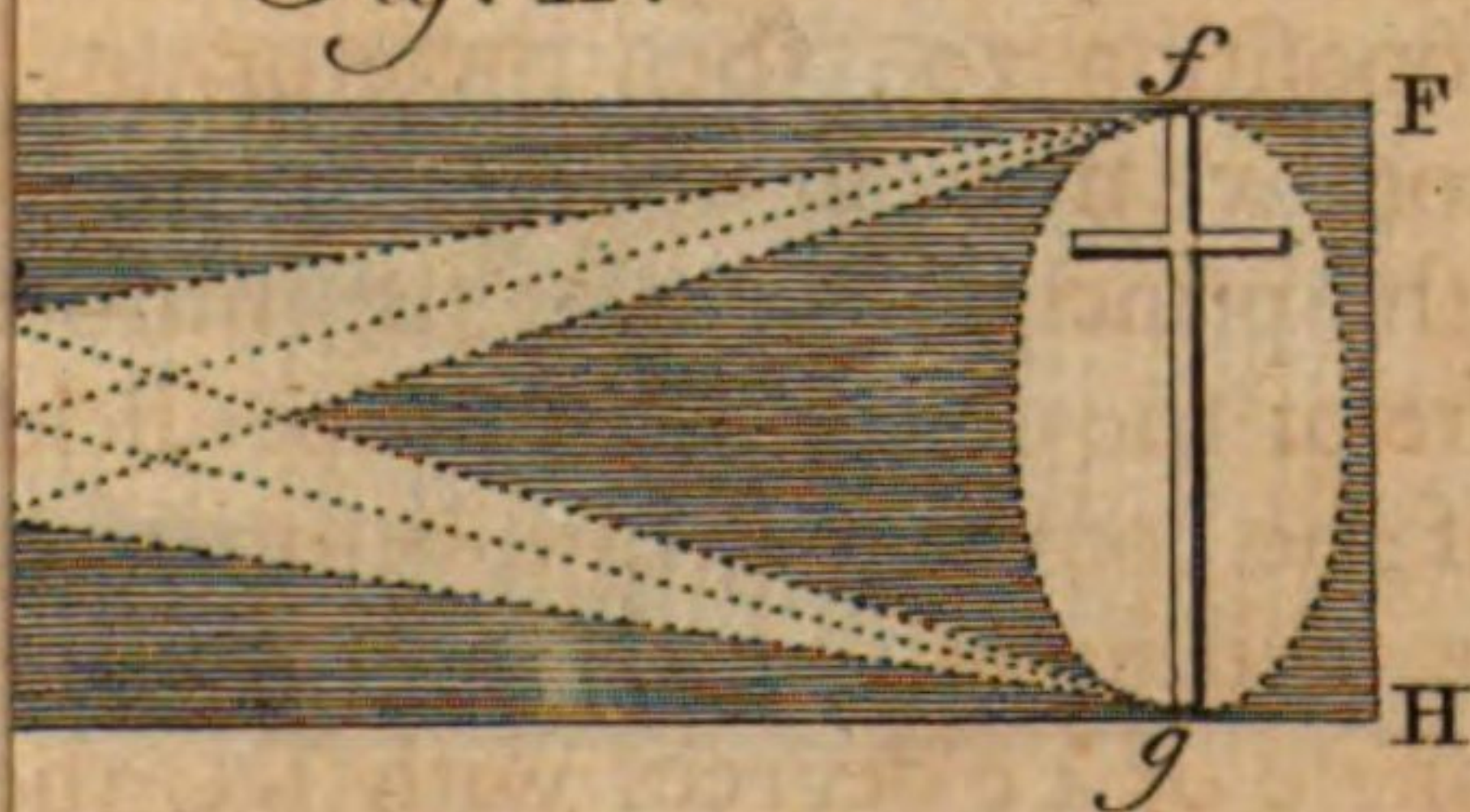


Fig. 11.



tube, or rather with the axis of the telescope; the rays will be reflected from thence, and instead of meeting in their respective focuses at mn , they will meet before the mirror at qs . Now since the picture mn would be perpendicular to DC , because the concave mirror GH is so, and consequently, since this picture would be behind the plane mirror KK and by the position of the mirror would make with it an angle of 45 degrees: it follows that when the rays, instead of going on so as to form the picture mn , are reflected and form the picture qs , this picture qs will be before the mirror and will make with it an angle of 45 degrees. Therefore mn and qs make with each other an angle of twice 45 or of 90 degrees, and consequently, since mn is perpendicular to the side of the tube or to the axis of the telescope, the reflected picture qs will be parallel to it. All this will be obvious from a similar case already explained in proposition 273. Now if this picture qs is in the principal focus of the convex lens LL ; this lens will be placed in respect of the picture just as the eye-glass is in an astronomical telescope, by proposition 172, and the eye O will see the picture distinctly through it.

The magnifying power of such a reflecting telescope as this, is determined in the same manner with the magnifying power of an astronomical telescope, in proposition 173. For the apparent diameter of the object seen with the telescope is to its apparent diameter seen without it, as the distance of the distinct picture from the object-mirror to the distance of it from the eye-glass. This can want but little proof, since we have already shewn in proposition 257, that the picture mn seen by the eye contiguous to the concave mirror GH would appear of the same size with the object. And by what has been said in proposition 173 it follows, that the picture $qs=mn$ when seen from the station of the object-glass, either with the glass or without it, will appear as much bigger than it does, when seen from the concave mirror, that is, as much bigger than the object appears to the naked eye, as the distance of the picture from the eye-glass is less than its distance from the mirror, or as its distance from the mirror is greater than its distance from the lens.

277. *Gregory's reflecting telescope, as it is now made, consists of two concave mirrors, a plano-convex lens, and a meniscus.*

In the tube $TYTY$, Plat. XV. fig. 2, a concave mirror LD is placed. Through TT the open end of the tube, rays, as IL , il , and CD , cd , are admitted from some distant object AB . The rays that are of the same beam, or that come from one and the same point in the object, are parallel, upon account of the remoteness of the object, by proposition 30. There-

Therefore KH the distinct inverted picture of the object will be formed in the principal focus of the mirrour LL, by propositions 210, 268. If f is the principal focus of the smaller concave mirrour EF, then KH is more remote from this mirrour than its principal focus, and the rays, which diverge from KH and fall at G and R upon the mirrour EF, will converge after refraction, by proposition 213; and consequently, if they pass on through a hole X in the first mirrour LD, they will form a second picture, which will be inverted in respect of the first, or which will be in the same position with the object AB. This picture would be somewhere about SS, if the rays that form it were made to converge only by being reflected from the mirrour EF: but before they come thither, whilst they are converging, they pass through the plano-convex lens MN, which is placed in a smaller tube joyned to the larger TYTY: the refraction of this lens makes the rays converge more, by proposition 74, and consequently they will meet in their respective focuses or will form the second picture at PV, before they come to SS. This erect picture is in the principal focus of the meniscal eye-glass SS, the convexity of which is greater than its concavity, so that its effect is the same with that of a convex eye-glass, by proposition 80, and consequently to an eye at O the erect picture PV will appear as a picture of the same sort does, in the common perspective of four glasses described in proposition 181.

CHAP. XX.

Of the different refrangibility of light.

WE have hitherto supposed that a particle of light, as it comes from the sun, is the least particle, into which light can be separated. But we must now correct this supposition by shewing that a particle of light, as it comes from the sun, is properly speaking a bundle of rays, which may be separated from one another. Therefore for the future by a ray of light we must be understood to mean not that collection of particles, which we have hitherto called by this name, but the least particles, into which light can be separated.

278. *Rays of light are said to be differently refrangible, when, at the same or equal angles of incidence, some are more turned out of the way than others.*

279. *Rays are said to be differently reflexible, if some are more easily reflected than others.*

280. *Light is called homogeneous, when all the rays are equally refrangible: it is called heterogeneous, when some rays are more refrangible than others.*
281. *The colours of homogeneous light are called primary or simple colours; and those of heterogeneous light are called secondary or mixed.*
282. *The rays of the sun are not all equally refrangible: and those rays, which have a different degree of refrangibility, have likewise a different colour.*

If a beam of light SF, Plat. XV. fig. 3, that comes from the sun, passes into a dark room through F a round hole in a window-shutter EG; this beam proceeding strait forwards and falling upon a paper at Y would there make a round picture of the sun, by proposition 105. This picture will be a confused one indeed, if the hole is a large one and there is no lens in the hole. However as this round spot of light is a picture of the sun, we shall hereafter, notwithstanding its confusion, call it by this name. Now if a glass prism ABC is placed between the hole in the window-shutter and the paper at Y, the rays of this beam by the refraction, which they suffer in the prism, will be bent from their strait course, and instead of going on so as to fall upon the paper at Y they will be turned upwards, and the picture of the sun produced by them will fall upon a paper MN that is placed above Y. If all the rays were equally bent upwards the picture would be a round one upon the paper MN, after the rays have been refracted, as well as when they passed strait forwards and fell upon a paper at Y. But this refracted picture PT is found to be oblong. The horizontal diameter or breadth of this oblong picture is equal to the diameter of the circular one Y: but the perpendicular diameter or height of the picture PT is much greater than its breadth. The refraction is made upwards and not in a horizontal direction: therefore no alteration ought to be made in the breadth or horizontal diameter of the picture; because no refraction, that is not in the direction of that diameter, can make the picture either broader or narrower. And if all the rays were equally refracted upwards, such a refraction would not change the length of the picture, as it is round, when it falls at Y, so it would be round, when it is refracted upwards by the prism and falls at PT. This oblong picture consists therefore of rays, which are differently refrangible: they all fall at equal angles of obliquity upon BC the first side of the prism, but in the refraction, some are more turned out of the way than others; those rays, which go to P the upper part of

M m m

the

the picture are the most refrangible, and those, which go to T the lower part of it, are least refrangible; the rest, which fall between P and T have intermediate degrees of refrangibility.

This oblong picture is of different colours in different parts of it. The most refrangible rays at P are violet, the least refrangible at T are red; the rays of intermediate refrangibility from the violet downwards to the red are indigo-coloured, blue, green, yellow, and orange. So that the whole picture is made up of rays of these seven different colours. We may from hence see the reason why the coloured picture consisting of differently refrangible rays should be oblong in such a manner that the two sides of it are right lines and the two ends semicircles. For it consists, as in fig. 4, of seven circles, the highest of which PAGQ is violet, the lowest STN is red, the five intermediate ones, BH, CI, DK, EL, OM, are indigo-coloured, blue, green, yellow, and orange. The white round picture Y, fig. 3, is formed by heterogeneous rays, that are of seven different sorts distinguished from one another by their different degrees of refrangibility and different colours. The refraction of the prism separates these rays from one another by refracting some of them more and others less. And consequently the refracted picture will consist of seven round pictures one below another. These round pictures are so near to each other that the highest of them APGR will mix itself with some of those below it, as with BH and CQI. This nearness of these several round pictures to each other will prevent their colours from being distinctly seen, it will likewise make the sides AS, GN, which are composed of small arcs of circles very close to one another, appear like right lines; but the two ends P and T will be semicircles.

If the centers of these circles continue at the same distance from one another and the circles themselves are made less, as *apg*, *bb*, *ci*, *dk*, *el*, *om*, *stn*, they will then be distinct or will not mix with each other; and as the colours of the several parts will by this means be kept separate, so the refracted rays instead of forming one continued oblong picture will form seven small circular ones placed in a line perpendicular to the horizon. This separation of the several parts in the refracted picture from each other is brought about, as in fig. 5, by making the hole F in the window-shutter very small, and by collecting the rays that come through it with a convex lens MN. For this will make a very small white picture of the sun at L if there is no prism *abc*: but the refraction of this prism, if it is placed a little beyond the lens, will separate the heterogeneous rays by refracting them upwards, and instead of one small round and white picture at L, there will be seven small round pictures at *pt*, of which

which *r* will be violet, *s* indigo, *u* blue, *x* green, *l* yellow, *y* orange, and *z* red.

That the prism ABC, fig. 3, does not make the rays diverge, so as to spread over the space PT, upon any other account but their different refrangibility, will be evident, if, as in fig. 6, a second prism DH is placed beyond the first *abc*. For if the rays, that come from S and pass through the hole F of the window-shutter EG, were by the first prism *abc* made to diverge and form the oblong picture PT upon any other account besides their different refrangibility; then supposing a second prism DH to be placed at right angles to the former the effect must be this; the first prism *abc* makes the rays diverge from one another in a line PT perpendicular to the horizon, and consequently the second prism DH must make them diverge from one another in a line parallel to the horizon, so that the second prism would encrease the breadth of the picture, as much as the first encreased its length, and as one prism alone makes the picture a long one both of them together would make it square as *prst*. But this second refraction does not alter the figure of the picture, but only the position of it: the second prism refracts the picture sideways; and those rays, which fell the highest at P after the first refraction, are refracted sideways the most by the second prism, those which fell the lowest at T, are refracted sideways the least, by which means the picture, though it continues oblong, will not be perpendicular to the horizon as PT was, but will be inclined so as to lie in the position *pt*. This makes it evident that the spreading of the rays by the first refraction was owing to their different refrangibility and to no other cause. It must be owing to their different refrangibility, because those, which were most refracted upwards by the first prism, are most refracted sideways by the second. It cannot be owing to any other cause, because if it was, the second prism would spread the rays in breadth as much as the first prism spreads them in length, and both prisms would make the picture square.

283. *Those rays of light, which are most refrangible, are likewise most reflexible.*

When a beam of light is admitted into a dark chamber through the hole F in the window-shutter EG, Plat. XV. fig. 7, and this beam falls upon a prism ACB, the sides of which AC and AB are equal, and the angle at A a right one; when the obliquity of these rays, as they are to pass out of the prism at its base BC, is less than 40 degrees, the greatest part of the beam will pass out, but some few rays will be reflected at the surface BC, by proposition 200. The rays, which pass through the base, form

an oblong coloured picture HK, by proposition 282, where MH is a more refrangible ray, and MK a less refrangible one. If the few rays of the beam, which are reflected from M in the direction MN, are made to pass through another prism XYV, they will likewise form an oblong coloured picture pt , where p is the most refrangible and t the least refrangible ray. This picture will be a very faint one, because there are but few rays reflected from M.

Now if the prism ACB is turned slowly round upon its axis in the direction ACB, the obliquity of the rays EM to the base BC will keep encreasing, till at last this obliquity may become so great, that no rays will pass out at M, but all of them will be reflected, by proposition 200. When this total reflection is made, the oblong picture pt , which was faint before, will become much brighter, because then not only a few rays but all the beam will be reflected thither. This total reflection will not be made all at once, but as the prism is turned slowly round upon its axis, the most refrangible rays MH will be first reflected, for the violet colour will disappear in the oblong picture HK, whilst all the other colours continue as bright as they were before; and when this colour disappears at HK, the same colour at p will become bright, and all the other colours at pt will continue as faint as they were before. When the prism is turned a little farther upon its axis, the indigo colour, which consists of rays that have the next greatest degree of refrangibility, will be reflected, so that this colour will disappear at HK and will become bright at pt . The same thing will happen to all the rays in their order; as the prism is turned round, each different sort of rays will be reflected sooner, as the rays have a greater degree of refrangibility or later as they have a less degree. The red rays at K, which are the least refrangible of all will be reflected last of all. From hence therefore it appears, that the rays of the sun are differently reflexible, and that those, which are most refrangible are likewise most reflexible.

284. *Rays of light, which differ in colour, differ likewise in the degree of their refrangibility.*

We have already seen, in proposition 282, that rays which are differently refrangible, are of different colours, the most refrangible rays of the suns light for instance, are violet, the least refrangible are red. We are now to prove the converse of this, that all violet rays whatever are more refrangible than all red ones. Let a stiff chart or piece of pasteboard DE, Plat. XV. fig. 8, be painted, half of it DG of a light purple colour, and the other half FE of a strong red. Now if this pasteboard is viewed
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through a prism BAC, *bac*, with its refracting angle at A upwards, the apparent place of the pasteboard seen through the prism will be raised above its real place, as is evident from the demonstration of proposition 137, upon Plat. XII. fig. 4. And so likewise, if the refracting angle A of the prism was downwards, the apparent place of the pasteboard seen through the prism would be below its real place. First therefore suppose the angle A to be upwards and the painted pasteboard, which is at DE, will appear at *de*, when it is seen through the prism. In this case the pasteboard seems to be broken in the middle, and the purple part *dg* appears something higher than the red part *fe*. But since the apparent place of the whole paper is raised by the refraction of the rays, as they pass through the prism, and the purple part is more raised than the red part, it follows that the purple rays are more refracted than the red ones. Secondly, suppose the angle A to be downwards, and then the painted pasteboard DE seen through the prism will appear at *δe*. And in this case again the pasteboard will seem to be broken in the middle, and the purple part *δγ* will appear something lower than the red part *φe*. But since the apparent place of the whole pasteboard is depressed by the refraction of the rays, as they pass through the prism, and the purple part is more depressed than the red part, it follows that the purple rays, as before, are more refracted than the red ones.

285. *Homogeneous light is refracted regularly without any dilatation or scattering of the rays.*

When the rays of any one particular colour in the oblong picture of the sun, as the green rays, for instance, are separated from one another in the manner already described in proposition 282; if some of these green rays which are homogeneous or are all equally refrangible, are transmitted through a very small round hole in a stiff pasteboard, and are refracted by a prism on the other side of the hole, the picture formed by these green rays after refraction upon a white paper held beyond the prism will not be oblong but circular, as the hole is through which they passed. Therefore this homogeneous light is not dilated nor are the rays of it scattered by this refraction.

286. *The confused appearance of objects, when they are seen through refracting bodies, is owing to the different refrangibility of light.*

If flies, or the letters of a small print, or any other minute objects are placed in heterogeneous light such as a direct beam of the suns, which has never been separated by any refraction into its homogeneous parts; these
objects

objects being viewed through a glass-prism will be seen confusedly, their edges will appear so misty that the smaller parts of minute animals cannot easily be distinguished from one another, and the letters of the small print cannot be read. But if the same objects are placed in a beam of homogeneous light, which is separated from all other rays of a different refrangibility in the manner already described in proposition 282, they will appear as distinct through a prism, as if they were viewed with the naked eye. Therefore we may conclude that this confusion is owing to the different refrangibility of those rays, which come from the objects: since objects never appear confused, when they are seen through refracting bodies, unless they are enlightened with several sorts of rays, which have different degrees of refrangibility.

287. *Where the mediums are the same, in rays of one and the same sort the sine of incidence will always bear the same proportion to the sine of the refracted angle, when the ray passes out of one of these given mediums into the other, whatever may be the obliquity of the incident ray.*

The violet rays, for instance, are more refrangible than the red ones; and consequently, when both the violet rays and the red rays are passing out of glass into air, if they fall at equal obliquities upon the surface of the glass, the violet rays will be more refracted from a perpendicular than the red ones, or the refracted angle in the violet rays will be bigger than in the red ones, by propositions 282, 284, 38. But if the proportion is once found, which the sine of incidence in each particular sort of rays bears to the sine of the refracted angle at any one obliquity as the rays are to pass out of glass into air; the same will always be the proportion of these sines to one another whatever is the obliquity or angle of incidence, when the rays are passing out of the same medium of glass into the same medium of air. Thus particularly, if at any small obliquity the sine of incidence is to the refracted sine in the violet rays as 50 to 77; the sine of incidence in the same sort of rays will be to the refracted sine as 50 to 77 at all greater obliquities. If the sine of incidence is to the refracted sine in the red rays as 50 to 78 at any one obliquity, this proportion will continue the same at all other obliquities whatever.

In proposition 40, it has been shewn, that when a ray of the sun is passing out of any given medium, as glass, into any other given medium, as air, the proportion of the sine of incidence to the refracted sine will not be changed by changing the obliquity of the incident ray. But then this was shewn upon a supposition that all rays as they come from the
sun

sun are equally refrangible: and consequently that proposition is not exactly true, since we have now shewn, in proposition 282, that rays of light, as they come from the sun, are some more and some less refrangible. But still proposition 40 will be true and the demonstration of it applicable to any one sort of rays, to the violet ones, or to the red ones in particular, which are equally refrangible; for in each particular sort of rays the supposition holds good, upon which that proposition was demonstrated. Therefore in each particular sort of rays the sine of incidence always bears the same proportion to the refracted sine at all obliquities whatever, when the refracting mediums are given.

288. *It is probable that any single ray of the least refrangible sort contains a greater quantity of matter than any single ray of the most refrangible sort.*

We have already seen, in propositions 282, 283, that at the same angles of incidence violet rays will be more refracted or more turned out of the way than red rays. And we have likewise seen, in propositions 37, 38, that rays are refracted when they pass out of one medium into another by being either more or less attracted in one medium than they are in the other. Now since, when all other circumstances are equal, when red rays and violet rays fall at equal obliquities and are to pass out of glass into air, so that the mediums and consequently the attractive force or cause of refraction is given, if the same cause can turn the violet rays more out of the way, or refract them more, than it does the red rays, these rays must have different moments; the most refrangible rays, or those which are most easily turned out of the way, have the least moment; and the least refrangible rays, or those which are most difficult to turn out of the way, have the greatest moment. But if all sorts of rays have the same velocity, as is probable from proposition 33, their respective quantities of matter will be as their moments, by proposition 13 of mechanics. And consequently any single ray of the most refrangible sort contains a less quantity of matter, than any single ray of the least refrangible sort.

It may be upon this account that a red colour or a pale purple is less pleasant to the eye than a blue, a green, or a yellow. The red rays strike the eye with so great a force as to be offensive to it; and the small force of the pale purple ones will produce too faint a sensation to be agreeable. The intermediate colours are therefore more pleasant to the eye, as the force of the rays is neither too great to be offensive, nor too small to produce a quick and lively sensation.

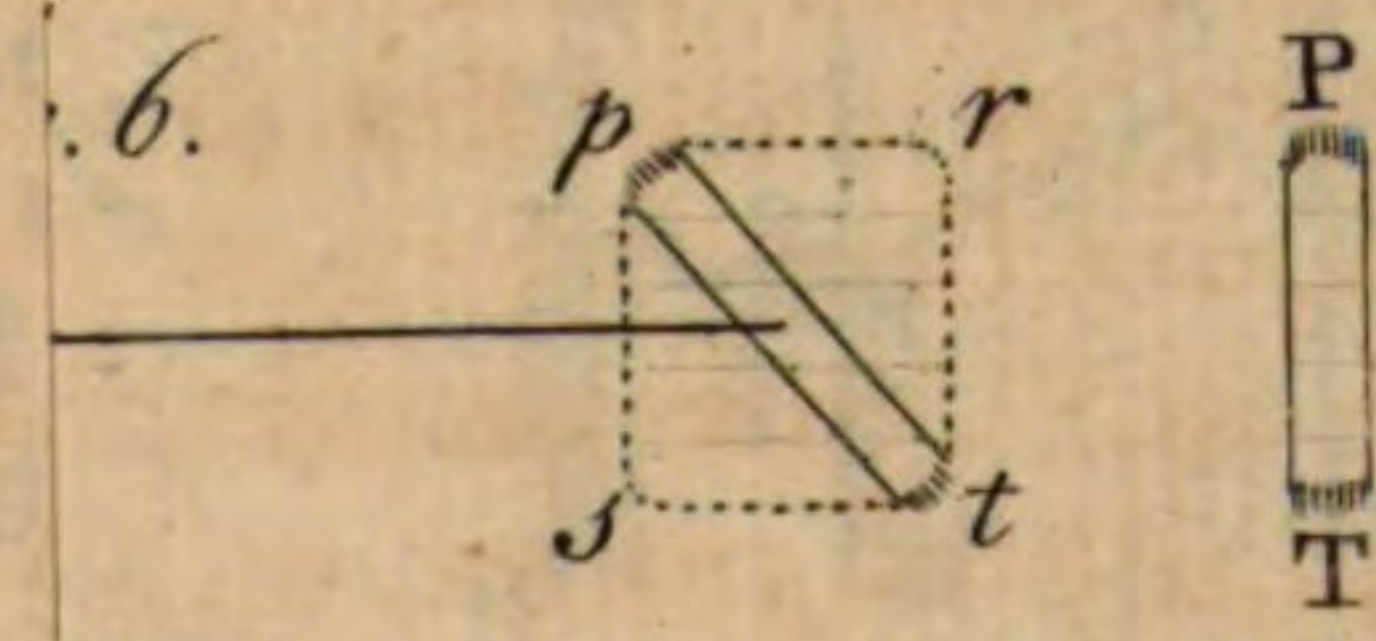
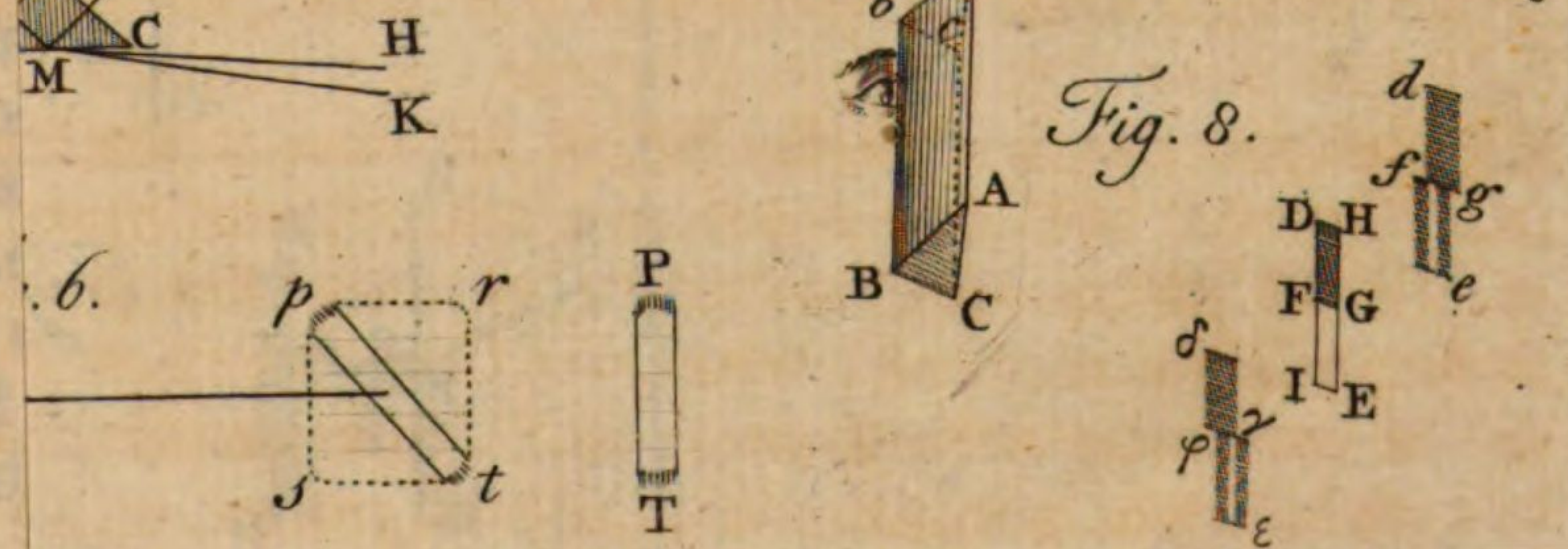
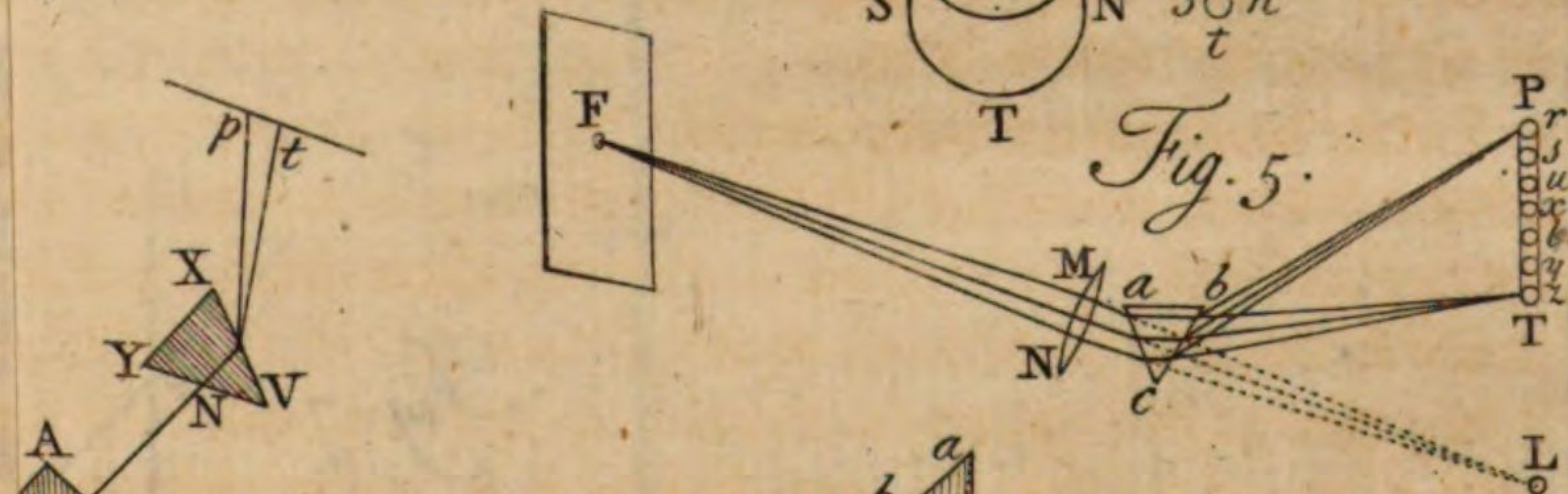
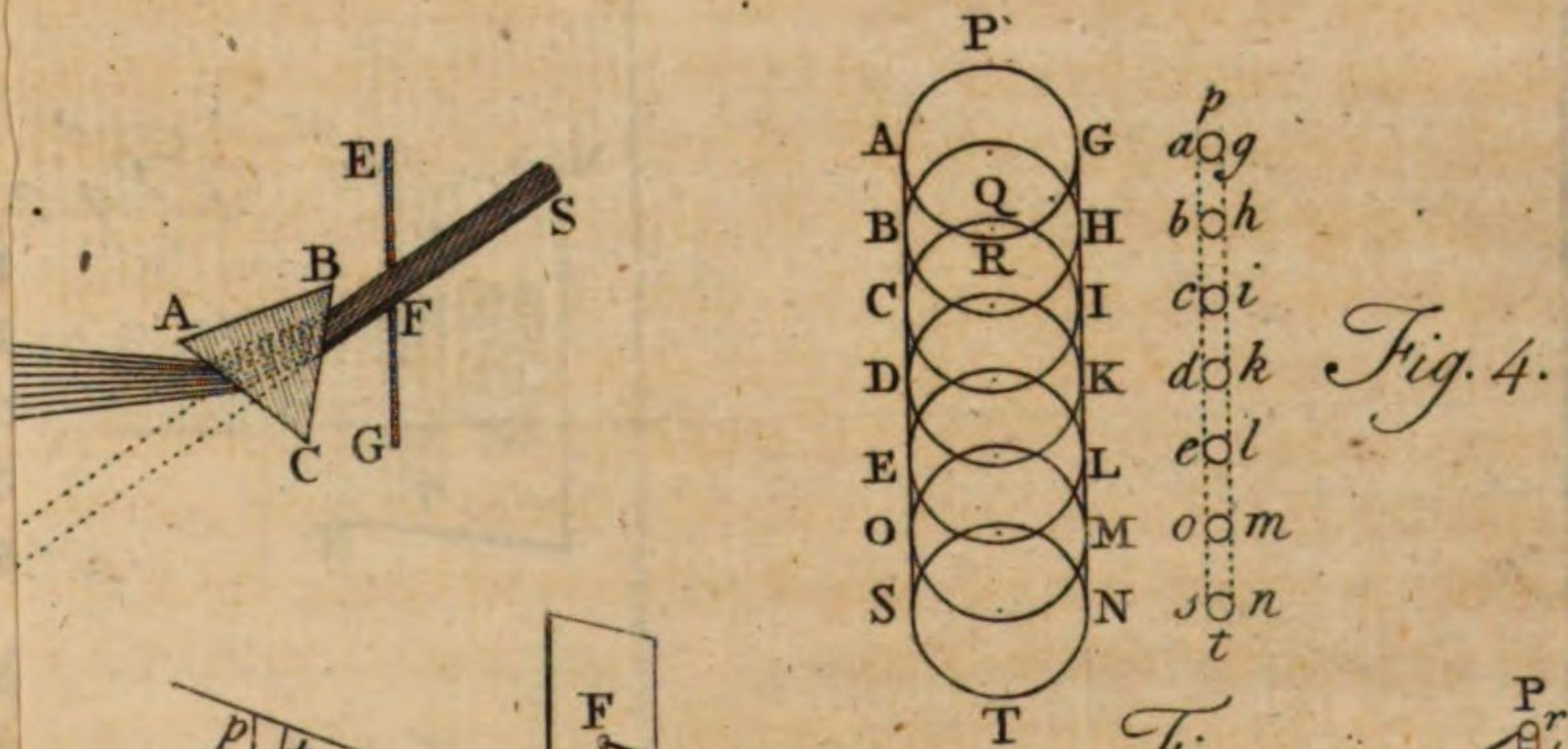
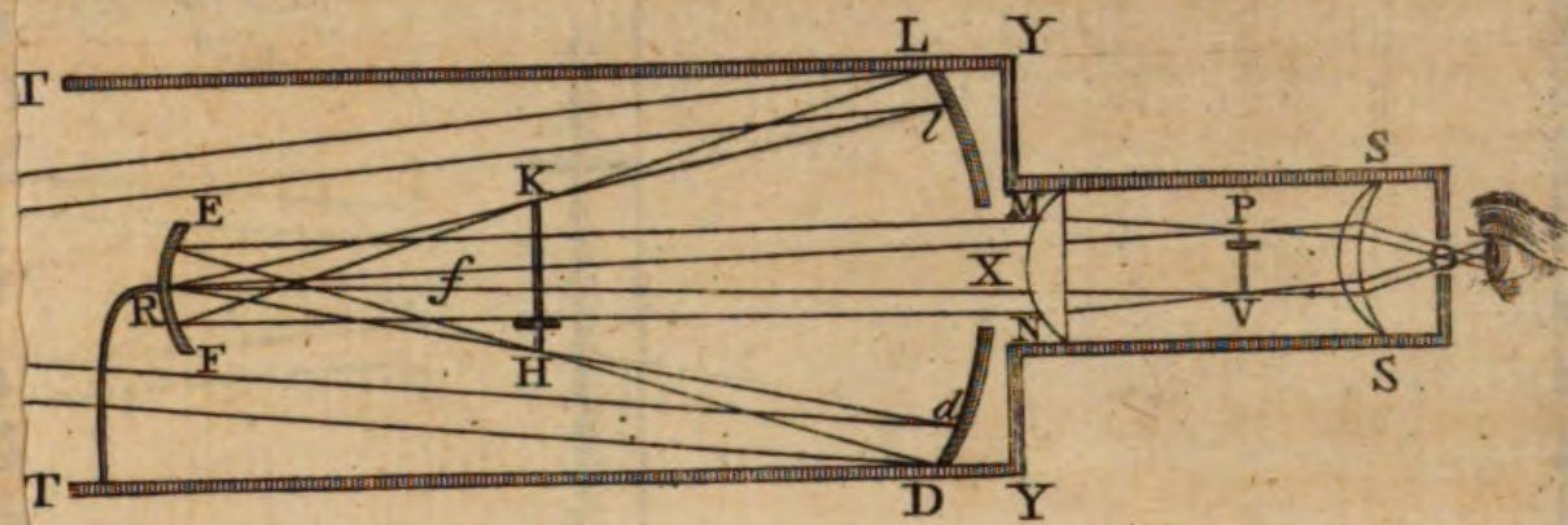
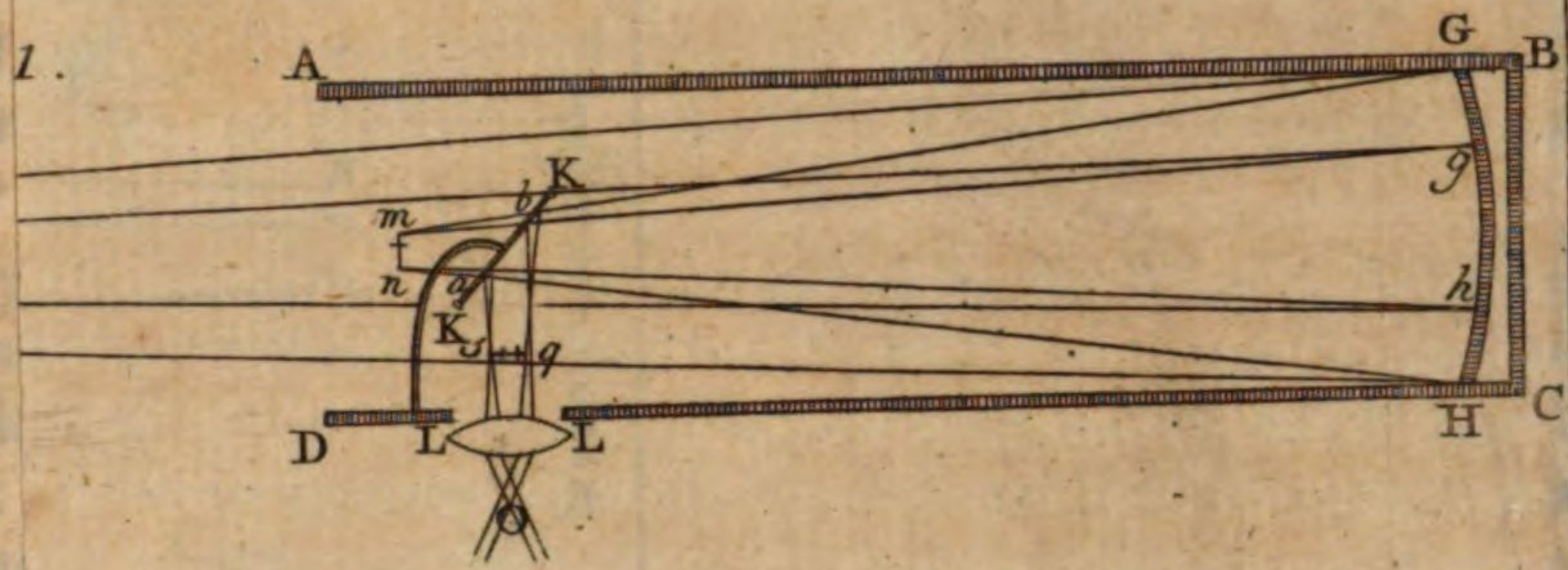
289. *The colours of homogeneous light are so invariable, that neither any refraction nor any reflection can alter them.*

If a beam of homogeneous light passes through a round hole in a paste-board, as in proposition 285, and then is refracted by a prism on the other side of the hole, this refraction will make no alteration in the colour of the rays: if they were red, or whatever was their colour, before they entered the prism, their colour will still be the same, when they have passed through it, and fall upon a white paper held beyond the prism. This proves the first part of the proposition, that the colours of homogeneous light are not to be changed by any refraction.

Red lead, which, as the name denotes, is red, when it is viewed in open day-light, or when heterogeneous rays fall upon it, will likewise be red, if it is placed in homogeneous red light: but red lead, when it is placed in any other sort of homogeneous light, will have the same colour with the rays that fall upon it and are reflected from it: if it is placed in yellow homogeneous light, it will be yellow, if in green light, it will be green, or if in blue light, it will be blue. Consequently the reflection of the rays from the red lead makes no alteration in their colour: for if it did, rays of any sort reflected from the lead would be of the same colour so that it would appear red in whatever sort of light it was placed. The same that is here said of red lead, is true of any other substance of any other colour. Grass, which is green, either in open day-light or in homogeneous green light, will not change the colour of any homogeneous rays by reflecting them, but will itself have the same colour with the rays, in which it is placed; it will be red in red light, or blue in blue light, or yellow in yellow light.

From hence we may conclude, by the way, that a body is of any particular colour, not because it reflects no other rays but those of that particular colour, but because it reflects those more copiously and others more sparingly. Red lead, as it appears red in red light, so in green light it appears green, or in blue light it appears blue: consequently it reflects rays of these sorts, and in the same manner it might be shewn to reflect all other sorts of rays. But then the red colour of red lead, when it is placed in red light is much brighter, than any other colour will be that it puts on by being placed in another sort of light: consequently it reflects red rays more copiously than any other sort of rays: and for this reason, when it is placed in open day-light, where it reflects all sorts of rays at once, the red rays are so much more numerous than the rest, as to make the whole mixture of their own colour.

PLATE XV.



THEORY OF COLOURS

When a substance is placed in the way of light, it will be coloured in proportion to the quantity of light which it absorbs.

If a substance is placed in the way of white light, it will be coloured in proportion to the quantity of light which it absorbs. If it absorbs all the light, it will be black. If it absorbs all the light except the blue, it will be yellow. If it absorbs all the light except the red, it will be green. If it absorbs all the light except the blue and red, it will be purple. The colour of any substance is determined by the quantity of light which it absorbs.

If a substance is placed in the way of light, it will be coloured in proportion to the quantity of light which it absorbs. If it absorbs all the light, it will be black. If it absorbs all the light except the blue, it will be yellow. If it absorbs all the light except the red, it will be green. If it absorbs all the light except the blue and red, it will be purple. The colour of any substance is determined by the quantity of light which it absorbs.



290. *Colours may be produced by composition, which shall in appearance be like the colours of homogeneous light: but then these compound colours will be altered by refraction.*

When, by means of two holes in the window-shutter of a dark room and of two prisms, two oblong coloured pictures are produced; if a circular piece of white paper is so placed that the red light of one picture and the yellow light of the other may fall upon it; this mixture will produce an orange colour, that in appearance will be like the primary orange colour. But between the simple and compound colour, though they are alike in appearance, there will be this difference: if the circular piece of paper, when it is enlightened with compound orange, is viewed through a prism, the rays will be found to be differently refrangible, and they will by the refraction of the prism be so separated from one another, that the paper seen through it will appear as two circles, one of which will be red and the other yellow: whereas, if the same paper, when it is enlightened with simple or primary orange, is viewed in like manner through a prism, the rays will be found to be equally refrangible, and the paper will appear through the prism, as it does to the naked eye, to be one orange-coloured circle distinctly terminated all round. After the same manner other homogeneous colours, as blue and yellow, when mixed together, will produce a new compound colour like the intermediate homogeneous green colour in appearance. But then the rays of this compound green will not be all of them equally refrangible, as the rays of the simple or primary green colour are.

291. *The whiteness of the suns light is compounded of all the primary colours mixed in a due proportion.*

Let the oblong coloured picture PT, Plat. XVI. fig. 1, fall upon the convex lens MN; and then all the rays, which are separated from one another at PT will be collected together by passing through the lens and will meet at its focus G, by proposition 84, in such a manner as to form a round picture of the sun upon a white paper DE. This round picture, which consists of rays of all sorts, of red, orange, yellow, blue, green, indigo, and violet, is white. And this whiteness is compounded of all the primary colours mixed together. None of the rays change their colour by being mixed with the rest; each sort retains the same colour after it is mixed with the rest that it had before: neither the red rays, nor the orange, nor the yellow, nor the blue, nor the green, nor the indigo, nor the violet, are made white by being mixed with the rest at the focus; but though none of the parts are white, yet the whole mixture is white.

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That the whiteness at the focus G arises from a mixture of all the primary colours, is evident. For if any of the colours are intercepted at the lens, the focus loses its whiteness, and becomes of that colour, which arises from a mixture of those, which are not intercepted. Thus if all the rays at PT are intercepted except the yellow, the orange, and the red, the focus will not be white, but will be orange-coloured. If all the rays are intercepted at PT except the blue, the green, and the yellow, the focus will then be green. The orange in one case and the green in the other case is the compound colour arising from a mixture of those rays, which are not intercepted. And in either case, if the rays, that were intercepted are again suffered to pass through the lens, the focus will recover its whiteness.

It may be more difficult to shew that the rays, when they are all of them mixed at the focus, retain their proper colours, and are none of them white though the compound mixture is white. To make this out, let the paper be removed from DE , where all the rays are mixed upon it at G , to de , where it will receive the rays, after they have crossed one another at the focus, and having got beyond it diverge again, by proposition 15. In this position of the paper, because the rays, that were mixed at the focus, have diverged from thence and are again separated from one another, the oblong coloured picture will appear again at tp , so that the red colour T , which was the lowest at the lens, will be the highest at the paper de . But though the colours are thus inverted by passing the focus, yet all of them appear at tp ; which would have been impossible, if each sort of rays by being mixed with the rest at the focus had lost their colour and had been made white. Nor indeed is the colour of any sort of rays at all changed by being mixed with the rest at the focus; but the same rays, that produced any particular colour in the oblong picture PT , are the rays that produce the same colour in the inverted picture tp ; as would be evident from intercepting any particular colour at PT : for if the green rays, for instance, are intercepted at PT there will be no green at tp : or if the red are intercepted at PT , there will then appear no red colour at tp : and the same thing will happen upon intercepting the rays of any other colour at PT , for then that colour will vanish at tp .

292. *Colours may be produced by composition that are neither exactly like any of the primary ones nor fully white.*

If the red rays of one coloured picture are mixed with the violet rays of another; according to the various proportions, in which they are mixed, various purples will be produced, such as are not like in appearance to the colour of any homogeneous light; and of these purples mixed with yellow and blue may be made other new colours.

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By mixing the coloured powders, which painters use, though the powders themselves resemble the primary colours, yet the mixture may be grey, or dun, or russet-brown, such as are the colours of a mans nail, of a mouse, of ashes, of ordinary stones, of mortar, of dust and dirt in the high-ways. Thus one part of red lead and five of viride æris compose a dun colour like that of a mouse. If to orpiment, which is yellow, a full bright purple powder used by painters is added, the mixture may be made of a pale red; and with the addition of a little viride æris, which, as the name imports, is green, and of a little blue bise, this pale red will change to grey or pale white, such as is the colour of ashes, or of wood newly cut, or of a mans skin.

These grey, dun, and russet colours are only imperfect sorts of white. And we may understand, why the mixture of these coloured powders should produce an imperfect white, and not a full bright one, from the following observation. All coloured powders suppress and stop great part of the light, that falls upon them: they reflect more of those rays from whence their colour arises than of any other sort; but they reflect even those more sparingly than white bodies do. Red lead, for instance, reflects fewer red rays than white paper does: for if red lead and white paper are both of them placed in homogeneous red light, the paper will appear of a brighter red than the lead. But if red lead suppresses many red rays, it may well be supposed to suppress many more rays of other colours. Since its redness is owing to its reflecting red rays more copiously and all other rays more sparingly, as was observed under proposition 289. From hence it follows, that, in a mixture of coloured powders, though they reflect rays of all sorts in a due proportion so that the compound light will not be more of one colour than another, but will be white; yet the whiteness will be much less bright than that of paper: because the mixture of powders suppresses and stops many rays, whereas the paper reflects almost all the rays, that fall upon it, and suppresses scarce any. Thus the whiteness in the mixture of powders and the whiteness in the paper are both of the same sort, and differ from one another only in degree or in the quantity of light. Therefore if some of this mixture of powders is placed in bright sun-shine and a piece of white paper is placed in the shade, the mixture by thus encreasing the light and the paper by thus diminishing it may be made to appear equally white.

293. *The colours of all bodies are either the simple colours of homogeneous light, or such compound colours as arise from a mixture of homogeneous light.*

Each sort of light has a peculiar colour of its own, which no refraction or reflection can change, by proposition 289. Therefore the colour of no natural body can be any other than either the colour of some sort of homogeneous light, or a compound colour arising from a mixture of the several sorts. For bodies appear coloured only by reflecting light; and no reflection can give any other colours to the rays but what they had before.

When in the foregoing propositions, or in any of those which follow, we speak of light and rays as coloured or endued with colours, we must be understood to speak not philosophically but according to vulgar conceptions. For properly the rays are not coloured. In them there is nothing else but a certain power and disposition to raise in the mind the idea of this or that colour. For as sound in a bell is nothing but a trembling motion, in the air it is nothing but that motion propagated from the bell, and in the mind it is an idea which this motion, when conveyed to the sensorium, excites in us: so colours in the object are nothing but such a modification, figure, and contexture of its parts, as makes it fit to reflect this or that sort of rays more copiously than others; in the rays they are nothing but the dispositions or powers of those rays to propagate this or that motion to the sensorium, and in the mind they are the ideas, which those motions excite there.

294. *The different refrangibility of light makes refracting telescopes imperfect.*

If we look through a refracting telescope at any object, the picture of this object should be distinct in the principal focus of the eye-glass, by proposition 172. But in a refracting telescope there will be no one distinct picture of the object any where. For suppose we were looking at the moon, the light, that comes from the moon, consists of different sorts of rays, and each sort has a different degree of refrangibility. The violet rays are refracted most, and the red rays least of all. Therefore, in passing through the object-glass, the violet rays will be made to converge more and will meet in their respective focuses, or will form a distinct picture, nearer to the object-glass than the red ones. Thus if there are seven different sorts of rays, instead of one picture of the moon, there will be seven; and each picture will be in a different place: the violet picture will be nearest to the object-glass, the red one will be farthest from it, and those of the intermediate colours will be between them. Now to see the moon perfectly through the telescope; the rays of all the several sorts of colours should be mixed together, and the picture composed of this mixture

ture should be distinct. But there is no distinct picture where all the colours are mixed. At the place of the green picture, for instance, that particular picture will be distinct, and the rays of all the colours will be mixed there: but then the picture made from the mixture will be confused. For the violet rays, and all those, which are more refrangible than the green ones, will have met in their respective focuses and will have begun to diverge from thence, before they come to the place of the green picture, and consequently at the place of the green picture, all the other pictures of the more refrangible colours will be confused. The red rays, and all those, which are less refrangible than the green ones, will be converging and will not be come to their respective focuses, when they get to the place of the green picture; and consequently, at the place of the green picture, all the other pictures of the less refrangible colours will be confused. The same is equally true in any other place. If the picture made by one sort of rays is distinct, those made by all the other sorts will be confused. Therefore the compound picture will always be imperfect, and the moon, or any other object for the same reason, will be seen imperfectly or confusedly through a refracting telescope.

295. *When two refracting telescopes of different lengths magnify equally, the shorter telescope of the two will be more imperfect than the longer.*

If the principal focal distance of the object-glass is 20 inches and that of the eye-glass is $\frac{1}{2}$ inch, such a telescope will magnify an object as much, as another telescope, in which the principal focal distance of the object-glass is 20 feet and that of the eye-glass is $\frac{1}{2}$ foot, by proposition 175. Now since the object appears equally magnified, through either telescope, it is plain that the distinct picture in each of them appears of the same size, when it is viewed through the eye-glass of its respective telescope. But the distinct picture, though it appears equal in each, is bigger in one of the telescopes than in the other. For the distinct picture in the longer telescope is 20 feet from the object-glass, and in the shorter is only 20 inches from it, by the construction: and since as 20 feet is to 20 inches, so is 12 to 1, the diameter of the distinct picture in the longer telescope will be 12 times the diameter of the distinct picture in the shorter, by prop. 88. But each picture through its respective eye-glass appears of the same size. Therefore the picture in the shorter telescope is magnified 12 times as much as that in the longer: for since it is 12 times less in itself it could not appear of the same size, unless it was 12 times as much magnified. Both these pictures, which I have hitherto called distinct, would
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be so, if the rays of light were all equally refrangible: but as the different rays are differently refrangible, they will both be confused or imperfect, by proposition 294. And this imperfection will appear the more through the eye-glass in proportion as the eye-glass magnifies the picture more. Therefore the picture will appear more imperfect in the shorter telescope than in the longer: for which reason the object seen through the shorter telescope will be seen more confusedly or more imperfectly than through the longer.

This is the reason referred to in proposition 175, why refracting telescopes that magnify an object very much will be useless if they are too short.

296. *Reflecting telescopes are more perfect than refracting ones.*

We have seen, in proposition 294, that the picture of an object produced by a refracting object-glass will be imperfect; upon account of the different refrangibility of light. But a picture produced by a reflecting mirror, will not be liable to the same imperfections. For though light, by proposition 283, consists of rays that are differently reflexible, yet reflection will never separate these rays from one another, unless where some of them are reflected, whilst others are transmitted in the manner described in proving proposition 283. But when all the rays, the red, orange, yellow, green, blue, indigo, and violet, fall together upon the concave surface of a polished mirror, as they all have equal angles of incidence and are all reflected, they will all have equal angles of reflection, by proposition 201, and consequently will all meet in a focus at the same distance.

From hence we see the reason why though refracting telescopes, that magnify much, must be long, by proposition 295, yet reflecting telescopes, that magnify as much, may be shorter. For the picture in a refracting telescope is an imperfect one, and will not bear magnifying much by the eye-glass, as was proved in proposition 295: but the picture in a reflecting telescope is more perfect; and consequently though it is small, as it will be, by proposition 256, when the concave object-mirror has a short principal focus, yet it will bear magnifying by the eye-glass as much as we please.

C H A P. XXI.

Of the colours of thin transparent plates.

297. *Water, air, glass, or any other transparent substance, when drawn out into thin plates, become coloured.*

WATER, when it is made tenacious by having soap mixed with it, may be blown up into a bubble A, Plat. XVI. fig. 2, such as children play with. If this bubble is set under a glass, so that the motion of the air may not affect it, then, as the water glides down the sides of it, and the top of it at A grows thinner, several colours will successively appear at A, and will spread themselves from thence in rings surrounding A and descending farther and farther down the sides of the bubble, till they vanish at BC in the same order in which they appeared. Thus, for instance, the first colour that appears at A the top of the bubble is red: this red spot spreads itself into a circular ring round A, and then the top of the bubble A becomes blue: this blue spot spreads itself in the same manner round A, and then A becomes red a second time. Before we go on to consider what other colours arise at A, we will observe what becomes of those, which arise first. The red, which first appeared at A, spreads itself into a circular ring round A; this ring grows larger, as the water glides down the sides of the bubble; so that the coloured ring glides down the bubble along with the water, till it sinks at last to BC and there encompasses the bubble. In like manner the blue, which arises at A after the red, spreads itself and descends down the bubble, as the red ring did. The colour, which arises next at A, is red a second time; this spreads itself in the same manner, and is succeeded by blue a second time. These are followed by a great variety of colours, which appear successively at A and spread themselves from thence in this order. Red, yellow, green, blue, purple; then again red, yellow, green, blue, violet; and lastly, red, yellow, white, blue. This last blue colour is succeeded at A by a black spot, which reflects scarce any light: this spot dilates itself, but not into a circular ring as the colours had done; it becomes broader and broader, till the bubble breaks.

A thin plate of water of the same sort with this bubble, but more lasting, may be otherwise procured. If a piece of plane polished glass is placed upon the object-glass of a long telescope; as in Plat. XVI. fig. 3, the plane surface of one glass and the convex one of the other will touch one another only at a single point; and if the interval between them is
filled

filled with water, as the glasses are pressed together, the same colours arise at the point of contact and spread themselves in circular rings round it in the same order, as in the soap-bubble. If BC, Plat. XVI. fig. 4, is a section of the plane glass, and DAE a section of the convex one; when they are pressed close together the thin plate of water, that fills the interval between them, will have a black spot at A, and this spot will be encompassed with rings of colours, in the same order, that they stand in that figure upon the line BC, on each side of A. If the colours are reckoned in the order, in which they stand on the plate of water, after the black spot appears at A; and we reckon them from the spot A towards the edges of the plate at B and C, then we must call blue the first colour. But if we reckon them in the order, in which they arose at A and spread themselves, then we must begin from B or C the edges of the plate and go on towards A, and in this reckoning we must call red the first colour.

If there is no water between the two glasses, then the interval will be filled with air, and this thin plate of air will have the same colours, that the plate of water had; with this difference only, that each of the coloured rings is larger in the plate of air than in the plate of water.

When glass is blown very thin at a lamp-furnace; thin plates of it thus formed will exhibit colours: and so likewise will thin plates of Muscovy-glass. Metals, when they are heated, send out to their surfaces scoria or vitrified parts, which cover the metals in form of a thin skin: and these scoria or thin plates cause colours upon the surface of the metal, such as are made to appear on polished steel by heating it, or on bell-metal by melting it first and then pouring it on the ground to cool in the air.

298. *When the thin plate is denser than the medium that surrounds it, the colours are more vivid, than they are, when the plate is rarer than that medium.*

A thin bubble is a plate of water encompassed with air: where the substance of the plate, which is water, is denser than the air, which is the medium that surrounds it. On the contrary, the plate of air between the two glasses BAC, DAE, Plat. XIV. fig. 4, is encompassed with glass: and here the substance of the plate is rarer than that of the circumambient medium. And the colours on the bubble of water are more vivid than those on the thin plate of air.

299. *When thin transparent plates reflect one sort of rays, they transmit the rest.*

If

If the plate of air between the two glasses BAC, DAE, Plat. XVI. fig. 4, is viewed by reflected light, the colours of it are those expressed on the upper part of the figure, from B to C, but if we look through it, that is, if we view it by transmitted light, or if the transmitted light falls upon a white paper, the colours that we see through the plate, or that fall on the paper, are those expressed on the lower part of the figure. Now any of the transmitted colours are what would arise from a mixture of all the remaining rays, after those of the reflected colour are separated from the suns heterogeneous light. Thus for instance, the fourth reflected colour from the black spot A inclusively is yellow, the transmitted colour is violet. The yellow rays and some of the orange and green, are reflected here, so that the mixture of the reflected light will be yellow. The mixture of the transmitted light therefore will be violet or rather such a purple, as is not exactly like any of the primary colours: for we observed, in proposition 292, that from red rays, violet, and blue new purples may be produced.

This is the case in some natural bodies, as well as in these transparent artificial plates; for if leave gold, which is made thin enough to transmit light, is held against the strong light of the suns rays, the gold, which is yellow, when seen by reflected light, will be blue, when thus seen by transmitted light.

The seventh reflected colour inclusively from the black spot is blue, the seventh transmitted colour is yellow. When the rays, which make the blue colour, are taken out of the suns heterogeneous light, the remaining rays will be yellow. Thus it happens likewise in some natural bodies; for an infusion of *lignum nephriticum*, which is blue, when seen by reflected light, is yellow, when seen by transmitted light.

The black spot A reflects scarce any light; and as rays of all colours are transmitted there, the transmitted colour is white, as it ought to be by proposition 291. The third reflected colour from the black spot inclusively is white. Therefore since, by proposition 291, all the rays are reflected there, no colour ought to be seen there, when we look through the plate; and accordingly that part of the plate is black.

Hence we see the reason why if there be two liquors of full colours in two different glass vessels, suppose red and blue, though each is transparent, when we look through it separately, yet we should not be able to see through both of them together, if one was held behind the other. For if the blue liquor for instance is held towards the light and the red towards the eye; since only blue rays pass through the first liquor, and come to the second; and since the second liquor will transmit no blue

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rays,

rays, but only red ones; it follows that no rays at all can come to the eye.

Indeed some transparent bodies appear of the same colour, whether we see them by reflected or transmitted light. Of this sort is most painted glass. But when this is the case, the coloured rays are reflected from the second surface of the body. Thus if a piece of painted glass is yellow, either when seen by reflected light or when seen by transmitted light; all the rays, but the yellow ones, are suppressed, as they pass through the glass; of these yellow rays most are transmitted at the second surface, the few which are reflected from thence will be sufficient to tinge all the light yellow, which is reflected from the first surface. This will be evident from making the body thick and pitching it on the backside; for by this means the reflected colour will be lost: whereas if it had been reflected from the first surface, the pitch at the second surface could not have altered it.

300. *The more dense the substance is, out of which a thin plate is made, the less is the thickness of the plate, where it reflects any certain colour.*

The colours are the same whether there is air or water between the two glasses BAC, DAE, only the coloured circles are smaller in the plate of water than in the plate of air. Thus the yellow, for instance, which is the fourth coloured circle from the black spot, is a less circle, or is nearer to the black spot, when there is a plate of water between the glasses, than when there is a plate of air between them. But the less the distance from the point of contact A, the closer the glasses are to one another, and consequently the thinner will be the plate that lies between them: consequently that part of a plate of water, where this yellow appears, is thinner than that part of a plate of water, where the same colour appears. And the same holds good in any other colour. But water is more dense than air. Therefore the more dense the substance is, out of which a thin plate is made, the less is the thickness of the plate, where it reflects any certain colour.

301. *The sort of colour, which is reflected from any part of a thin plate, depends only upon the thickness of the plate itself in that part: but the same colour will be made less vivid by encreasing the density of the medium, with which the plate is encompassed.*

The colours upon any part of a thin plate of Muscovy-glass are the same in sort, whether the plate is dry or wetted with water. Therefore the sort of colour in any part depends not upon the medium that encompasses

passes the plate, but upon the thickness of the plate itself; since the colours are the same, when the plate is dry and encompassed with air, or wet and so encompassed with water. But the same colours are more faint, when the plate is wet, than when it is dry; and consequently, the brightness of the colours does depend upon the medium that encompasses the plate, and the denser that medium is the fainter will be the colours; they are more faint when the plate is covered with water than when it is dry and so is surrounded with air.

302. *The rays of light have alternate fits of easy reflection and easy transmission, which return at equal intervals.*

Let GF, Plat. XVI. fig. 5, be a beam of homogeneous light consisting all of one sort of rays, as suppose all the rays that compose the beam were red ones. Then if these rays fall upon a thin plate of air between the two glasses BAC, DAE, at A there will be a dark spot, and all the rays will be transmitted; round this spot there will be a red ring, where all the rays are reflected; round this red ring there will be a dark ring, where all the rays are transmitted. And if the thickness of the plate, where all the rays are reflected in the ring nearest to A, is called 1, the thickness, where the dark ring appears and all the rays are transmitted, will be 2. Again, at that part of the plate where the thickness is 3, all the rays will be transmitted: at the thickness 4, they will be all reflected. And thus alternately, as expressed by the lines in the figure, the rays will be reflected, in all parts of the plate, where the thickness is expressed by any of the uneven numbers 1, 3, 5, 7, 9, &c. and will be transmitted, where the thickness is expressed by any of the even numbers 2, 4, 6, 8, 10, &c.

Now as the plate is the same in all parts; the cause of this alternate reflection and transmission must be in the rays themselves. And their dispositions to be thus alternately, reflected and transmitted are what we call fits of easy reflection and easy transmission.

The rays that are in a fit of easy reflection penetrate as far as the second surface of the plate. For if the second surface of a thin plate of Muscovy-glass is wetted, the colours caused by the alternate reflection grow fainter. Whereas if the reflection was made at the first surface, wetting the second could not affect the colours. But since those rays, which have passed from the first surface of the plate to the second, where the thickness of it is 1, are reflected, and those, which have passed from the first surface to the second, where the thickness of it is 2, are transmitted; and then again those, which have thus passed from one surface to the other, where the thickness is 3 are reflected; and those, which

have passed in the same manner, where the thickness is 4, are transmitted, it follows that the fits of easy reflection and transmission return at equal intervals. So that, if a ray was to set out from A in the line AB, fig. 6, and was to be in a fit of easy reflection, when it had moved from A to *c*; it would be in a fit of easy transmission, when it had moved to twice that distance from A, or when it was got to *d*: at *e*, or the distance 3 from A, it will be in a fit of easy reflection: at *f*, or the distance 4, in a fit of easy transmission: at *g*, or 5, in a fit of easy reflection: at B, or 6, in a fit of easy transmission, and thus in the farther progress of the ray the same fits will return at equal intervals.

Thus if the thickness of the plate of air, where the rays of any homogeneous colour are all reflected is equal to *Ac* or 1, and the rays are in a fit of easy reflection, when they come to the second surface of the plate; then where the thickness of the plate is *Ad* or 2, the rays, will be in a fit of easy transmission, when they come to the second surface, and consequently, will all pass through that surface. Again where the thickness is *Ae* or 3, the rays when they come to the second surface will be in a fit of easy reflection, and will all be reflected, where the thickness is *Af* or 4, the fit of easy transmission will be returned when the rays come to the second surface, so that all of them will be transmitted. And in like manner, by such fits returning at equal intervals, the rays will be reflected, where the thickness is expressed by the numbers 1, 3, 5, 7, 9, &c. and will be transmitted, where it is expressed by 2, 4, 6, 8, 10, &c.

303. *When a thin coloured plate is viewed obliquely the colours of every part in the plate will be altered.*

When a bubble of water or a plate of air between two glasses BAC, DAE, Plat. XVI. fig. 4, is viewed obliquely, the coloured rings dilate themselves: and consequently a ring of any one colour by being dilated gets into that part of the plate, where a ring of some other colour appeared, when the plate was viewed directly.

304. *If the plate is denser than the medium that encompasses it, the colours of it, when viewed obliquely, change less, than they would, if the plate was rarer than the medium that encompasses it.*

A bubble of water is a thin plate denser than the air, that encompasses it: and a plate of air between the two glasses BAC, DAE, Plat. XVI. fig. 4, is rarer than the glass that encompasses it. Upon viewing each of these thin plates obliquely, the coloured rings on the plate of water dilate less

less than those on the plate of air. Therefore since, by proposition 303, it is by this dilatation of the rings, that a ring of one colour gets into a part of the plate, where a ring of some other colour appeared, when the plate was viewed directly; that is, since it is by this dilatation of the rings that the several parts of the plates change their colours, it follows, that any part of a plate of water encompassed with air changes colour less, upon being viewed obliquely, than any part of a plate of air encompassed with glass.

305. *When the medium, which encompasses a coloured transparent plate, is given, the colours change less upon altering the situation of the eye, as the substance is more dense out of which that plate is made.*

The matter, out of which a bubble of water is made, is not so dense, as that out of which a bubble of glass is made, and glass is not so dense as the scoria or glassy skin thrown out by metals, when they are heated. Now any of these plates either of water, or glass, or metalline substance, when they are encompassed with the same medium air, will change their colour a little upon being viewed obliquely. But the plate of water changes the most, the plate of glass less than that, and the scoria of metals least of all.

CHAP. XXII.

Of the opakeness, transparency and colours of natural bodies.

306. *The opakeness of bodies is owing to the many reflections and refractions, which the rays of light suffer within those bodies.*

THE smallest parts of almost all natural bodies are transparent, as will readily be granted by those, who have been used to look through microscopes. A piece of leaf-gold is transparent, if it is held up against the hole of a window-shutter in a dark room. And any other substance, however opaque it may seem in the open air, will appear transparent by the same means, when it is made of a sufficient thinness. Even metals become transparent, if they are dissolved in a proper menstruum, as gold in aqua regia, or silver in aqua fortis, and by being thus dissolved are reduced to very small particles. But since even in opaque bodies every single particle transmits light or is transparent, the whole would likewise transmit light, unless the rays, when they are to pass through all the particles, which make up the whole, were so turned out of the way by in-

innumerable refractions and reflections, as to be stopped and suppressed in their passage. That this is the reason why bodies, that consist of transparent particles, should be opaque, is evident; since opaque bodies, when they are reduced to a sufficient thinness become transparent: for then there will be but few particles lying beyond one another for the light to pass through; and as the rays will suffer fewer refractions and reflections, some of them may get through a thin plate, though all of them would be suppressed in a thicker mass of the same substance.

307. *The medium, with which the pores of opaque bodies are filled, is not of the same density with the particles of those bodies.*

Bodies consist of transparent particles, and their opakeness is owing to the many reflexions and refractions, which the light suffers within them, by proposition 306. Now if the interstices between the particles of any body were filled with a medium of the same density with the particles, the light would neither be refracted nor reflected, as it passed out of the particles into the interstices and out of the interstices into the pores, by propositions 37, 38, 200, but would pass through the body, and the body would be transparent. Consequently in an opaque body, where the light is suppressed by the refractions and reflections, which it suffers, the particles, that compose the body, and the medium, that fills the pores or interstices between the particles, must be of different densities.

Hence we may see the reason why paper when it has been dipped in water or oyl, is more transparent, than when it is dry. For when the paper is thoroughly wetted with water or oyl, the pores of it are filled with a medium that is nearly of the same density with its particles. On the contrary, though oyl of turpentine and water are both of them transparent, when they are separate, yet if they are shaken together so as to mix but imperfectly, the mixture becomes much less transparent. Because the parts of each fluid are separated from one another, and those of the other fluid, which are of a different density, get in between them.

308. *The parts of bodies and their interstices must not be less than of a certain definite bigness to render them opaque and coloured.*

The most opaque bodies become transparent, when their particles are subtilly divided. As metals such as gold or silver, which are opaque in large masses, become transparent, when the former is dissolved in aqua regia and the latter in aqua fortis. And, in proposition 297, we observed that at the top of a bubble of water, where the water is extremely thin, there is a black spot, which reflects scarce any light at all; though the water
is

is encompassed with air, which is a medium of a different density. Consequently if the diameter of the particles, of which any natural substance consists, was no greater than the thickness of the bubble, where it reflects no light, but transmits all, such a body would be transparent notwithstanding the interstices, that are between its particles, were filled with a medium, the density of which is different from theirs.

In like manner, we observed, that when a thin plate of air lies between two pieces of glass BAC, DAE, Plat. XVI. fig. 4, there is a dark spot, which reflects no light and transmits all, not only at the point A where the glasses touch one another, but also round that point to some distance where the glasses are very near to one another. From hence we may conclude that though the particles of any natural substance were as dense as glass, and the medium which fills their interstices was as rare as air, yet if these interstices were no bigger than the interval between the two glasses BAC, DAE, at that place where all light is transmitted, such a body would be transparent.

The transparency of water seems to be owing to the causes here mentioned, to the smallness of its parts, or of its pores, or of both. For we are sure that the pores of water are filled with air, because the air may be drawn out from the water in an air pump; and consequently, as the pores are filled with a medium of a different density from the parts, the mixture ought to be opaque, like such a mixture of water and oil of turpentine, as was mentioned in proposition 307. But the smallness either of the parts, or of the interstices, or of both will prevent the mixture from being opaque.

Since therefore all bodies will be transparent, if either their parts or their interstices are too small, it follows that the parts and likewise the pores of such bodies, as are not transparent but opaque and coloured, must not be less than of a certain and determinate bigness.

309. *The colours of natural bodies depend upon the size of their particles.*

Different parts of thin transparent plates, according to the different thickness of them, are of different colours, by proposition 297. Now if any part of such a thin plate of glass, for instance, where it appears of one uniform colour, should be split into threads or broken into small particles; all these particles would make a heap of powder of the same colour. And the small particles of natural bodies, since they are transparent, like so many fragments of a thin plate, must exhibit colours in the same manner.

310. *The parts of bodies, on which their colours depend, are much denser than the medium, which fills their pores.*

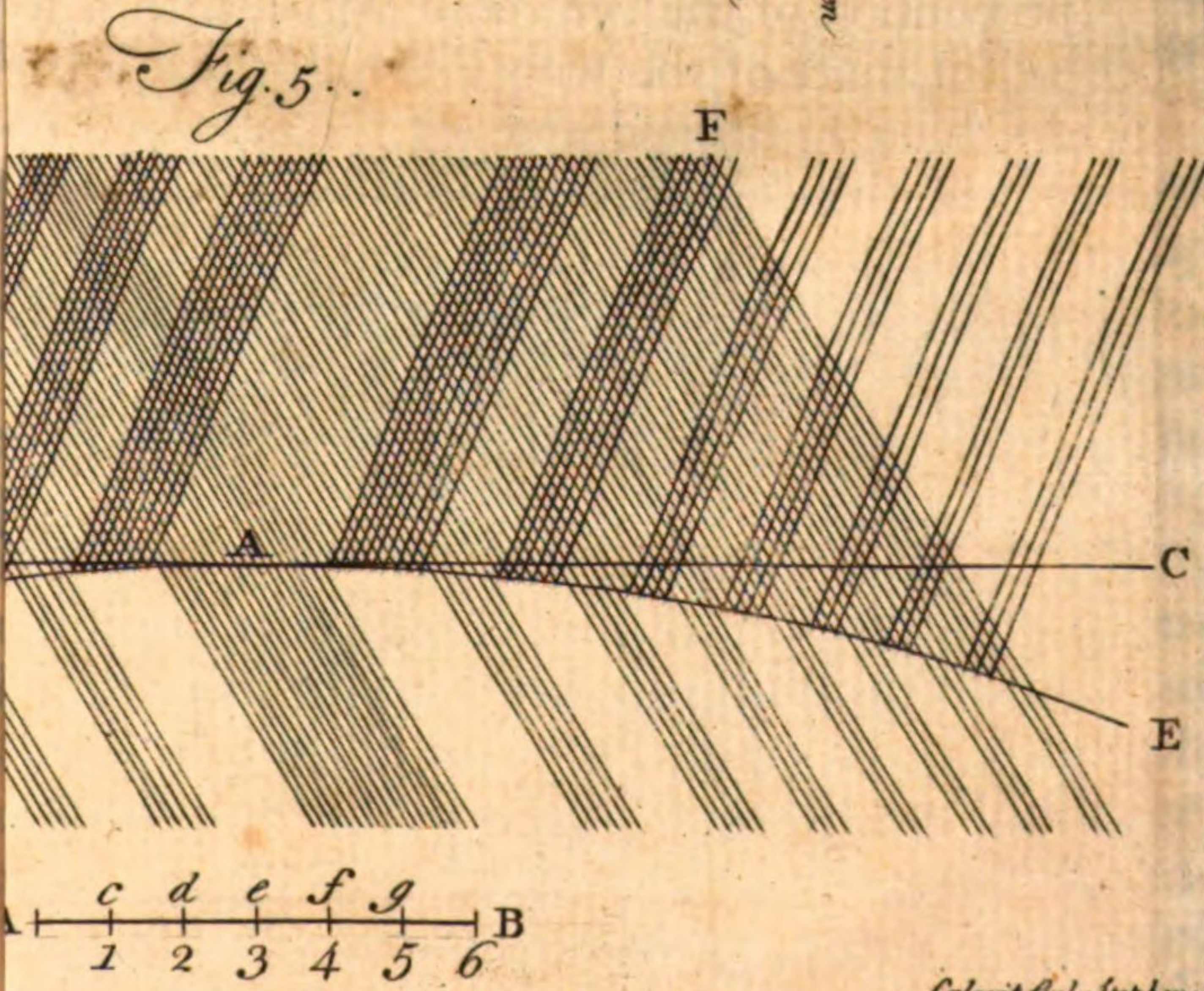
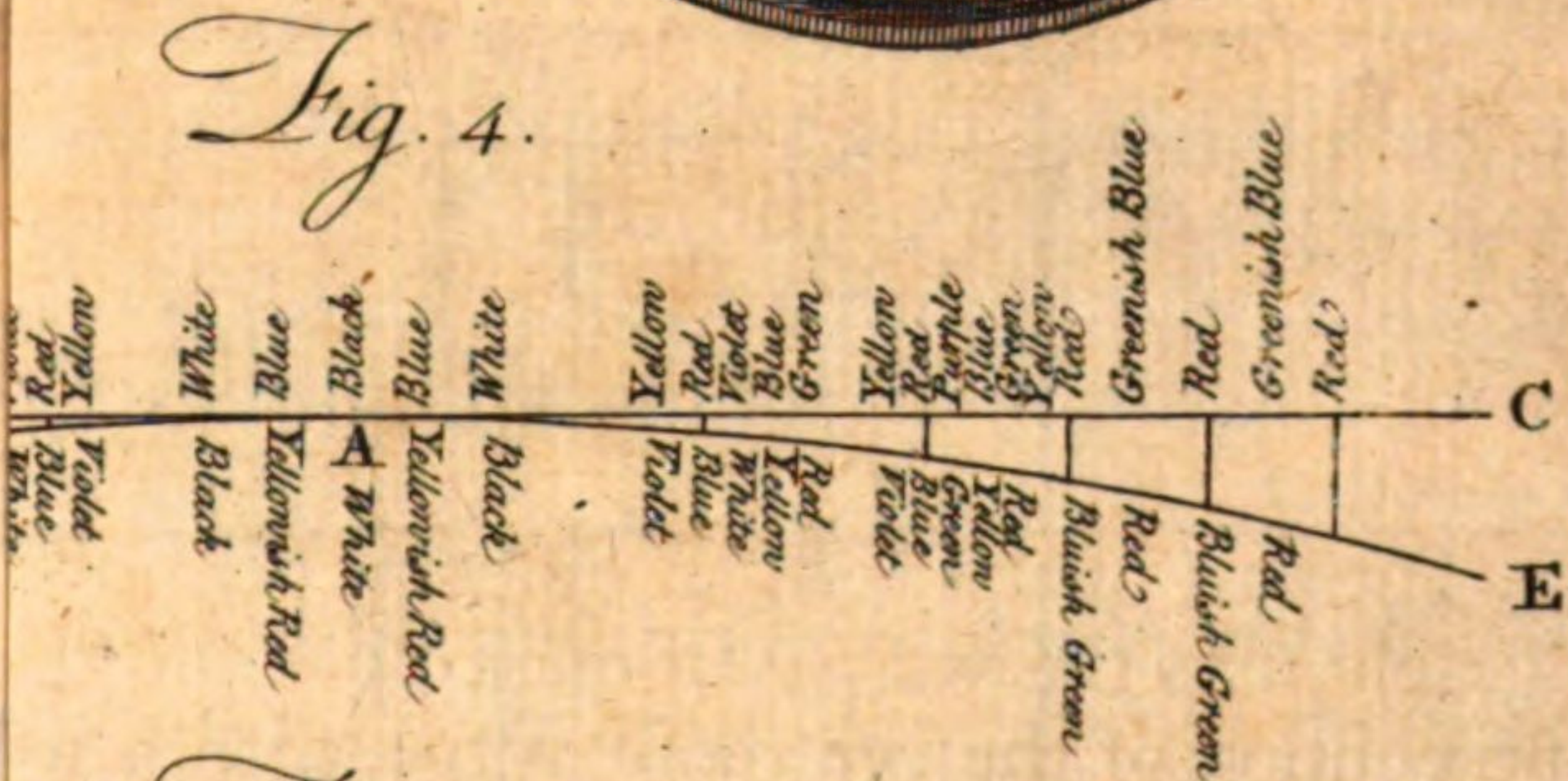
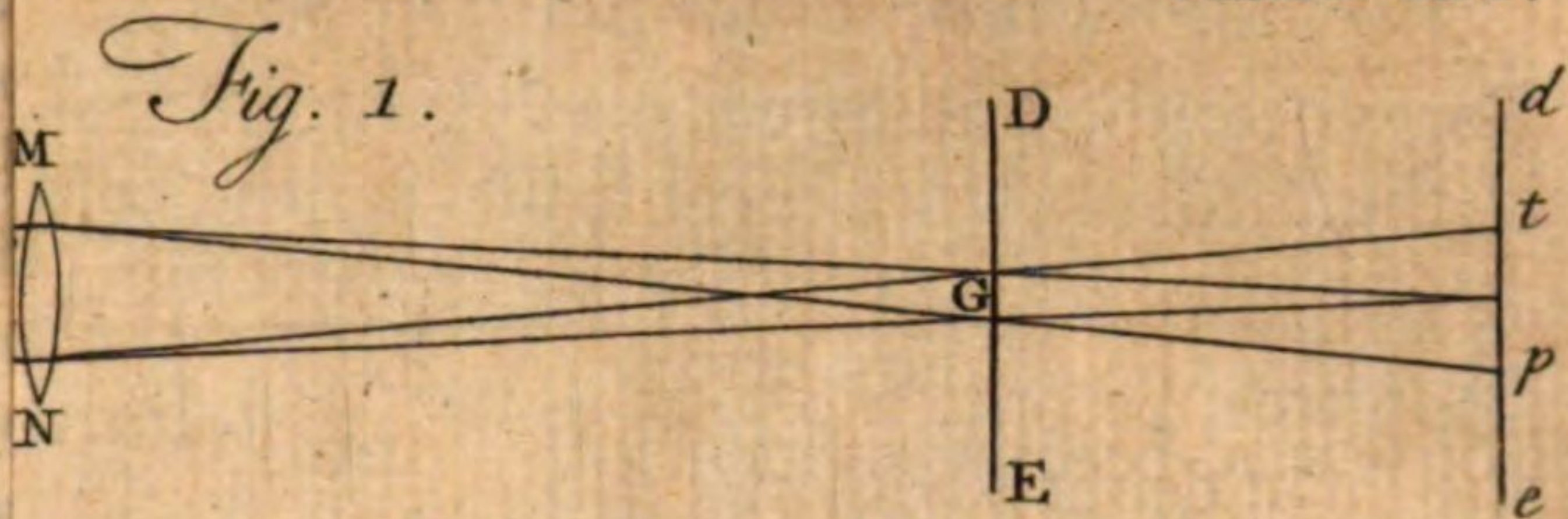
For where the transparent plate or particle consists of a rarer substance than the medium, that encompasses it, the colours are less vivid than those of natural bodies commonly are, by proposition 289. For this reason it is that the colours of silks or cloths, when they are wetted with oyl or water, become more faint; because these liquors are more nearly of the same density with the particles, than the medium is, which fills the interstices when they are dry. Besides, the colours upon a transparent plate change very sensibly, unless the plate consists of a substance much denser than the medium that encompasses it, by propositions 304, 305. But most natural bodies are of the same colour in whatever position of the eye they are viewed. Therefore their transparent particles, upon which their colours depend, are much denser than the medium, which encompasses those particles or fills the interstices between them.

Nor is the case otherwise even in those bodies which do change colour upon being viewed obliquely, such as changeable silks, or the feathers of a peacocks tail or of a pidgeons neck. For this change of colour, upon the situation of the eye being changed, is no reason for concluding that the medium, which fills the interstices or pores, is more nearly of the same density with the particles, upon which the colours depend, in these bodies than in others. Since the change of colour is plainly owing to our seeing a different part of the body in different positions of the eye. Thus in changeable silks the warp is of one colour and the woof of another, and in one position of the eye more of the warp is seen, and in another position of it more of the woof is seen. In like manner, if a pidgeons neck appears blue in one position of the eye, and crimson in another, it is because in these different positions we see different parts of the same feathers.

311. *We cannot from the colour of a body make any conjecture about the size of the particles, upon which its colours depend.*

Suppose, from the appearance of the colour in any yellow body, that we had determined its yellow to be of the same sort with that, which is next to the black spot in a plate of air or water, or glass. The thickness of a plate, where it appears of this colour, is different according to the different density of the substance out of which that plate is made, by proposition 300. The thickness of a plate of air, where it appears of this colour, is greater than that of a plate of water, where it appears of the same colour, and much greater still than that of a plate of glass. Suppose

PLATE XVI.



pose therefore farther, that we were able to determine exactly what is the thickness of a plate of air or water or glass, where each of them is tinged with the same yellow colour, that any natural body exhibits: yet we cannot determine whether the diameter of the particles, upon which this body's colour depends, is equal to the thickness of the plate of air, or of water, or of glass, unless we could first determine whether the density of those particles is equal to the density of air, or to that of water, or to that of glass: since the particles must be larger, if their density is equal to the density of air, than if it is equal to the density of water, and larger if it is equal to that of water, than if it is equal to that of glass. And indeed we have good reason to conclude that the density of the parts, upon which the colours of natural bodies depend, is greater even than that of glass; and consequently that the diameter of those parts is much less than the thickness of a plate of glass, where it appears of the same colour with the body. For upon being viewed obliquely thin plates of glass change colour, whereas natural bodies do not: and the colour of natural bodies is made more unchangeable than that of thin plates of glass by their particles being more dense than glass is, by propositions 305, 310.

C H A P. XXIII.

Of the rainbow.

312. *When the rays of the sun fall upon a drop of rain and enter into it, some of them after one reflection and two refractions may come to the eye of a spectator, who has his back towards the sun and his face towards the drop.*

IF XY, Plat. XVII. fig. 1, is a drop of rain, and the sun shines upon it in any lines *sf*, *sd*, *sa*, &c. most of the rays will enter into the drop; some few of them only will be reflected from the first surface, by propositions 198, 200. Those rays, which are reflected from thence, do not come under our present consideration, because they are never refracted at all. The greatest part of the rays then enter the drop, and those passing on to the second surface will most of them be transmitted through the drop, by propositions 198, 200. But neither do those rays, which are thus transmitted, fall under our present consideration, since they are not reflected. For the rays, which are described in the proposition, are such, as are twice refracted and once reflected. However, at the second surface, or hinder part of the drop, at *pg* some few rays will be reflected, by propositions 198, 200, whilst the rest are transmitted: those rays proceed

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in some such lines as nr , nq , and coming out of the drop in the lines rv , qt , may fall upon the eye of a spectator, who is placed any where in those lines, with his face towards the drop, and consequently with his back towards the sun, which is supposed to shine upon the drop in the lines sf , sd , sa , &c. These rays are twice refracted and once reflected; they are refracted, when they pass out of the air into the drop, they are reflected from the second surface, and are refracted again, when they pass out of the drop into the air.

313. *When rays of light reflected from a drop of rain come to the eye, those are called effectual, which are able to excite a sensation.*

314. *When rays of light come out of a drop of rain, they will not be effectual, unless they are parallel and contiguous.*

There are but few rays, that can come to the eye at all, by proposition 312. For the greatest part of those rays which enter the drop xy , Plat. XVII. fig. 1, between x and a , pass out of the drop through the hinder surface pg ; only few are reflected from thence and come out through the nearer surface between a and y . Now such rays as emerge, or come out of the drop, between a and y will be ineffectual, unless they are parallel to one another as rv and qt are: because such rays as come out diverging from one another, will be so far asunder, when they come to the eye, that all of them cannot enter the pupill; and the very few, that can enter it, will not be sufficient to excite any sensation. But even rays, which are parallel, as rv , qt , will not be effectual, unless there are several of them contiguous or very near to one another. The two rays rv and qt alone will not be perceived, though both of them enter the eye: for so very few rays are not sufficient to excite a sensation.

315. *When rays of light come out of a drop of rain after one reflection, those will be effectual, which are reflected from the same point, and which entered the drop near to one another.*

Any rays as sb and cd , Plat. XVII. fig. 2, when they have passed out of the air into a drop of water will be refracted towards the perpendiculars bl , dl , by proposition 37; and as the ray sb falls farther from the axis av than the ray cd , sb will be more refracted than cd , as was shewn in a similar case in proposition 57; so that these rays, though parallel to one another at their incidence may describe the lines be , and de after refraction, and be both of them reflected from one and the same point e . Now all rays, which are thus reflected from one and the same point, when they have

have described the lines ef , eg , and after reflection emerge at f and g , will be so refracted, when they pass out of the drop into the air, as to describe the lines fb , gi , parallel to one another. If these rays were to return from e in the lines eb , ed , and were to emerge at b and d , they would be refracted into the lines of their incidence bs , dc by proposition 39. But if these rays, instead of being returned in the lines eb , ed , are reflected from the same point e in the lines eg , ef , the lines of reflection eg and ef will be inclined both to one another and to the surface of the drop, just as much as the lines eb and ed are. First eb and eg make just the same angle with the surface of the drop, for the angle bex , which eb makes with the surface of the drop, is the complement of incidence, and the angle gey , which eg makes with the surface, is the complement of reflection; and these two are equal to one another, by proposition 201. In the same manner we might prove that ed and ef make equal angles with the surface of the drop. Secondly the angle bed is equal to the angle feg , or the reflected rays eg , ef , and the incident rays be , de are equally inclined to each other. For the angle of incidence bel is equal to the angle of reflection gel , and the angle of incidence del is equal to the angle of reflection fel , by proposition 201. Consequently the difference between the angles of incidence is equal to the difference between the angles of reflection, or $bel - del = gel - fel$ or $bed = gef$. Since therefore either the lines eg , ef or the lines eb , ed are equally inclined both to one another and to the surface of the drop; the rays will be refracted in the same manner, whether they were to return in the lines eb , ed , or are reflected in the lines eg , ef . But if they were to return in the lines eb , ed , the refraction, when they emerge at b and d would make them parallel. Therefore if they are reflected from one and the same point e in the lines eg , ef , the refraction, when they emerge at g and f , will likewise make them parallel.

But though such rays, as are reflected from the same point in the hinder part of a drop of rain, are parallel to one another, when they emerge, and so have one condition that is requisite towards making them effectual; yet there is another condition necessary: for, by proposition 314, rays, that are effectual, must be contiguous, as well as parallel. And though rays, which enter the drop in different places, may be parallel, when they emerge, those only will be contiguous, which enter it nearly at the same place.

Let xy , fig. 1, be a drop of rain, ag the axis or diameter of the drop and sa a ray of light, that comes from the sun and enters the drop at the point a . This ray sa , because it is perpendicular to both the surfaces, will pass strait through the drop in the line agh without being refracted, by pro-

position 41. But any collateral rays that fall about sb , as they pass through the drop, will be made to converge to their axis, and passing out at n will meet the axis at b , by proposition 82. Rays which fall farther from the axis than sb , such as those which fall about sc , will likewise be made to converge; but then their focus will be nearer to the drop than b , by proposition 57. Suppose therefore i to be the focus to which the rays that fall about sc will converge, any ray sc , when it has described the line co within the drop and is tending to the focus i , will pass out of the drop at the point o . The rays, that fall upon the drop about sd more remote still from the axis, will converge to a focus still nearer than i , as suppose at k , by proposition 57. These rays therefore go out of the drop at p . The rays, that fall still more remote from the axis as se , will converge to a focus nearer than k , as suppose at l ; and the ray se , when it has described the line eo within the drop and is tending to l , will pass out at the point o . The rays, that fall still more remote from the axis, will converge to a focus still nearer. Thus the ray sf will after refraction converge to a focus at m , which is nearer than l , and having described the line fn within the drop, it will pass out at the point n . Now here we may observe that as any rays sb or sc , fall farther above the axis sa , the points n , or o , where they pass out behind the drop, will be farther above g , or that as the incident ray rises from the axis sa , the arc gno encreases: till we come to some ray sd , which passes out of the drop at p , and this is the highest point where any ray, that falls upon the quadrant or quarter ax , can pass out: for any rays se , or sf , that fall higher than sd , will not pass out in any point above p , but at the points o , or n , which are below it. Consequently though the arc $gnop$ encreases, whilst the distance of the incident ray from the axis sa encreased, till we came to the ray sd ; yet afterwards the higher the ray falls above the axis sa , this arc $pong$ will decrease.

We have hitherto spoken of the points on the hinder part of the drop, where the rays pass out of it; but this was for the sake of determining the points from whence those rays are reflected, which do not pass out behind the drop. For, in explaining the rain-bow, we have no farther reason to consider those rays, which go through the drop; since they can never come to the eye of a spectator placed any where in the lines rv or qt with his face towards the drop. Now as there are many rays, which pass out of the drop between g and p , so, by proposition 200, some few rays will be reflected from thence; and consequently the several points between g and p , which are the points, where some of the rays pass out of the drop, are likewise the points of reflection for the rest, which do
not

not pass out. Therefore in respect of those rays, which are reflected, we may call gp the arc of reflection, and may say that this arc of reflection encreases, as the distance of the incident ray from the axis sa encreases, till we come to the ray sd , the arc of reflection is gn for the ray sb , it is go for the ray sc , and gp for the ray sd . But after this, as the distance of the incident ray from the axis sa encreases, the arc of reflection decreases; for og less than pg is the arc of reflection for the ray se , and ng is the arc of reflection for the ray sf .

From hence it is obvious, that some one ray, which falls above sd , may be reflected from the same point with some other ray, which falls below sd . Thus for instance the ray sb will be reflected from the point n , and the ray sf will be reflected from the same point; and consequently, when the reflected rays nr , nq are refracted as they pass out of the drop at r , and q , they will be parallel, by what has been shewn in the former part of this proposition. But since the intermediate rays, which enter the drop between sf and sb , are not reflected from the same point n , these two rays alone will be parallel to one another, when they come out of the drop; and the intermediate rays will not be parallel to them. And consequently these rays rv , qt , though they are parallel, after they emerge at r and q , will not be contiguous, and for that reason will not be effectual, by proposition 314. The ray sd is reflected from p , which has been shewn to be the limit of the arc of reflection; such rays, as fall just above sd and just below sd , will be reflected from nearly the same point p , as appears from what has been already shewn. These rays therefore, will be parallel, because they are reflected from the same point p , and they will likewise be contiguous, because they all of them enter the drop at one and the same place very near to d . Consequently such rays, as enter the drop at d and are reflected from p the limit of the arc of reflection, will be effectual, by proposition 314, since when they emerge at the fore part of the drop between a and y they will be both parallel and contiguous.

If we can make out hereafter that the rain bow is produced, by the rays of the sun, which are thus reflected from drops of rain, as they fall whilst the sun shines upon them, this proposition may serve to shew us that this appearance is not produced by any rays, that fall upon any part and are reflected from any part of those drops: since this appearance cannot be produced by any rays but those, which are effectual; and effectual rays must always enter each drop at one certain place in the fore-part of it, and must likewise be reflected from one certain place in the hinder surface.

316. *When rays, that are effectual, emerge from a drop of rain after one reflection and two refractions; those, which are most refrangible, will, at their emerſion, make a leſs angle with the incident rays, than thoſe do, which are leaſt refrangible: and by this means the rays of different colours will be ſeparated from one another.*

Let fb and gi , Plat. XVII. fig. 2, be effectual violet rays emerging from the drop at fg , and fn , gp , effectual red rays emerging from the ſame drop at the ſame place. Now though all the violet rays are parallel to one another by propoſition 314, becauſe they are ſuppoſed effectual; and though all the red rays are likewiſe parallel to one another for the ſame reaſon; yet the violet rays will not be parallel to the red rays. Theſe rays, as they have different colours, and different degrees of refrangibility, will diverge from one another, any violet ray gi , which emerges at g , will diverge from any red ray gp , which emerges at the ſame place. Now both the violet ray gi and the red ray gp , as they paſs out of the drop of water into the air, will be refracted from the perpendicular lo . But the violet ray is more refrangible than the red one, by propoſition 282, and for that reaſon gi , or the refracted violet ray, will make a greater angle with the perpendicular than gp the refracted red ray; or the angle igo will be greater than the angle pgo . Suppoſe the incident ray sb to be continued in the direction sk , and the violet ray ig to be continued backwards in the direction ik , till it meets the incident ray at k . Suppoſe likewiſe the red ray pg to be continued backwards in the ſame manner, till it meets the incident ray at w . The angle iks is that which the violet ray, or moſt refrangible ray at its emerſion, makes with the incident ray: and the angle pws is that which the red ray, or leaſt refrangible ray at its emerſion, makes with the incident ray. The angle iks is leſs than the angle pws . For, in the triangle gwk , gws or pws is the external angle at the baſe, and gkw or iks is one of the internal oppoſite angles: and either internal oppoſite angle is leſs than the external angle at the baſe. Euc. b. I. prop. 16. What has been ſhewn to be true of the rays gi and gp might be ſhewn in the ſame manner of the rays fb and fn , or of any other rays that emerge reſpectively parallel to gi and gp . But all the effectual violet rays are parallel to gi , and all the effectual red rays are parallel to gp , by propoſition 314. Therefore the effectual violet rays at their emerſion make a leſs angle with the incident ones than the effectual red ones. And for the ſame reaſon in all the other ſorts of rays thoſe, which are moſt refrangible, at their emerſion from a drop of rain after one reflection, will make a leſs angle with the incident rays, than thoſe do, which are leſs refrangible.

Or

Or otherwise. When the rays gi and gp emerge at the same point g , as they both come out of water into air and consequently are refracted from a perpendicular, instead of going strait forwards in the line eg continued, they will both be turned round upon the point g from the perpendicular go . Now it is easy to conceive that either of these lines might be turned in this manner upon the point g as upon a center, till they became parallel to sb the incident ray. But if either of these lines or rays were refracted so much from go as to become parallel to sb , the ray so much refracted would, after emerfion, make no angle with sk , because it would be parallel to it. And consequently that ray, which is most turned round upon the point g , or that ray, which is most refrangible, will after emerfion be nearest parallel to the incident ray, or will make the least angle with it. The same may be proved of all other rays emerging parallel to gi and gp respectively, or of all effectual rays; those, which are most refrangible, will after emerfion make a less angle with the incident rays, than those do, which are least refrangible.

But since the effectual rays of different colours make different angles with sk at their emerfion, they will be separated from one another: so that if the eye was placed in the beam $fghi$ it would receive only rays of one colour from the drop $xagy$, and if it was placed in the beam $fgnp$, it would receive only rays of some other colour.

The angle swp , which the least refrangible or red rays make with the incident ones, when they emerge so as to be effectual, is found by calculation to be 42 degrees 2 minutes. And the angle ski , which the most refrangible rays make with the incident ones, when they emerge so as to be effectual, is found to be 40 degrees 17 minutes. The rays, which have the intermediate degrees of refrangibility, make with the incident ones intermediate angles between 42 degrees 2 minutes and 40 degrees 17 minutes.

317. *If a line is supposed to be drawn from the center of the sun through the eye of the spectator, the angle, which any effectual ray after two refractions and one reflection makes with the incident ray, will be equal to the angle, which it makes with that line.*

Let the eye of the spectator be at i , Plat. XVII. fig. 2, and let qt be the line supposed to be drawn from the center of the sun through the eye of the spectator; the angle git , which any effectual ray makes with this line, will be equal to the angle iks , which the same ray makes with the incident ray sb or sk . If sb is a ray coming from the center of the sun, then since qt is supposed to be drawn from the same point, these two lines, upon account

count of the remoteness of the point from whence they are drawn, may be looked upon as parallel to one another, as will easily appear from proposition 30. But the right line *ki* crossing these two parallel lines will make the alternate angles equal. Euc. b. I. prop. 29. Therefore *kit* or *git* is equal to *ski*.

318. *When the sun shines upon the drops of rain as they are falling; the rays, that come from those drops to the eye of a spectator after one reflection and two refractions, produce the primary rainbow.*

If the sun shines upon the rain as it falls there are commonly seen two bows, as AFB, CHD, Plat. XVII. fig. 4, or if the cloud and rain does not reach over that whole side of the sky where the bows appear, then only a part of one or of both bows is seen in that place, where the rain falls. Of these two bows the innermost AFB is the more vivid of the two, and this is called the primary bow. The outer part TFY of the primary bow is red, the inner part VEX is violet, the intermediate parts, reckoning from the red to the violet, are orange, yellow, green, blue, and indigo. Suppose the spectators eye to be at O, and let LOP be an imaginary line drawn from the center of the sun through the eye of the spectator. If a beam of light S coming from the sun falls upon any drop F; and the rays, that emerge at F in the line FO so as to be effectual, make an angle FOP of 42 degrees 2 minutes with the line LP, then these effectual rays make an angle of 42 degrees 2 minutes with the incident rays, by proposition 317, and consequently these rays will be red, by proposition 316; so that the drop F will appear red. All the other rays, which emerge at F, and would be effectual if they fell upon the eye, are refracted more than the red ones and consequently will pass above the eye. If a beam of light S falls upon the drop E, and the rays, that emerge at E in the line EO so as to be effectual, make an angle EOP of 40 degrees 17 minutes with the line LP; then these effectual rays make likewise an angle of 40 degrees 17 minutes with the incident rays, by proposition 317, and the drop E will appear of a violet colour, by proposition 316. All the other rays, which emerge at E, and would be effectual if they came to the eye, are refracted less than the violet ones, and therefore pass below the eye. The intermediate drops between F and E will for the same reasons be of the intermediate colours.

Thus we have shewn why a set of drops from F to E, as they are falling, should appear of the primary colours, red, orange, yellow, green, blue, indigo, and violet. It is not necessary that the several drops, which produce these colours, should all of them fall at exactly the same distance from

from the eye. The angle FOP, for instance, is the same, whether the distance of the drop from the eye is OF, or whether it is in any other part of the line OF something nearer to the eye. And whilst the angle FOP is the same, the angle made by the emerging and incident rays, and consequently the colour of the drop will be the same. This is equally true of any other drop. So that although in the figure the drops F and E are represented as falling perpendicularly one under the other, yet this is not necessary in order to produce the bow.

But the coloured line FE, which we have already accounted for, is only the breadth of the bow. It still remains to be shewn, why not only the drop F should appear red, but why all the other drops quite from A to B in the arc ATFYB should appear of the same colour. Now it is evident that wherever a drop of rain is placed, if the angle, which the effectual rays make with the line LP, is equal to the angle FOP, that is, if the angle, which the effectual rays make with the incident rays, is 42 degrees 2 minutes, any of those drops will be red for the same reason that the drop F is of this colour.

If FOP was to turn round upon the line OP, so that one end of this line should always be at the eye and the other be at P opposite to the sun; such a motion of this figure would be like that of a pair of compasses turning round upon one of the legs OP with the opening FOP. In this revolution the drop F would describe a circle, P would be the center and ATFYB would be an arc in this circle. Now since in this motion of the line and drop OF the angle made by FO with OP, that is, the angle FOP, continues the same; if the sun was to shine upon this drop as it revolves, the effectual rays would make the same angle with the incident rays, in whatever part of the arc ATFYB the drop was to be, by proposition 317. Therefore whether the drop is at A, or at T, or at Y, or at B, or wherever else it is in this whole arc, it would appear red as it does at F. The drops of rain, as they fall, are not indeed turned round in this manner: but then, as innumerable of them are falling at once in right lines from the cloud, whilst one drop is at F, there will be others at Y, at T, at B, at A, and in every other part of the arc ATFYB: and all these drops will be red for the same reason that the drop F would have been red, if it had been in the same place. Therefore, when the sun shines upon the rain as it falls, there will be a red arc ATFYB opposite to the sun. In the same manner, because the drop E is violet, we might prove that any other drop, which, whilst it is falling, is in any part of the arc AVEXB, will be violet, and consequently, at the same time that the red arc ATFYB appears, there will likewise be a violet arc AVEXB, below or within it. FE is

the distance between these two coloured arcs; and from what has been said it follows that the intermediate space between these two arcs will be filled up with arcs of the intermediate colours, orange, yellow, blue, green and indigo. All these coloured arcs together make up the primary rainbow.

319. *The primary rainbow is never a greater arc than a semicircle.*

Since the line LOP is drawn from the sun through the eye of the spectator, and since P, Plat. XVII. fig. 4, as has been shewn in proposition 318, is the center of the rainbow; it follows that the center of the rainbow is always opposite to the sun. The angle FOP is an angle of 42 degrees 2 minutes, as was observed under proposition 316; or F the highest part of the bow is 42 degrees 2 minutes from P the center of it. If the sun is more than 42 degrees 2 minutes high, P the center of the rainbow, which is opposite to the sun, will be more than 42 degrees 2 minutes below the horizon; and consequently F the top of the bow, which is only 42 degrees 2 minutes from P, will be below the horizon; that is, when the sun is more than 42 degrees 2 minutes high, no primary rainbow will be seen. If the sun is something less than 42 degrees 2 minutes high, then P will be something less than 42 degrees 2 minutes below the horizon; and consequently F, which is only 42 degrees 2 minutes from P, will be just above the horizon, that is, a small part of the bow at this height of the sun will appear close to the ground opposite to the sun. If the sun is 20 degrees high, then P will be 20 degrees below the horizon; and F the top of the bow being 42 degrees 2 minutes from P will be 22 degrees 2 minutes above the horizon: therefore at this height of the sun, the bow will be an arc of a circle, whose center is below the horizon; and consequently that arc of the circle, which is above the horizon, or the bow, will be less than a semicircle. If the sun is in the horizon, then P the center of the bow will be in the opposite part of the horizon, F the top of the bow will be 42 degrees 2 minutes above the horizon, and the bow itself, because the horizon passes through the center of it will be a semicircle. More than a semicircle can never appear; because if the bow was more than a semicircle, P the center of it must be above the horizon: but P is always opposite to the sun, therefore P cannot be above the horizon, unless the sun is below it: and when the sun is set or is below the horizon, it cannot shine upon the drops of rain, as they fall, and consequently when the sun is below the horizon, no bow at all can be seen.

320, *When the rays of the sun fall upon a drop of rain, some of them after two reflections and two refractions may come to the eye of a spectator, who has his back towards the sun and his face towards the drop.*

If

If bgw , Plat. XVII, fig. 3, is a drop of rain, and parallel rays coming from the sun, as zv , yw , fall upon the lower part of it, they will be refracted towards the perpendiculars vl , wl , as they enter into it, by proposition 37, and will describe some such lines as vb , wi . At b and i great part of these rays will pass out of the drop; but some of them will be reflected from thence in the lines bf , ig , by proposition 200. At f and g again, great part of the rays, that were reflected thither, will pass out of the drop, as those do, which were described in proposition 312. But these rays will not come to the eye of a spectator at o . However here again all the rays will not pass out, but some few will be reflected from f and g in some such lines as fd , gb , by proposition 200, and these, when they emerge out of the drop of water into the air at b and d , will be refracted from the perpendiculars, and describing the lines dt , bo , may come to the eye of a spectator, who has his back towards the sun and his face towards the drop.

321. *Those rays, which are parallel to one another after they have been once refracted and once reflected in a drop of rain, will be effectual, when they emerge after two refractions and two reflections.*

No rays can be effectual, by proposition 314, unless they are contiguous, and parallel. From what was said, under proposition 315, it appears, that when rays come out of a drop of rain contiguous to one another, either after one or after two reflections, they must enter the drop nearly at one and the same place. And if such rays, as are contiguous, are parallel after the first reflection, they will emerge parallel, and therefore will be effectual. Let zv and yw be contiguous rays, which come from the sun and are parallel to one another, when they fall upon the lower part of the drop bgw , Plat. XVII. fig. 3. Suppose these rays to be refracted at v and w and to be reflected at b and i ; if they are parallel to one another, as bf , gi , after this first reflection, then, after they are reflected a second time from f and g , and refracted a second time as they emerge at d and b , they will go out of the drop parallel to one another in the lines dt and bo and will therefore be effectual.

The rays zv , yw , are refracted towards the perpendiculars vl , wl , when they enter the drop, by proposition 37, and will be made to converge, by proposition 48. As these rays are very oblique, their focus will not be far from the surface vw . If this focus is at k , the rays, after they have passed the focus, will diverge from thence in the directions kb , ki , and if ki is the principal focal distance of the concave reflecting surface bi , the reflected rays bf , ig , will be parallel, by proposition 211. These rays bf , ig , are reflected again from the concave surface fg , and will meet in

a focus at e , so that ge , will be the principal focal distance of this reflecting surface fg , by proposition 210. And because bi and fg are parts of the same sphere the principal focal distances ge , and ki will be equal to one another. When the rays have passed the focus e , they will diverge from thence in the lines ed , eb : and we are to shew that, when they emerge at d and b and are refracted there, they will become parallel.

Now if the rays vk , wk , when they have met at k , were to be turned back again in the directions kv , kw , and were to emerge at v and w , they would be refracted into the lines of their incidence vz , wy , by proposition 40, and therefore would be parallel. But since ge is equal to ik , as has already been shewn; the rays ed , eb , that diverge from e , fall in the same manner upon the drop at d and b , as the rays kv , kw , would fall upon it at v and w ; and ed eb are just as much inclined to the refracting surface db , as kv , kw , would be to the surface vw . From hence it follows that the rays ed , eb , emerging at d and b will be refracted in the same manner, and will have the same direction in respect of one another, as kv , kw , would have. But kv and kw would be parallel after refraction. Therefore ed and eb will emerge in lines dp , bo , so as to be parallel to one another, and consequently so as to be effectual.

322. *When rays, that are effectual, emerge from a drop of rain after two reflections and two refractions, those, which are most refrangible, will at their emersion make a greater angle with the incident rays, than those do, which are least refrangible: and by this means the rays of different colours will be separated from one another.*

If rays of different colours, which, by proposition 282, are differently refrangible, emerge at any point b , Plat. XVII. fig. 3, these rays will not be all of them equally refracted from the perpendicular. Thus if bo is a red ray, which is of all others the least refrangible, and bm is a violet ray, which is of all others the most refrangible, when these two rays emerge at b the violet ray will be refracted more from the perpendicular bx than the red ray, and the refracted angle xbm will be greater than the refracted angle xbo . From hence it follows that these two rays after emersion will diverge from one another. In like manner, the rays, that emerge at d , will diverge from one another; a red ray will emerge in the line dp , a violet ray in the line dt . So that though all the effectual red rays of the beam $bdmt$ are parallel to one another, and all the effectual red rays of the beam $bdop$ are likewise parallel to one another, yet the violet rays will not be parallel to the red ones, but the violet beam will diverge from the red beam. Thus the rays of different colours will be separated from one another.

This

This will appear farther, if we consider, what the proposition affirms, that any violet or most refrangible ray will make a greater angle with the incident rays, than any red or least refrangible ray makes with the same incident rays. Thus if yw is an incident ray, bm a violet ray emerging from the point b , and bo a red ray emerging from the same point; the angle, which the violet ray makes with the incident one, is ym , and that, which the red ray makes with it, is yo . Now ym is a greater angle than yo . For in the triangle brs the internal angle brs is less than bsy the external angle at the base. Euc. b. I. prop. 16. But ym is the complement of brs or of bry to two right ones, and yo is the complement of bsy to two right ones. Therefore, since bry is less than bsy , the complement of bry to two right angles will be greater than the complement of bsy to two right angles; or ym will be greater than yo .

Or otherwise. Both the rays bo and bm , when they are refracted in passing out of the drop at b , are turned round upon the point b from the perpendicular bx . Now either of these lines bo or bm might be turned round in this manner, till it made a right angle with yw . Consequently that ray, which is most turned round upon b or which is most refracted, will make an angle with yw that will be nearer to a right one, than that ray makes with it, which is least turned round upon b or which is least refracted. Therefore that ray, which is most refracted will make a greater angle with the incident ray, than that which is least refracted.

But since the emerging rays, as they are differently refrangible, make different angles with the same incident ray yw , the refraction, which they suffer at emerfion, will separate them from one another.

The angle ym , which the most refrangible or violet rays make with the incident ones, is found by calculation to be 54 degrees 7 minutes: and the angle yo , which the least refrangible or red rays make with the incident ones, is found to be 50 degrees 57 minutes: the angles, which the rays of the intermediate colours, indigo, blue, green, yellow, and orange make with the incident rays, are intermediate angles between 54 degrees 7 minutes and 50 degrees 57 minutes.

323. *If a line is supposed to be drawn from the center of the sun through the eye of the spectator; the angle, which after two refractions and two reflections any effectual ray makes with the incident ray, will be equal to the angle, which it makes with that line.*

If yw , Plat. XVII. fig. 3, is an incident ray, bo an effectual ray, and qn a line drawn from the center of the sun through o the eye of the spectator; the angle yo , which the effectual ray makes with the incident ray,

is equal to *son* the angle, which the same effectual ray makes with the line *qn*. For *yw* and *qn* considered as drawn from the center of the sun are parallel, by proposition 30, *bo* crosses them, and consequently makes the alternate angles *yso*, *son* equal to one another. Euc. b. I. prop. 29.

324. *When the sun shines upon the drops of rain as they are falling; the rays, that come from those drops to the eye of a spectator after two reflections and two refractions, produce the secondary rainbow.*

The secondary rainbow is the outermost CHD, Plat. XVII. fig. 4. When the sun shines upon a drop of rain H; and the rays HO, which emerge at H so as to be effectual, make an angle HOP of 54 degrees 7 minutes with LOP a line drawn from the sun through the eye of the spectator, the same effectual rays will make likewise an angle of 54 degrees 7 minutes with the incident rays S, by proposition 323; and the rays which emerge at this angle are violet ones, by what was observed under proposition 322. Therefore, if the spectators eye is at O, none but violet rays will enter it: for as all the other rays make a less angle with OP, by propositions 322, 323, they will fall above the spectators eye. In like manner, if the effectual rays, that emerge from the drop G, make an angle of 50 degrees 57 minutes with the line OP, they will likewise make the same angle with the incident rays S, by proposition 323, and consequently from the drop G to the spectators eye at O no rays will come but red ones, by proposition 322; for all the other rays making a greater angle with the line OP, by propositions 322, 323, will fall below the eye at O. For the same reason the rays emerging from the intermediate drops between H and G, and coming to the spectators eye at O, will emerge at intermediate angles, and therefore, by proposition 322, will have the intermediate colours. Thus, if there are seven drops from H to G inclusively, their colours will be violet, indigo, blue, green, yellow, orange and red. This coloured line is the breadth of the secondary rainbow.

Now if HOP was to turn round upon the line OP, like a pair of compasses upon one of the legs OP with the opening HOP, it is plane from the supposition that, in such a revolution of the drop H, the angle HOP would be the same, and consequently the emerging rays would make the same angle with the incident ones. But in such a revolution the drop would describe a circle of which P would be the center and CNHRD an arc. Consequently, since, when the drop is at N, or at R, or any where else in that arc, the emerging rays make the same angle with the incident ones, as when the drop is at H, the colour of the drop will

will be the same to an eye placed at O, whether the drop is at N, or at H, or at R, or any where else in that arc. Now though the drop does not thus turn round as it falls, and does not pass through the several parts of this arc, yet, since there are drops of rain falling every where at the same time, when one drop is at H, there will be another at R, another at N, and others in all parts of the arc, and these drops will all of them be violet-coloured, for the same reason that the drop H would have been of this colour if it had been in any of those places. In like manner as the drop G is red, when it is at G, it would likewise be red in any part of the arc CWGQD; and so will any other drop, when, as it is falling, it comes to any part of that arc. Thus as the sun shines upon the rain, whilst it falls, there will be two arcs produced, a violet-coloured one, CNHRD, and a red one CWGQD; and for the same reasons the intermediate space between those two arcs will be filled up with arcs of the intermediate colours. All these arcs together make up the secondary rainbow.

325. *The colours of the secondary rainbow are fainter than those of the primary rainbow; and are ranged in the contrary order.*

The primary rainbow is produced by such rays, as have been only once reflected; the secondary rainbow is produced by such rays, as have been twice reflected, by propositions 318, 324. But at every reflection some rays pass out of the drop of rain without being reflected, by proposition 200: so that the oftener the rays are reflected, the fewer of them are left. Therefore the colours of the secondary bow are produced by fewer rays, and consequently will be fainter, than the colours of the primary bow.

In the primary bow, reckoning from the outside of it, the colours are ranged in this order, red, orange, yellow, green, blue, indigo, violet. In the secondary bow, reckoning from the outside, the colours are violet, indigo, blue, green, yellow, orange, red. So that the red, which is the outermost or highest colour in the primary bow, is the innermost or lowest colour in the secondary one.

Now the violet rays, when they emerge so as to be effectual after one reflection, make a less angle with the incident rays, than the red ones, by proposition 316; consequently the violet rays make a less angle with the lines OP, Plat. XVII. fig. 4, than the red ones, by proposition 317. But in the primary rainbow the rays are only once reflected, and the angle, which the effectual rays make with OP, is the distance of the coloured drop from P the center of the bow, by proposition 318. Therefore the violet drops or violet arc in the primary bow will be nearer to
the

the center of the bow, than the red drops or red arc; that is, the innermost colour in the primary bow will be violet, and the outermost colour will be red. And for the same reason through the whole primary bow every colour will be nearer to the center P, as the rays of that colour are more refrangible.

But the violet rays, when they emerge so as to be effectual after two reflections, make a greater angle with the incident rays, than the red ones, by proposition 322; consequently the violet rays will make a greater angle with the line OP, than the red ones, by proposition 323. But in the secondary rainbow the rays are twice reflected, and the angle, which the effectual rays make with OP, is the distance of the coloured drop from P the center of the bow, by proposition 324. Therefore the violet drops or violet arc in the secondary bow will be farther from the center of the bow, than the red drops or red arc; that is, the outermost colour in the secondary bow will be violet and the innermost colour will be red. And for the same reason through the whole secondary bow every colour will be farther from the center P, as the rays of that colour are more refrangible.

326. *The secondary rainbow is never a greater arc than a semicircle.*

This is proved in the same manner as proposition 319. The only difference between the primary and secondary rainbows in this respect is this: in the primary bow the rays of the highest colour make an angle of 42 degrees 2 minutes with the line OP, or the highest colour is 42 degrees 2 minutes from P the center of the bow: and for this reason the bow does not begin to appear, till the suns height is less than 42 degrees 2 minutes; and, when the sun is in the horizon on one side, the top of the bow will be 42 degrees 2 minutes high on the opposite side of the heavens. But the rays of the highest colour in the secondary bow make an angle of 54 degrees 7 minutes with the line OP, or the highest colour is 54 degrees 7 minutes from P the center of the bow: for this reason the secondary bow will begin to appear, when the height of the sun is less than 54 degrees 7 minutes; and, when the sun is in the horizon on one side, the center of the secondary bow will be in the horizon and the top of the bow will be 54 degrees 7 minutes above it.

The end of optics.

PLATE XVII.

Fig. 1.

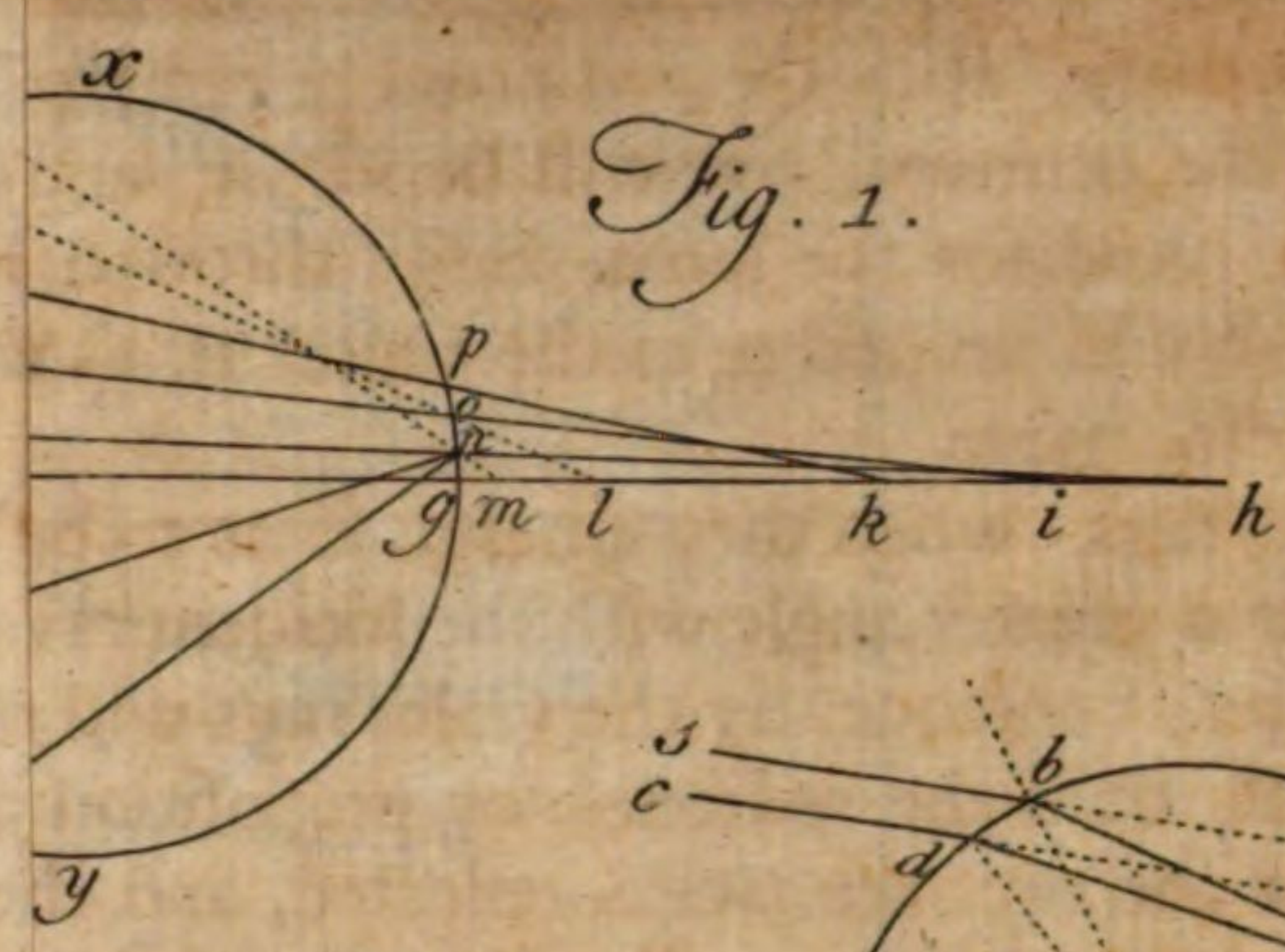
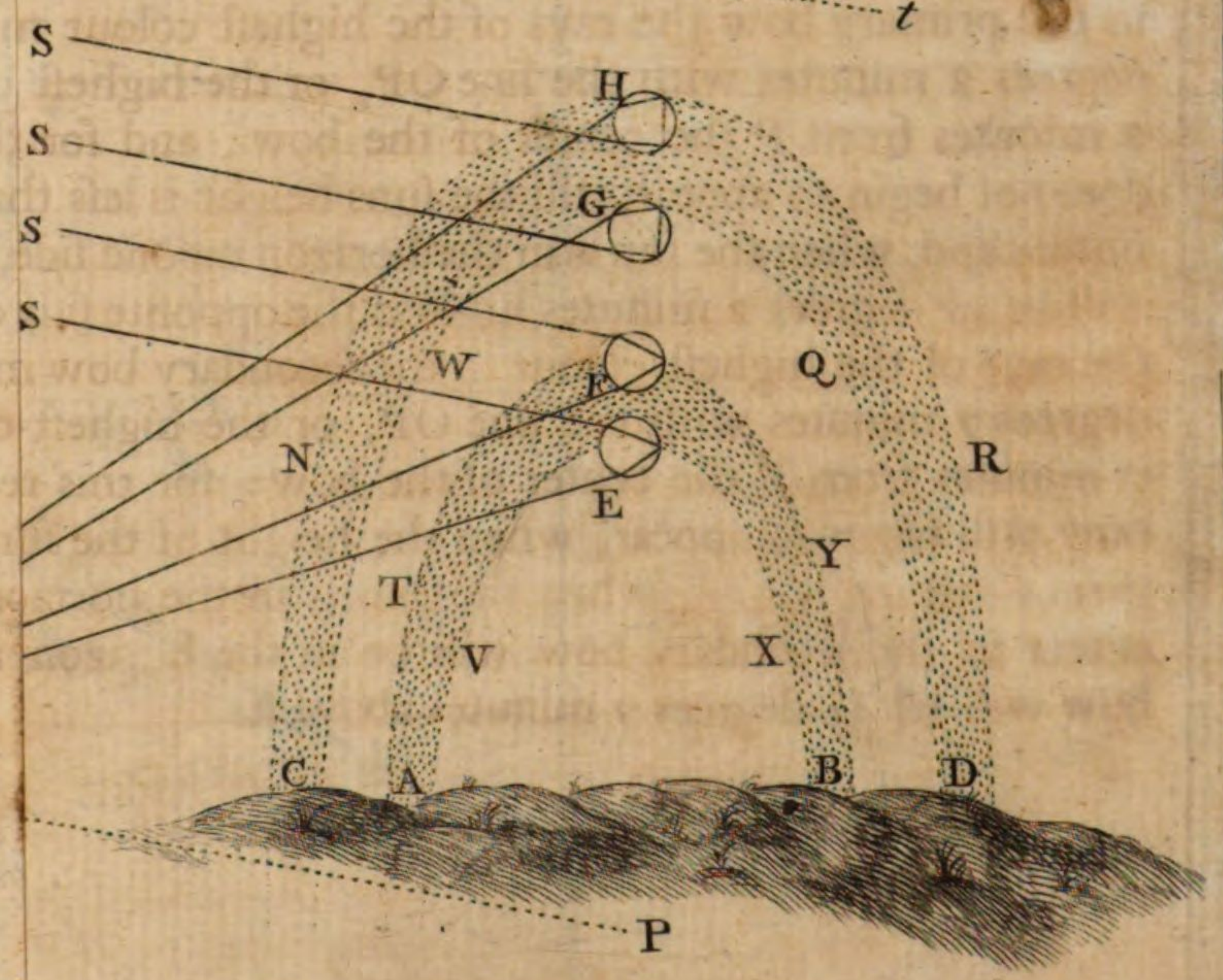
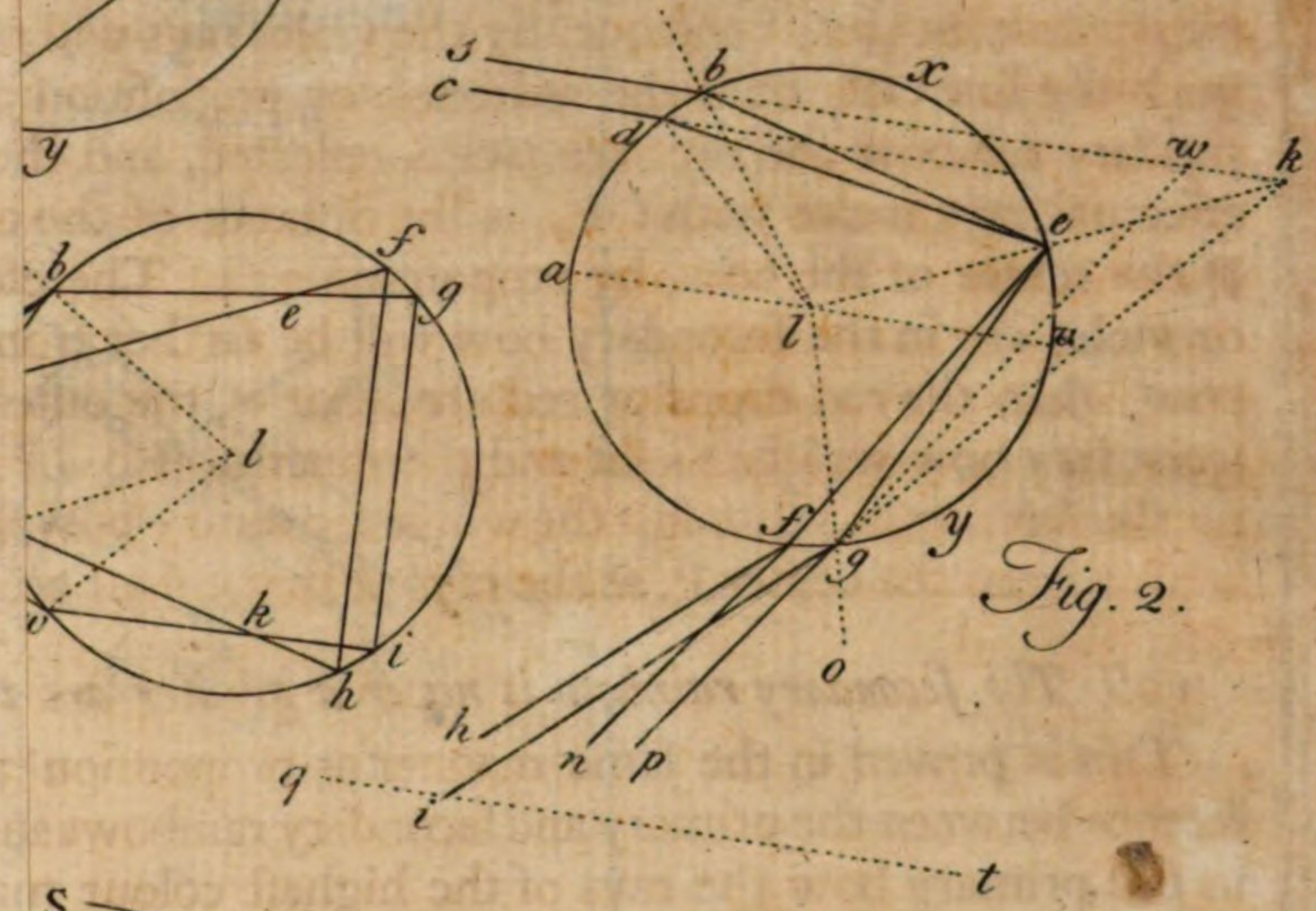


Fig. 2.





Rutherford, Thomas,

A System Of Natural Philosophy Being A Course of Lectures In Mechanics,
Optics, Hydrostatics, and Astronomy; Which are read in St Johns College
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